Free Vibration of an SDF system

The equation of motion equilibrium of free vibration of an SDF system is given

$$m\ddot{u} + c\dot{u} + ku = 0$$

1 Undamped Free Vibration

1.1 Solution of Equation of Motion of Equilibrium

Setting $c = 0$, the equation becomes

$$\begin{cases} m\ddot{u} + ku = 0 \\ u|_{t=0} = u(0) \quad \dot{u}|_{t=0} = \dot{u}(0) \end{cases}$$

The solution of the equation has the form $u = e^{st}$, where the constant s is unknown. Substitution into the equation gives

$$(ms^2 + k)e^{st} = 0$$

Then $ms^2 + k = 0 \Rightarrow s_{1,2} = \pm i\omega_n$, $\omega_n = \sqrt{k/m}$, the general solution of the equation is

$$u(t) = A_1 e^{st} + A_2 e^{-st} \text{, or } u(t) = A\cos\omega_n t + B\sin\omega_n t$$

And the differentiation of the expression yields $\dot{u}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t$

Set $t = 0$, there is

$$\begin{cases} u(0) = A \\ \dot{u}(0) = \omega_n B \end{cases} \Rightarrow \begin{cases} A = u(0) \\ B = \dot{u}(0)/\omega_n \end{cases}$$

So there is

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

1.2 Definition

Natural circular frequency of vibration

$$\omega_n = \sqrt{k/m}$$

simple harmonic motion

the motion which has the form $u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$

Natural period of vibration

$$T_n = \frac{2\pi}{\omega_n}$$

Natural cyclic frequency of vibration

$$f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi}$$

Amplitude of motion

$$u_0 = \sqrt{[u(0)]^2 + [\frac{\dot{u}(0)}{\omega_n}]^2}$$

1.3 Discussion

(1). The natural vibration properties ω_n, T_n, and f_n depend only on the mass and stiffness of the structure, they are independent of the initial displacement and velocity

(2). Alternative form of these properties

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}}, \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}, \quad T_n = 2\pi\sqrt{\frac{\delta_{st}}{g}},$$

where $\delta_{st} = mg/k$ is the static deflection of the mass m suspended from a spring of stiffness k (3). The amplitude u_o depends on the initial displacement and velocity.

2 Viscously Damped Free Vibration

the equation is

$$\begin{cases} m\ddot{u} + c\dot{u} + ku = 0 \\ u\,|_{t=0} = u(0) \quad \dot{u}\,|_{t=0} = \dot{u}(0) \end{cases}$$

Make $\omega_n = \sqrt{\frac{k}{m}}$, $\varsigma = \frac{c}{2m\omega_n} = \frac{c}{c_{cr}}$, where $c_{cr} = 2m\omega_n = 2\sqrt{km} = \frac{2k}{\omega_n}$

$$\ddot{u} + 2\varsigma\omega_n\dot{u} + \omega_n^2 u = 0$$

2.1 Types of Motion

If $c \ge c_{cr}$ or $\varsigma \ge 1$, the system returns to its equilibrium position without oscillating

If $c < c_{cr}$ or $\varsigma < 1$, the system oscillates about its equilibrium position with progressively decreasing amplitude

2.2 Solution of the Equation for Underdamped System

If $\varsigma < 1$, set $u = e^{st}$, and

$$(s^2 + 2\varsigma\omega_n s + \omega_n^2)e^{st} = 0 \Rightarrow s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

The **characteristic equation** gives two roots

$$s_{1,2} = \omega_n(-\varsigma \pm i\sqrt{1-\varsigma^2})$$

Hense the general solution is

$$u(t) = e^{-\varsigma\omega_n t}(A\cos\omega_D t + B\sin\omega_D t), \text{ where } \omega_D = \omega_n\sqrt{1-\varsigma^2}$$

so

$$\dot{u}(t) = -\varsigma\omega_n e^{-\varsigma\omega_n t}(A\cos\omega_D t + B\sin\omega_D t) + \omega_n e^{-\varsigma\omega_n t}(-A\sin\omega_n t + B\cos\omega_n t)$$

When $t = 0$,

$$\begin{cases} u(0) = A \\ \dot{u}(0) = -\varsigma\omega_n A + B\omega_n \end{cases} \Rightarrow \begin{cases} A = u(0) \\ B = \dfrac{\dot{u}(0) + \varsigma\omega_n u(0)}{\omega_n} \end{cases}$$

The solution of the equation is

$$u(t) = e^{-\varsigma\omega_n t}[u(0)\cos\omega_D t + \frac{\dot{u}(0) + \varsigma\omega_n u(0)}{\omega_D}\sin\omega_D t]$$

2.3 Definition

Critical damping coefficient

$$c_{cr} = 2m\omega_n = 2\sqrt{km} = \frac{2k}{\omega_n}$$

Damping ratio (fraction of critical damping)

$$\varsigma = \frac{c}{2m\omega_n} = \frac{c}{c_{cr}}$$

Underdamped system

A damped system in which $c < c_{cr}$ or $\varsigma < 1$

Natural frequency of damped vibration

$$\omega_D = \omega_n \sqrt{1-\varsigma^2} < \omega_n$$

Natural period of damped vibration

$$T_D = \frac{T_n}{\sqrt{1-\varsigma^2}} > T_n$$

Displacement amplitude of the damped system

$$u_D = \pm \rho e^{-\varsigma\omega_n t}, \text{ where } \rho = \sqrt{[u(0)]^2 + [\frac{\dot{u}(0) + \varsigma\omega_n u(0)}{\omega_D}]^2}$$

2.4 Discussion

(1). Damping has the effect of lowering the natural frequency from ω_n to ω_D and lengthening the natural period from T_n to T_D. **These effects are negligible for damping ratios below 0.2(20%)**

(2). Decay of motion is found from the expression of amplitude of free vibration.

The ratio of the displacement at time *t* to its value at time $t + T_D$ is independent of *t*, which is given by

$$\frac{u(t)}{u(t+T_D)} = \exp(\varsigma\omega_n T_D) = \exp(\frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}})$$

The natural logarithm of this ratio is

$$\delta = \ln\frac{u_i}{u_{i+1}} = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}} \approx 2\pi\varsigma(\varsigma < 0.2)$$

2.5 Energy in Free Vibration

2.5.1 Input Energy

$$E_I = \frac{1}{2}k[u(0)]^2 + \frac{1}{2}m[\dot{u}(0)]^2$$

2.5.2 Energy at Any Instant

The total energy in a vibrating system is made up of two parts, kinetic energy E_K and potential energy equal to the strain energy E_S, which is expressed by

$$E_K = \frac{1}{2}m[\dot{u}(t)]^2 \qquad E_S = \frac{1}{2}k[u(t)]^2$$

For $u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$, the upper energy is derived by

$$E_K(t) = \frac{1}{2}m\omega_n^2[-u(0)\sin\omega_n t + \frac{\dot{u}(0)}{\omega_n}\cos\omega_n t]^2$$

$$E_S(t) = \frac{1}{2}k[u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t]^2$$

The total energy is

$$E_K + E_S = \frac{1}{2}m\omega_n^2[-u(0)\sin\omega_n t + \frac{\dot{u}(0)}{\omega_n}\cos\omega_n t]^2 + \frac{1}{2}k[u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t]^2$$

$$= \frac{1}{2}k[\frac{\dot{u}(0)^2}{\omega_n^2} + u(0)^2] = \frac{1}{2}ku(0)^2 + \frac{1}{2}m\dot{u}(0)^2 = E_I$$

2.5.3 Energy Dissipated in Viscous Damping

$$E_D = \int f_D du = \int_0^{t_1}(c\dot{u})\dot{u}dt = \int_0^{t_1} c\dot{u}^2 dt$$