华东师范大学2003数学分析

To my parents

1 简答题

1.1 $\lim\limits_{x\to 1}\dfrac{\sin^2(1-x)}{(x-1)^2(x+2)}=?$

解答.

$$原式 = \lim_{x\to 1}\frac{\sin^2(x-1)}{(x-1)^2}\cdot\frac{1}{x+2}=\frac{1}{3}.$$

1.2 $y=\arccos\dfrac{1}{1+x^2}$, 则 $y'=?$

解答.

$$\begin{aligned} y' &= -\frac{1}{1-\left(\frac{1}{1+x^2}\right)^2}\cdot\frac{-2x}{(1+x^2)^2} \\ &= \frac{2x}{(1+x^2)\sqrt{x^4+2x^2}} \\ &= \frac{2\cdot sgn(x)}{(1+x^2)\sqrt{x^2+2}}. \end{aligned}$$

1.3 $\displaystyle\int \ln^2 x dx=?$

解答.

$$\begin{aligned} 原式 &= x\ln^2 x-\int 2\ln x dx \\ &= x\ln^2 x-2\left(x\ln x-\int dx\right) \\ &= x\ln^2 x-2x\ln x+2x+C. \end{aligned}$$

1.4 $z=y^x\sin\dfrac{x}{y}$, 则 $dz=?$

解答.

$$\begin{aligned} z &= e^{x\ln y}\sin\frac{x}{y} \\ z_x &= y^x\cdot\ln y\cdot\sin\frac{x}{y}+y^x\cdot\cos\frac{x}{y}\cdot\frac{1}{y} \\ z_y &= y^x\cdot\frac{x}{y}\cdot\sin\frac{x}{y}+y^x\cos\frac{x}{y}\cdot\frac{-x}{y^2} \\ dz &= z_xdx+z_ydy \\ &= y^{x-1}\sin\frac{x}{y}\left(y\ln ydx+xdy\right)+y^{x-2}\cos\frac{x}{y}\left(ydx-xdy\right). \end{aligned}$$

1.5 $D=\{(x,y);\ x^2+y^2\le 1\}$, 则 $\iint_D e^{x^2+y^2}dxdy=?$

解答.

$$原式 = \int_0^1 2\pi re^{r^2}dr=\pi\left(e-1\right).$$

1.6 $L=\{(x,y);\ x^2+y^2=1\}$, 方向为顺时针方向, 则 $\int_L xdy-ydx=?$

解答.

$$原式 = -2\iint_D dxdy=-2\pi.$$

2 判断题

2.1 若 $\lim\limits_{n\to\infty}x_n=0$, 则 $\lim\limits_{n\to\infty}\sqrt[n]{x_n}=0$.

解答. 错. 比如说

$$\lim_{n\to\infty}\frac{1}{n}=1,$$

但

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}=1\left(1\le\sqrt[n]{n}\le 1+\sqrt{\frac{2}{n-1}}\right).$$

2.2 若 $f(x)$ 在 $(0,\infty)$ 上可导, 且导函数 $f'(x)$ 有界, 则 $f(x)$ 在 $(0,\infty)$ 上一致连续.

解答. 对. 实际上, 若$|f(x)| \le M$, 则 $\forall\, \varepsilon > 0$, $\exists\, \delta = \varepsilon/M > 0, s.t.$

$$|x - x'| < \delta \Rightarrow |f(x) - f(x')| = |f'(\xi)|\,|x - x'| \le M\delta = \varepsilon.$$

2.3 若 $f(x)$ 在 $[a,b]$ 上可积, $F(x) = \int_a^x f(t)dt$ 在 $x_0 \in (a,b)$ 上可导, 则 $F'(x_0) = f(x_0)$.

解答. 错. 比如说

$$f(x) = \begin{cases} 1, & x = 0, \\ x, & x \in [-1,0) \cup (0,1]. \end{cases}$$

则 $f(x)$ 可积且

$$F(x) = \int_{-1}^x f(t)dt = -\frac{1}{2} + \frac{x^2}{2},\ x \in [-1,1],$$

但

$$F'(0) = 0 \ne f(0).$$

2.4 若 $\sum_{n=1}^{\infty}(a_{2n-1} + a_{2n})$ 收敛, 且 $\lim_{n\to\infty} a_n = 0$, 则 $\sum_{n=1}^{\infty} a_n$ 收敛.

解答. 对. 实际上由假设, $\forall\, \varepsilon > 0$, 当 $N = N(\varepsilon)$ 充分大时,

- N 是偶数, P 是偶数,

$$\left|\sum_{n=N+1}^{N+P} a_n\right| < \varepsilon;$$

- N 是偶数, P 是奇数,

$$\left|\sum_{n=N+1}^{N+P} a_n\right| \le \left|\sum_{n=N+1}^{N+P-1} a_n\right| + |a_{N+P}| < \varepsilon;$$

- N 是奇数, P 是偶数,

$$\left|\sum_{n=N+1}^{N+P} a_n\right| \le |a_{N+1}| + \left|\sum_{n=N+2}^{N+P-1} a_n\right| + |a_{N+P}| < \varepsilon;$$

- N 是奇数, P 是奇数,

$$\left|\sum_{n=N+1}^{N+P} a_n\right| \le |a_{N+1}| + \left|\sum_{n=N+2}^{N+P} a_n\right| < \varepsilon;$$

3 求极限 $\lim\limits_{t\to x}\left(\dfrac{\sin t}{\sin x}\right)^{\frac{x}{\sin t-\sin x}}$. 记此极限为 $f(x)$, 求函数 $f(x)$ 的间断点, 并判别间断点类型.

解答. 当 $x \ne k\pi$, $k \in \mathbb{Z}$ 时,

$$\begin{aligned}\lim_{t\to x}\left(\frac{\sin t}{\sin x}\right)^{\frac{x}{\sin t-\sin x}} &= Exp\left(x\lim_{t\to x}\frac{1}{\sin t-\sin x}\ln\frac{\sin t}{\sin x}\right)\\ &= Exp\left(x\lim t\to x\frac{1}{\cos t}\cdot\frac{\sin x}{\sin t}\cdot\frac{\cos t}{\sin x}\right)\\ &= Exp\left(x\lim_{t\to x}\frac{1}{\sin t}\right)\\ &= e^{\frac{x}{\sin x}}.\end{aligned}$$

故而

$$f(x) = e^{\frac{x}{\sin x}},\ x \ne k\pi,\ k \in \mathbb{Z},$$

其间断点为 $\{k\pi\}_{k\in\mathbb{Z}}$, 且 0 是可去间断点, $\{k\pi\}_{k\ne 0}$ 是第二类间断点.

4 设 $f'(x)$ 在 $[0,a]$ 上连续, 且 $f(0)=0$. 证明:

$$\left|\int_0^a f(x)dx\right| \le \frac{Ma^2}{2},$$

其中 $M = \sup\limits_{0\le x\le a}|f'(x)|$.

证明. 对 $\forall\, x \in [0,a]$,

$$|f(x)| = |f(0) + f'(\xi_x)x| \le Mx,$$

故

$$\left|\int_0^a f(x)dx\right| \le \int_0^a |f(x)|\,dx \le \int_0^a Mxdx = \frac{Ma^2}{2}.$$

5 若函数 $f(x,y)$ 在 $\mathbb{R}^2$ 上对 x 连续, 且存在 $L>0$, 对 $\forall\, x,\ y',\ y'' \in \mathbb{R}$,

$$|f(x,y') - f(x,y'')| \le L\,|y'-y''|\,.$$

求证: $f(x,y)$ 在 $\mathbb{R}^2$ 上连续.

证明. 由

$$\begin{aligned} |f(x',y') - f(x,y)| &\le |f(x',y') - f(x',y)| + |f(x',y) - f(x,y)| \\ &\le L\,|y'-y| + |f(x',y) - f(x,y)|\,, \end{aligned}$$

及所给条件便得结论.

6 求下列积分

$$I = \iint_S f(x,y,z)dS\ (a>0),$$

其中 $S = \{(x,y,z);\ x^2+y^2+z^2 = a^2\}$,

$$f(x,y,z) = \begin{cases} x^2+y^2, & z \ge \sqrt{x^2+y^2}, \\ 0, & z < \sqrt{x^2+y^2}. \end{cases}$$

解答.

$$\text{原式} = \iint_{x^2+y^2\le\frac{a^2}{2}} (x^2+y^2)\frac{a}{\sqrt{a^2-(x^2+y^2)}}dxdy$$

$$
\begin{aligned}
&= a\int_0^{\frac{a}{\sqrt{2}}} 2\pi r\cdot r^2\cdot\frac{a}{\sqrt{a^2-r^2}}dr\\
&= 2\pi a^2\int_0^{\frac{a}{\sqrt{2}}}\frac{r^3}{\sqrt{a^2-r^2}}dr\\
&= 2\pi a^5\int_0^{\frac{\pi}{4}}\sin^{\theta} d\theta\\
&= \frac{\pi a^5}{6}\left(8-5\sqrt{2}\right),
\end{aligned}
$$

其中最后一步是因为

$$
\begin{aligned}
\int \sin^3\theta d\theta &= \int\left(1-\cos^2\theta\right)d\theta\\
&= -\cos\theta+\frac{1}{3}\cos^3\theta+C.
\end{aligned}
$$

7 设 $0<r<1$, $x\in\mathbb{R}$.

7.1 求证: $\dfrac{1-r^2}{1-2r\cos x+r^2}=1+2\sum_{n=1}^{\infty}r^n\cos nx.$

7.2 求证: $\int_0^{\pi}\ln(1-2r\cos x+r^2)dx=0.$

证明. 7.1 由

$$
2\cos nx\cdot\cos x=\cos(n+1)x+\cos(n-1)x,
$$

知

$$
2r\cdot r^n\cos nx\cdot\cos x=r^{n+1}\cos(n+1)x+r^2\cdot r^{n-1}\cos(n-1)x,
$$

故而若记

$$
f(x)=\sum_{n=1}^{\infty}r^n\cos nx,
$$

则

$$
2r\cos x\cdot f(x)=\left[f(x)-r\cos x\right]+r^2\left[f(x)+1\right],
$$

即

$$f(x) = \frac{-r^2 + r\cos x}{1 - 2r\cos x + r^2},$$

于是

$$1 + 2\sum_{n=1}^{\infty} r^n \cos nx = 1 + 2f(x) = \frac{1 - r^2}{1 - 2r\cos x + r^2}.$$

7.2

$$\begin{aligned}
& \int_0^{\pi} \ln(1 - 2r\cos x + r^2)dx \\
= \ & x\ln(1 - 2r\cos x + r^2)|_0^{\pi} - \int_0^{\pi} \frac{2rx\sin x}{1 - 2r\cos x + r^2}dx \\
= \ & 2\pi\ln(1 + r) - \frac{2r}{1 - r^2}\int_0^{\pi} x\sin x\left(1 + 2\sum_{n=1}^{\infty} r^n\cos nx\right)dx \\
= \ & 2\pi\ln(1 + r) - \frac{2r}{1 - r^2} \\
& \cdot\left\{\int_0^{\pi} x\sin x dx + \sum_{n=1}^{\infty} r^n \int_0^{\pi} x\left[\sin(1 + n)x + \sin(1 - n)x\right]dx\right\} \\
= \ & 2\pi\ln(1 + r) - \frac{2r}{1 - r^2}\left[\pi + \sum_{n=1}^{\infty} r^n \cdot \pi \cdot \frac{(-1)^n}{n + 1}\sum_{n=2}^{\infty} r^n \cdot \pi \cdot \frac{(-1)^{n-2}}{n - 1}\right] \\
& \left(m \in \mathbb{Z} - \{0\} \Rightarrow \int_0^{\pi} x\sin mx dx = \pi \cdot \frac{(-1)^{m-1}}{m}\right) \\
= \ & 2\pi\ln(1 + r) - \frac{2\pi r}{1 - r^2}\left\{-1 - \frac{1}{r}\left[\ln(1 + r) - r\right] + \ln(1 + r)\right\} \\
& \left(\ln(1 + r) = \sum_{n=0}^{\infty}(-1)^n \frac{r^{n+1}}{n + 1}\right) \\
= \ & 0
\end{aligned}$$

8 $a > 0,\ b > 0,\ a_1 = a,\ a_2 = b,\ a_{n+2} = 2 + \dfrac{1}{a_{n+1}^2} + \dfrac{1}{a_n^2},\ n = 1, 2, \cdots.$ 求证: $\{a_n\}$ 收敛.

证明. 由通项公式知

$$2 \le a_n \le 2 + \frac{1}{2^2} \cdot 2 = \frac{5}{2},$$

而

$$\begin{aligned} |a_{n+2} - a_{n+1}| &\le \left|\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}\right| + \left|\frac{1}{a_n^2} - \frac{1}{a_{n-1}^2}\right| \\ &\le \frac{5}{16}\left(|a_{n+1} - a_n| + |a_n - a_{n-1}|\right). \end{aligned}$$

故若记

$$b_n \equiv |a_{n+1} - a_n| \ge 0,$$

则

$$b_{n+1} \le \frac{5}{16}\left(b_n + b_{n-1}\right).$$

我们断言:

$$\exists\ 0 < r < 1,\ C = C(a, b, r) > 0,\ \ s.t.\ 0 \le b_n \le C_r r^n. \tag{1}$$

一旦断言成立, 则 $\sum_{n=1}^{\infty} b_n$ 收敛, 而 $\{a_n\}$ 收敛.

往证 (1). 实际上, 任取 $0 < r < 1$ 满足

$$\frac{5}{16} r + 1 \le r^2 \ \left(\text{比如 } r = \frac{15}{17}\right),$$

再取

$$C = C(a, b, r) = \max\left\{\frac{b_1}{r}, \frac{b_2}{r^2}\right\},$$

则由数学归纳法易知 (1) 成立.