中科院2003数分

To my parents

1 设A, B为常数,求

$$\lim_{x\to 0^+}\left(e^{\frac{A}{x}}+e^{\frac{B}{x}}\right).$$

解答. 由对称性不妨设$A\leq B$,则

$$原式=\lim_{x\to 0^+}e^{\frac{B}{x}}\left(e^{\frac{A-B}{x}}+1\right)=\begin{cases}0, & if \quad B<0,\\ 1, & if \quad B=0, A<B,\\ 2, & if \quad B=0, A=B,\\ +\infty, & if \quad B>0.\end{cases}$$

2 确定λ取何值时,函数

$$f(x)=\begin{cases}x^\lambda\sin\dfrac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

在$x=0$处连续,可导,导数连续.

解答.

- 当$\lambda>0$时, $\lim_{x\to 0}f(x)=0$,从而$f(x)$在$x=0$处连续.
- 当$\lambda>1$时,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}x^{\lambda-1}\sin\frac{1}{x}=0,$$

从而$f(x)$在$x=0$处可导.

- 当$\lambda>2$时,

$$x\neq 0\Rightarrow f'(x)=\lambda x^{\lambda-1}\sin\frac{1}{x}-x^{\lambda-2}\cos\frac{1}{x},$$

而

$$\lim_{x\to 0}f'(x)=0=f'(0),$$

即$f'(x)$在$x=0$处连续.

3 设$0<x<y<1$或者$1<x<y$,则 $\frac{y}{x}>\frac{y^x}{x^y}$.

证明.

- 对函数$f(t)=t-\ln(t+1),\ t>-1$,由

$$f'(t)=1-\frac{1}{t+1}=\frac{t}{t+1}\begin{cases}<0, & -1<t<0,\\ >0, & t>0.\end{cases}$$

知

$$-1<t<0 \text{ 或者 } t>0\Rightarrow f(t)>0,$$

即

$$-1<t<0 \text{ 或者 } t>0\Rightarrow t>\ln(t+1).$$

- 对函数$g(t)=\frac{\ln t}{1-t},\ t>0$,由

$$g'(t)=\frac{t(1-t)+\ln t}{(1-t)^2}=\frac{\frac{1}{t}-1-\ln\left(\frac{1}{t}-1+1\right)}{(1-t)^2}>0,$$

知当$0<x<y<1$或者$1<x<y$时,

$$\frac{\ln x}{1-x}=g(x)<g(y)=\frac{\ln y}{1-y},$$

即

$$x^{1-y}<y^{1-x},\quad \frac{y}{x}>\frac{y^x}{x^y}.$$

4 计算积分

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}.$$

解答.

$$\begin{aligned}\text{原式} &= \int_{-\pi}^{\pi}\frac{d\theta}{2-\cos\theta}=\int_{-\pi}^{0}+\int_0^{\pi}\frac{d\theta}{2-\cos\theta}\\ &= \int_0^{\pi}\frac{d\theta}{2+\cos\theta}+\int_0^{\pi}\frac{d\theta}{2-\cos\theta}\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\pi} \frac{4d\theta}{4-\cos^2\theta} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4d\theta}{4-\sin^2\theta} = \int_0^{\frac{\pi}{2}} \frac{8d\theta}{4-\sin^2\theta} \\
&= \int_0^{\frac{\pi}{2}} \frac{8d\theta}{3+\cos^2\theta} = \int_0^{\frac{\pi}{2}} \frac{8\sec^2\theta d\theta}{3\sec^2\theta+1} = \int_0^{\frac{\pi}{2}} \frac{8d\tan\theta}{3\tan^2\theta+4} \\
&= \int_0^{\infty} \frac{8dx}{3x^2+4} = \frac{8}{4}\cdot\frac{2}{\sqrt{3}}\cdot\int_0^{\infty}\frac{d\frac{\sqrt{3}x}{2}}{\left(\frac{\sqrt{3}x}{2}\right)^2+1} \\
&= \frac{2\sqrt{3}}{3}\pi.
\end{aligned}
$$

5 设$f(x)$在$[a,b]$上连续,在(a,b)内可导,且 $f(a)=0$, $f(x)>0$,当$x\in(a,b]$. 求证: 不存在常数$M>0$使得$0\leq f'(x)\leq Mf(x)$,当 $x\in(a,b]$.

证明. *用反证法.若*

$$\exists\ M>0,\ s.t.\ x\in(a,b]\Rightarrow 0\leq f'(x)\leq Mf(x).$$

则

$$x\in(a,b]\Rightarrow d\ln f(x)\leq Mdt.$$

于$(x,b]$上求积而有

$$\ln f(b)-\ln f(x)\leq M(b-x),$$

即

$$e^{\ln f(b)-M(b-x)}\leq f(x).$$

又f连续,而

$$0<e^{\ln f(b)-M(b-a)}\leq f(a)=0.$$

矛盾.

6 计算积分

$$\iiint_V \frac{2z}{\sqrt{x^2+y^2}}dv,$$

其中V是平面图形

$$D=\left\{(x,y,z);\ x=0,\ y\geq 0,\ y^2+z^2\leq 1,\ 2y-z\leq 1\right\}$$

绕z轴旋转一周所生成的图形.

解答. 用平面$z=t\in[-1,1]$去割D而分片求积,有

$$\begin{aligned}
\text{原式} &= \int_{-1}^{\frac{3}{5}} zdz \iint_{x^2+y^2\leq\left(\frac{z+1}{2}\right)^2} \frac{1}{\sqrt{x^2+y^2}}dxdy \\
&\quad + \int_{\frac{3}{5}}^{1} zdz \iint_{x^2+y^2\leq 1-z^2} \frac{1}{\sqrt{x^2+y^2}}dxdy \\
&= \int_{-1}^{\frac{3}{5}} z\cdot 2\pi\cdot\frac{z+1}{2}dz + \int_{\frac{3}{5}}^{1} z\cdot 2\pi\cdot\sqrt{1-z^2}dz \\
&= \frac{32\pi}{375}+\frac{128\pi}{375} \\
&= \frac{32\pi}{75}.
\end{aligned}$$

7 求椭球面

$$\frac{x^2}{96}+y^2+z^2=1$$

上距平面$3x+4y+12z=228$最近和最远的点.

解答.

- 由椭球面方程

$$\frac{x^2}{96}+y^2+z^2=1$$

知

$$\frac{x}{48}dx+2ydy+2zdz=0,$$

而其上点(x,y,z)的法方向为

$$\left(\frac{x}{96},y,z\right),$$

- 在所要(yào)求的点(x,y,z)上,有

$$\frac{\frac{x}{96}}{3}=\frac{y}{4}=\frac{z}{12},$$

而

$$(x,y,z)=\left(9,\frac{1}{8},\frac{3}{8}\right)\text{ 或者 }\left(-9,-\frac{1}{8},-\frac{3}{8}\right).$$

又

$$\frac{\left|3\cdot 9+4\cdot\frac{1}{8}+12\cdot\frac{3}{8}-228\right|}{\sqrt{3^2+4^2+12^2}}=\frac{196}{13},$$

$$\frac{\left|3\cdot(-9)+4\cdot\left(\frac{1}{8}\right)+12\cdot\left(-\frac{3}{8}\right)-228\right|}{\sqrt{3^2+4^2+12^2}}=\frac{260}{13}=20,$$

故最近点为$\left(9,\frac{1}{8},\frac{3}{8}\right)$, 最远点为$\left(-9,-\frac{1}{8},-\frac{3}{8}\right)$.

8 设函数$f(x)$在$x=0$连续,并且

$$\lim_{x\to 0}\frac{f(2x)-f(x)}{x}=A,$$

求证:$f'(0)$存在,并且$f'(0)=A$.

证明. 由题意,

$$\forall\ \varepsilon>0,\ \exists\ \delta>0,\ s.t.\ |x|<\delta\Rightarrow\left|\frac{f(2x)-f(x)}{x}-A\right|<\frac{\varepsilon}{2}.$$

于是

$$\begin{array}{ccccc}-\frac{\varepsilon}{2}\cdot\frac{|x|}{2} & < & f(x)-f\left(\frac{x}{2}\right) & < & \frac{\varepsilon}{2}\cdot\frac{|x|}{2},\\ -\frac{\varepsilon}{2}\cdot\frac{|x|}{2^2} & < & f\left(\frac{x}{2}\right)-f\left(\frac{x}{2^2}\right) & < & \frac{\varepsilon}{2}\cdot\frac{|x|}{2^2},\\ & & \cdots & & \end{array}$$

求和而有

$$-\frac{\varepsilon}{2}\cdot|x|\cdot\sum_{k=1}^{n}\frac{1}{2^k}<f(x)-f\left(\frac{x}{x^n}\right)-Ax\sum_{k=1}^{n}\frac{1}{2^k}<\frac{\varepsilon}{2}\cdot|x|\cdot\sum_{k=1}^{n}\frac{1}{2^k},$$

令$n\to\infty$,得

$$-\varepsilon|x|<f(x)-f(0)-Ax<\varepsilon|x|,$$

即

$$\left|\frac{f(x)-f(0)}{x}-A\right|<\varepsilon,$$

而$f'(0)$存在,并且$f'(0)=A$.

9 设$f(x)$是$[-1,1]$上的连续函数,则

$$\lim_{y\to0^+}\int_{-1}^{1}\frac{yf(x)}{x^2+y^2}dx=\pi f(0).$$

证明. • 对$\forall\ \eta\in(0,1)$,先写下:

$$\begin{aligned}&\left|\int_{-1}^{1}\frac{yf(x)}{x^2+y^2}dx-\pi f(0)\right|\\ \leq\ &\left|\int_{-1}^{-\eta}+\int_{\eta}^{1}\frac{yf(x)}{x^2+y^2}dx\right|\\ &+\left|\int_{-\eta}^{\eta}\frac{yf(x)}{x^2+y^2}dx-\int_{-\eta}^{\eta}\frac{yf(0)}{x^2+y^2}dx\right|\\ &+\left|\int_{-\eta}^{\eta}\frac{yf(0)}{x^2+y^2}dx-\pi f(0)\right|\\ =:\ &I_1+I_2+I_3.\end{aligned}$$

• 然后记 $\sup_{x\in[-1,1]}|f(x)|=M<\infty$, 设$|y|\leq1$.对任意固定的$\varepsilon>0$,

★ 由$f(x)$在$x=0$处的连续性,

$$\exists\ \eta>0,\ s.t.\ |x|\leq\eta\Rightarrow|f(x)-f(0)|<\frac{\varepsilon}{3\pi},$$

而

$$\begin{aligned} I_2 &\leq \max_{|x|\leq\eta}|f(x)-f(0)|\cdot\int_{-\eta}^{\eta}\frac{|y|}{x^2+|y|^2}dx \\ &= \max_{|x|\leq\eta}|f(x)-f(0)|\cdot 2\cdot\int_0^{\eta}\frac{d\frac{x}{|y|}}{1+\left(\frac{x}{|y|}\right)^2} \\ &\leq \pi\cdot\max_{|x|\leq\eta}|f(x)-f(0)| \\ &\leq \frac{\varepsilon}{3}. \end{aligned}$$

★ 对上述固定的$\eta>0$,

$$\begin{aligned} I_1 &\leq 2M\,|y|\cdot\int_{\eta}^{1}\frac{dx}{\eta^2} \\ &= \frac{2M(1-\eta)}{\eta^2}\,|y| \\ &< \frac{\varepsilon}{3}, \end{aligned}$$

当$|y|<\delta_1=\delta_1(\varepsilon)=\delta_1(\varepsilon,\eta)=\dfrac{\varepsilon\eta^2}{3M(1-\eta)}$时.

★ 同样地,

$$I_3 \leq |f(0)|\cdot\left|2\arctan\frac{\eta}{|y|}-\pi\right|<\frac{\varepsilon}{3},$$

当$|y|<\delta_2=\delta_2(\varepsilon)=\delta_2(\eta)$时.

从而$\forall\ \varepsilon>0,\ \exists\ \delta=\min\{\delta_1,\delta_2\}>0,\ s.t.$

$$|y|<\delta\Rightarrow\left|\int_{-1}^{1}\frac{yf(x)}{x^2+y^2}dx-\pi f(0)\right|<\varepsilon.$$

即

$$\lim_{y\to0^+}\int_{-1}^{1}\frac{yf(x)}{x^2+y^2}dx=\pi f(0).$$