兰州大学2007数分

To my parents

1 1.1 $\lim\limits_{x\to 0^+}\left(\dfrac{(1+x)^{1/x}}{e}\right)^{1/x}$

解答.

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{(1+x)^{1/x}}{e}\right)^{1/x} &= Exp\left[\lim_{x\to 0^+}\frac{\frac{1}{x}\ln(1+x)-1}{x}\right]\\ &= Exp\left[\lim_{x\to 0^+}\frac{x-(1+x)\ln(1+x)}{(1+x)x^2}\right]\\ &= Exp\left[\lim_{x\to 0^+}\frac{1-(1-\ln(1+x))}{2x}\right]\\ &= Exp\left[\lim_{x\to 0^+}\frac{-\ln(1+x)}{2x}\right]\\ &= e^{-\frac{1}{2}}.\end{aligned}$$

1.2 $\lim\limits_{n\to\infty}\ln\sqrt[n]{\left(1+\dfrac{1^2}{n^2}\right)\left(1+\dfrac{2^2}{n^2}\right)\cdots\left(1+\dfrac{n^2}{n^2}\right)}$

解答.

$$\begin{aligned}&\lim_{n\to\infty}\ln\sqrt[n]{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)}\\ =&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left[1+\left(\frac{k}{n}\right)^2\right]\\ =&\int_0^1\ln(1+x^2)dx\\ =&x\ln(1+x^2)|_0^1-\int_0^1\frac{2x^2}{1+x^2}dx\\ =&\ln 2+\frac{\pi}{2}-2.\end{aligned}$$

1.3 $\lim\limits_{(x,y)\to(0,0)}\left(|x|^\alpha \sin\frac{1}{y}+|y|^\beta\cos\frac{1}{x}\right)$

解答.

$$\begin{aligned}&\lim_{(x,y)\to(0,0)}\left(|x|^\alpha \sin\frac{1}{y}+|y|^\beta\cos\frac{1}{x}\right)\\=&\lim_{(x,y)\to(0,0)}|x|^\alpha \sin\frac{1}{y}+\lim_{(x,y)\to(0,0)}|y|^\beta\cos\frac{1}{x}\\=&0+0\\=&0.\end{aligned}$$

1.4 $\int_e^{e^2}\frac{\ln\ln x}{x\ln x}dx$

解答.

$$\int_e^{e^2}\frac{\ln\ln x}{x\ln x}dx=\int_1^2\frac{\ln t}{t}dt=\frac{(\ln t)^2}{2}|_1^2=\frac{(\ln 2)^2}{2}.$$

1.5 $\oint_C\frac{(x+y)dx-(x-y)dy}{x^2+y^2}$,其中$C$为圆周$x^2+y^2=a^2$的逆时针方向.

解答.

$$\begin{aligned}&\oint_C\frac{(x+y)dx-(x-y)dy}{x^2+y^2}\\=&\int_0^{2\pi}\frac{a(\cos\theta+\sin\theta)(-a\sin\theta)-a(\cos\theta-\sin\theta)(a\cos\theta)}{a^2}d\theta\\=&\int_0^{2\pi}(-1)d\theta=-2\pi.\end{aligned}$$

2 讨论级数$\sum\limits_{n=1}^\infty\frac{1}{n^p(\ln n)^q}$的收敛性,其中$p>0,q>0$.

解答. 1) $p>1$

由 $\lim_{n\to\infty}\frac{1/[n^p(\ln n)^q]}{1/n^p}=\lim_{n\to\infty}\frac{1}{(\ln n)^q}=0$ 及比较判别法知原级数收敛;

2) $p=1$

由 $\int_3^\infty \frac{1}{x(\ln x)^q}=\int_{\ln 3}^\infty \frac{dt}{t^q}$ 及积分判别法知

a) $0<q\le 1$ 时级数发散;

b) $q>1$ 时级数收敛;

3) $p<1$

取 $r\in(p,1)$ 由 $\lim_{n\to\infty}\frac{1/[n^p(\ln x)^q]}{n^r}=\lim_{n\to\infty}\frac{n^{r-p}}{(\ln x)^q}=\lim_{x\to\infty}\frac{e^{(r-p)x}}{x^q}=+\infty$ 及比较判别法知原级数发散.

注记. $p<1$ 时用到极限 $\lim_{x\to\infty}\frac{e^{\alpha x}}{x^\beta}=+\infty$,这是因为

$$\frac{e^{\alpha x}}{x^\beta}>\frac{1}{([\beta]+2)!}\frac{(\alpha x)^{[\beta]+2}}{x^\beta}>\frac{\alpha^{[\beta]+2}}{([\beta]+2)!}x\quad (x>1).$$

3 证明 $2<e<3$.

证明. 由 e^x 的 $Taylor$ 展式 $e^x=\sum_{k=1}^\infty\frac{x^k}{k!}$ 知

$$2=1+1<e=\sum_{k=1}^\infty\frac{x^k}{k!}<1+1+\sum_{k=2}^\infty\frac{1}{2^{k-1}}=3.$$

4 设 $f(x,y)$ 关于 x,y 均是一元连续函数.

1) 举例说明 $f(x,y)$ 可以不是二元连续函数.

2) 证明当 $f(x,y)$ 关于 x 单调时, $f(x,y)$ 是二元连续函数.

证明. 1) 取

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

即有之.

2) $\forall (x_0,y_0)\in\mathbb{R}^2$,由题意,

a) $f(x,y_0)$在$x=x_0$处连续,而有

$$\forall\varepsilon>0,\exists\delta_1>0,\ s.t.\ |x-x_0|\leq\delta_1\Rightarrow|f(x,y_0)-f(x_0,y_0)|<\frac{\varepsilon}{2}$$

b) $f(x_0-\delta_1,y),f(x_0+\delta_1,y)$都在$y=y_0$处连续,而有对上述任意的$\varepsilon$,

$$\exists\delta_2>0,\ s.t.\ |y-y_0|\leq\delta_2\Rightarrow\begin{cases}|f(x_0-\delta_1,y)-f(x_0-\delta_1,y_0)|<\frac{\varepsilon}{2}\\ |f(x_0+\delta_1,y)-f(x_0+\delta_1,y_0)|<\frac{\varepsilon}{2}\end{cases}$$

于是对$\forall(x,y)\in[x_0-\delta_1,x_0-\delta_1]\times[y_0-\delta_2,y_0+\delta_2]$,

- 若$f(\cdot,y)$递增,则

$$f(x,y)\leq f(x_0+\delta,y)<f(x_0+\delta_1,y_0)+\frac{\varepsilon}{2}<f(x_0,y_0)+\varepsilon$$

$$f(x,y)\geq f(x_0-\delta,y)>f(x_0-\delta_1,y_0)-\frac{\varepsilon}{2}>f(x_0,y_0)-\varepsilon$$

- 若$f(\cdot,y)$递减,则

$$f(x,y)\geq f(x_0+\delta,y)>f(x_0+\delta_1,y_0)-\frac{\varepsilon}{2}>f(x_0,y_0)-\varepsilon$$

$$f(x,y)\leq f(x_0-\delta,y)<f(x_0-\delta_1,y_0)+\frac{\varepsilon}{2}<f(x_0,y_0)+\varepsilon$$

于是有之.

5 令$x_1=a,x_2=b,x_{n+2}=\dfrac{x_{n+1}+x_n}{2},n=1,2,3,\cdots$.试求$\lim\limits_{n\to\infty}x_n$.

解答. 由题意,

$$x_{n+1}-x_n=-\frac{1}{2}(x_n-x_{n-1})=\cdots=(-\frac{1}{2})^{n-1}(x_2-x_1)=(-\frac{1}{2})^{n-1}(b-a)$$

于是

$$|x_{n+p}-x_n|\le\left|\sum_{k=n}^{n+p-1}(x_{k+1}-x_k)\right|\le(b-a)\sum_{k=n}^{n+p-1}(\frac{1}{2})^{n-1}\le\frac{1}{2^{n-2}}(b-a)\to0\ (n\to\infty)$$

$\{x_n\}$是$Cauchy$列,$\{x_n\}$收敛,设极限为x.

再由

$$x_{k+1}-x_k=\left(-\frac{1}{2}\right)^{k-1}(b-a)$$

知

$$x_{n+1}=x_1+\sum_{k=1}^{n}(x_{k+1}-x_k)=a+(b-a)\sum_{k=1}^{n}\left(-\frac{1}{2}\right)^{k-1}$$

两边$n\to\infty$取极限,有

$$x=a+(b-a)\frac{1}{1-(-\frac{1}{2})}=\frac{a+2b}{3}.$$

6 设$f(x)$是$[0,1]$上的连续函数,0是$f(x)$在$[0,1]$上的唯一零点,并且$f'_+(0)>0$. 试证明$\int_0^1\frac{x}{f(x)}dx$收敛.

证明. 由题意及连续函数的介值性,下列两种情况有切仅有一种发生:

1) $f(x)>0,\forall x\in(0,1]$

2) $f(x)<0,\forall x\in(0,1]$

又$f'_+(0)>0$,即$\lim\limits_{x\to0^+}\frac{f(x)}{x}>0$,从而$\exists\delta>0,x\in(0,\delta]\Rightarrow\frac{f(x)}{x}>0$即$f(x)>0$.

故而情况1)发生,即$f(x)>0,\forall x\in(0,1]$,从而$\int_0^1\frac{x}{f(x)}dx$只以0为瑕点.

再由

$$\lim_{x\to 0^+}\frac{x/f(x)}{1/\sqrt{x}}=\lim_{x\to 0^+}\frac{x}{f(x)}\sqrt{x}=\frac{1}{f'_+(0)}\cdot 0=0$$

及比较判别法知 $\int_0^1 \frac{x}{f(x)}dx$ 收敛.

7 求 $I=\iiint_\Omega z\cos(x^2+y^2)dxdydz$,其中

$$\Omega=\{(x,y,z)\in\mathbb{R}^3 \mid z\geq 0, x^2+y^2+z^2\leq 1\}.$$

解答.

$$\begin{aligned}
I &= \iiint_\Omega z\cos(x^2+y^2)dxdydz\\
&= \int_0^1\left(\iint_{x^2+y^2\leq 1-z^2} z\cos(x^2+y^2)dxdy\right)dz\\
&= 2\pi\int_0^1\left(z\int_0^{\sqrt{1-z^2}} r\cos r^2 dr\right)dz\\
&= \pi\int_0^1 z\sin(1-z^2)dz\\
&= \frac{\pi}{2}(1-\cos 1)\\
&= \pi\sin^2\frac{1}{2}\ \square
\end{aligned}$$

8 对 $x\in\mathbb{R}$,记 $\rho(x)=min_{n\in\mathbb{Z}}|x-n|$,其中 $\mathbb{Z}$ 是整数集.试证明:

1) $\rho(x)$ 是 $\mathbb{R}$ 上周期为1的连续函数;

2) $f(x)=\sum_{n=0}^{\infty}\frac{\rho(10^n x)}{10^n}$ 是 $\mathbb{R}$ 上的连续函数.

证明. *1)* $\forall x_0\in\mathbb{R}, \exists n\in\mathbb{Z},\ s.t.\ n\leq x_0<n+1.$

a) $x_0=n$

此时,$\rho(x_0)=0; x_0+1=n+1, \rho(x+1)=0=\rho(x_0)$. 且当$x\in(x_0-\frac{1}{2}, x_0+\frac{1}{2})$时, $\rho(x)=x-x_0$,而有

$$\lim_{x\to x_0}\rho(x)=\lim_{x\to x_0}(x-x_0)=0=\rho(x_0)$$

b) $x\ n<x_0\leq n+\frac{1}{2}$

此时,$\rho(x_0)=x-n; n+1<x_0+1\leq(n+1)+\frac{1}{2}, \rho(x_0+1)=(x_0+1)-(n_0+1)=x_0-n=\rho(x_0)$, 取

$$\delta=min\{x_0-n, n+\frac{1}{2}-x_0\}>0,$$

当$x\in(x_0-\delta, x_0+\delta)$时,$\rho(x)=x-n$,而也有

$$\lim_{x\to x_0}\rho(x)=\lim_{x\to x_0}(x-n)=x_0-n=\rho(x_0)$$

c) $n+\frac{1}{2}<x_0<n+1$

此时,$\rho(x_0)=n+1-x_0; n+1+\frac{1}{2}<x_0+1\leq n+2, \rho(x_0+1)=(n+2)-(x_0+1)=(n+1)-x_0=\rho(x_0)$, 取

$$\delta=min\{x_0-n-\frac{1}{2}, n+1-x_0\}>0,$$

当$x\in(x_0-\delta, x_0+\delta)$时,$\rho(x)=n+1-x$,而也有

$$\lim_{x\to x_0}\rho(x)=\lim_{x\to x_0}(n+1-x)=n+1-x_0=\rho(x_0)$$

从而f是$\mathbb{R}$上周期为1的连续函数.

2) 明显的,

$$\left|\frac{\rho(10^nx)}{10^n}\right|\leq\frac{1}{10^n}$$

由$Weierstrass$判别法知函数项级数$\sum_{n=0}^{\infty}\frac{\rho(10^nx)}{10^n}$一致收敛.

再由1),$\frac{\rho(10^nx)}{10^n}$是连续的,而 (表递进)$f(x)$作为一致收敛的,各项连续的函数项级数的和,是连续的.