浙大2010数分

To my parents

1 计算下列极限和积分

1.1 $\lim_{n\to\infty}\sum_{k=n^2}^{(n+1)^2}\frac{1}{\sqrt{k}}$

解答. 由

$$\frac{2n+2}{n+1}\leq\sum_{k=n^2}^{(n+1)^2}\frac{1}{\sqrt{k}}\leq\frac{2n+2}{n}$$

知

$$\text{原式}=2.$$

1.2 $\iint_{[0,\pi]\times[0,1]} y\sin(xy)dxdy$

解答.

$$\text{原式}=\int_0^1 dy\int_0^\pi y\sin(xy)dx=\int_0^1[1-\cos(\pi y)]\,dy=1.$$

1.3 $\lim_{x\to 0}\frac{e^x\sin x-x(1+x)}{\sin^3 x}$

解答.

$$\begin{aligned}\text{原式}&=\lim_{x\to0}\frac{e^x\sin x-x-x^2}{x^3}\\&=\lim_{x\to0}\frac{e^x(\sin x+\cos x)-1-2x}{3x^2}\\&=\lim_{x\to0}\frac{2e^x\cos x-2}{6x}\\&=\frac13\lim_{x\to0}e^x(-\sin x+\cos x)\\&=\frac13.\end{aligned}$$

1.4 计算

$$\iint_{\Sigma} z dx dy$$

其中Σ是三角形 $\{(x,y,z);\ x,y,z\geq 0,\ x+y+z=1\}$,其法方向与$(1,1,1)$相同.

解答. 由*Gauss*公式,

$$原式 = \iiint_{\substack{x,y,z\geq 0\\ x+y+z\leq 1}} dxdydz = \frac{1}{3}.$$

1.5 $\displaystyle\int_0^{2\pi}\sqrt{1+\sin x}dx$

解答.

$$\begin{aligned}
原式 &= \int_0^{2\pi}\frac{|\cos x|}{\sqrt{1-\sin x}}dx\\
&= \int_0^{\frac{\pi}{2}}+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}+\int_{\frac{3\pi}{2}}^{2\pi}\frac{|\cos x|}{\sqrt{1-\sin x}}dx\\
&= -2\left(1-\sin x\right)^{\frac{1}{2}}\Big|_0^{\frac{\pi}{2}}+2\left(1-\sin x\right)^{\frac{1}{2}}\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}-2\left(1-\sin x\right)^{\frac{1}{2}}\Big|_{\frac{3\pi}{2}}^{2\pi}\\
&= 2+2\sqrt{2}-2\left(1-\sqrt{2}\right)\\
&= 4\sqrt{2}.
\end{aligned}$$

1.6 $\displaystyle\int_0^1\frac{\ln(1+x)}{1+x^2}dx$

解答.

$$\begin{aligned}
原式 &= \int_0^{\frac{\pi}{4}}\ln(1+\tan\theta)d\theta\ \ (x=\tan\theta)\\
&= \int_0^{\frac{\pi}{4}}\ln(\sin\theta+\cos\theta)d\theta-\int_0^{\frac{\pi}{4}}\ln\cos\theta d\theta\\
&= \int_0^{\frac{\pi}{4}}\ln\left[\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right]d\theta-\int_0^{\frac{\pi}{4}}\ln\cos\theta d\theta
\end{aligned}$$

$$\begin{aligned} &= \frac{\pi}{8}\ln 2 + \int_{-\frac{\pi}{4}}^{0} \ln\cos\alpha d\alpha \ \left(\theta - \frac{\pi}{4} = \alpha\right) \\ &\quad - \int_0^{\frac{\pi}{4}} \ln\cos\theta d\theta \\ &= \frac{\pi}{8}\ln 2. \end{aligned}$$

2 设$a_n = \sin a_{n-1}$,$n \geq 2$,且$a_1 > 0$.计算

$$\lim_{n\to\infty}\sqrt{\frac{n}{3}}a_n.$$

解答. • $\lim\limits_{n\to\infty} a_n = 0$.

事实上,

$$|a_n| \leq |\sin|a_{n-1}|| \leq |a_{n-1}|,$$

而$\lim\limits_{n\to\infty}|a_n| = A$存在.于

$$a_n = \sin a_{n-1}$$

两边令$n\to\infty$,有

$$A = \lim_{n\to\infty}|a_n| = \lim_{n\to\infty} \pm\sin|a_{n-1}| = \pm\sin A.$$

而$A = 0$.

• $$\lim_{n\to\infty}\sqrt{\frac{n}{3}}a_n = \begin{cases} 1, & 若 \quad 2k\pi < a_1 < 2k\pi + \pi, \\ 0, & 若 \quad a_1 = k\pi, \\ -1, & 若 \quad 2k\pi + \pi < a_1 < 2k\pi + 2\pi, \end{cases} \quad k = 1, 2, \cdots.$$

事实上,

★ 当$a_1 = k\pi$时,$a_n = 0$,而$\lim\limits_{n\to\infty}\sqrt{\frac{n}{3}}a_n = 0$.

★ 当$a_1 \neq k\pi$时,

$$a_n \begin{cases} > 0, & 若 \quad 2k\pi < a_1 < 2k\pi + \pi, \\ < 0, & 若 \quad 2k\pi + \pi < a_1 < 2k\pi + 2\pi. \end{cases}$$

故此时,仅须证

$$\lim_{n\to\infty}\frac{n}{3}a_n^2=1.$$

而这可通过Stolz公式立即得到:

$$\begin{aligned}
\lim_{n\to\infty}\frac{n}{3}a_n^2 &= \frac{1}{3}\lim_{n\to\infty}\frac{n}{\frac{1}{a_n^2}}\\
&= \frac{1}{3}\lim_{n\to\infty}\frac{1}{\frac{1}{a_n^2}-\frac{1}{a_{n-1}^2}}\\
&= \frac{1}{3}\lim_{n\to\infty}\frac{1}{\frac{1}{\sin^2 a_{n-1}}-\frac{1}{a_{n-1}^2}}\\
&= \frac{1}{3}\lim_{n\to\infty}\frac{a_{n-1}^2\sin^2 a_{n-1}}{a_{n-1}^2-\sin^2 a_{n-1}}\\
&= \frac{1}{3}\lim_{x\to 0}\frac{x^2\sin^2 x}{x^2-\sin^2 x}\\
&= \frac{1}{3}\lim_{x\to 0}\frac{x^3}{x-\sin x}\cdot\frac{x}{x+\sin x}\cdot\frac{\sin^2 x}{x^2}\\
&= \frac{1}{6}\lim_{x\to 0}\frac{x^3}{x-\sin x}\\
&= \frac{1}{6}\lim_{x\to 0}\frac{3x^2}{1-\cos x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{2x}{\sin x}\\
&= 1.
\end{aligned}$$

3 设函数$f(x)$在$(-\infty,+\infty)$上连续,n为 奇数.证:若

$$\lim_{n\to+\infty}\frac{f(x)}{x^n}=\lim_{n\to-\infty}\frac{f(x)}{x^n}=1.$$

则方程$f(x)+x^n=0$有实根.

证明. 由题意,

$$\exists\ A>0,\ s.t.\ \left\{\begin{array}{lllll} x\le -A & \Rightarrow & \frac{f(x)}{x^n}\ge\frac{1}{2} & \Rightarrow & f(x)\le\frac{1}{2}x^n,\\ x\ge A & \Rightarrow & \frac{f(x)}{x^n}\ge\frac{1}{2} & \Rightarrow & f(x)\ge\frac{1}{2}x^n.\end{array}\right.$$

而
$$f(A)+A^n \geq \frac{3}{2}A^n > 0 > -\frac{1}{2}A^n \geq f(-A)+(-A)^n,$$
于是由连续函数介值定理,$\exists\ \xi\in(-A,A),\ s.t.$
$$f(\xi)+\xi^n=0.$$

4 证明
$$\int_0^\infty \frac{\sin xy}{y}dy$$
在$[\delta,+\infty)$上一致连续(其中$\delta>0$).

证明. • $\displaystyle\int_0^\infty \frac{\sin xy}{y}dy$ 一致收敛于$[\delta,\infty)$.

事实上,因为
$$\lim_{y\to 0}\frac{\sin xy}{y}=\lim_{y\to 0}\frac{\sin xy}{xy}\cdot x=x,$$
及
$$\int_0^\infty \frac{\sin xy}{y}dy=\int_0^1+\int_1^\infty \frac{\sin xy}{y}dy,$$
只需验证反常积分
$$\int_1^\infty \frac{\sin xy}{y}dy$$
一致收敛,而用Dirichlet判别法:

★ $\displaystyle\left|\int_1^A \sin xydy\right| \leq \frac{2}{x} \leq \frac{2}{\delta} < \infty,$

★ $\displaystyle\frac{1}{y}$关于y递减且 $\displaystyle\lim_{y\to\infty}\frac{1}{y}=0.$

• $\displaystyle\int_0^\infty \frac{\sin xy}{y}dy$ 一致连续于$[\delta,\infty)$.

由$\displaystyle\int_0^\infty \frac{\sin xy}{y}dy$一致收敛于$[\delta,\infty)$知对任意固定的$\varepsilon>0$,
$$\exists\ A>0,\ s.t.\ x\in[\delta,\infty) \Rightarrow \left|\int_A^\infty \frac{\sin xy}{y}dy\right| < \frac{\varepsilon}{3},$$

而有

$$
\begin{aligned}
& \left|\int_0^\infty \frac{\sin xy}{y} - \frac{\sin x'y}{y} dy\right| \\
\leq & \left|\int_0^A \frac{\sin xy}{y} - \frac{\sin x'y}{y} dy\right| + \left|\int_A^\infty \frac{\sin xy}{y} dy\right| + \left|\int_A^\infty \frac{\sin x'y}{y} dy\right| \\
= & \left|\int_0^A (x-x')\cos(\xi_{xy}y)\,dy\right| + \frac{2\varepsilon}{3} \quad (Lagrange\text{中值定理}) \\
\leq & A|x-x'| + \frac{2\varepsilon}{3} \\
\leq & \varepsilon, \text{ 当 } |x-x'| < \frac{\varepsilon}{3A} \text{ 时.}
\end{aligned}
$$

即$\int_0^\infty \frac{\sin xy}{y} dy$关于$x$在$[\delta, \infty)$上一致收敛.

5 设$f(x)$连续.证明$Possion$公式:

$$\int_{x^2+y^2+z^2=1} f(ax+by+cz)ds = 2\pi \int_{-1}^1 f\left(\sqrt{a^2+b^2+c^2}t\right) dt.$$

证明. • 当$a=b=c=0$时,显然成立$Possion$公式.

• 否则,用平面$\frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}=t\,(t\in\mathbb{R})}$去割球面$x^2+y^2+z^2=1$,而分片求积, 由余面积公式,

$$
\begin{aligned}
\int_{x^2+y^2+z^2=1} f(ax+by+cz)ds &= \int_{-1}^1 dt \int_{\substack{x^2+y^2+z^2=1 \\ ax+by+cz=\sqrt{a^2+b^2+c^2}t}} f \\
&= 2\pi \int_{-1}^1 f\left(\sqrt{a^2+b^2+c^2}t\right) dt.
\end{aligned}
$$

6 设$\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$为 实数序列,满足

(1) $\lim_{n\to+\infty} |b_n| = \infty$.

(2) $\left\{\frac{1}{|b_n|}\sum_{i=1}^{n-1}|b_{i+1}-b_i|\right\}_{n\geq 1}$ 有界.

证明:若
$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$
存在,则
$$\lim_{n\to\infty}\frac{a_n}{b_n}$$
也存在.

证明. 记
$$c_n=\frac{a_{n+1}-a_n}{b_{n+1}-b_n},$$
不妨假设 $\lim_{n\to\infty}c_n=c=0$.若不然, 用a_n-cb_n代替a_n,而
$$\lim_{n\to\infty}\frac{(a_{n+1}-cb_{n+1})-(a_n-cb_n)}{b_{n+1}-b_n}=0,$$
$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{a_n-cb_n}{b_n}+c.$$
往证 $\lim_{n\to\infty}\frac{a_n}{b_n}=0$. 由

- $\left\{\frac{1}{|b_n|}\sum_{i=1}^{n-1}|b_{i+1}-b_i|\right\}_{n\geq 1}$ 有界,而设界为$M>0$.
- $\lim_{n\to\infty}c_n=0$知对任意固定的$\varepsilon>0$,
$$\exists\ N_1>0,\ s.t.\ n\geq N_1\Rightarrow|c_n|<\frac{\varepsilon}{2M}.$$
- $\lim_{n\to\infty}b_n=0$知对上述N_1,
$$\exists\ N>0,\ n\geq N\Rightarrow\frac{|a_{N_1}|}{|b_n|}<\frac{\varepsilon}{2}.$$

现有
$$a_n\ =\ a_{n-1}+c_{n-1}(b_n-b_{n-1})$$

$$
\begin{aligned}
&= \cdots \\
&= a_{N_1} + \sum_{i=N_1}^{n-1} c_i \left(b_{i+1} - b_i\right),
\end{aligned}
$$

而

$$
\begin{aligned}
\left|\frac{a_n}{b_n}\right| &\leq \frac{|a_{N_1}|}{|b_n|} + \max_{N_1 \leq i \leq n-1} \cdot \frac{\sum_{i=N_1}^{n-1} |b_{i+1} - b_i|}{|b_n|} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2M} \cdot M \\
&= \varepsilon, \text{ 当 } n > N \text{ 时}.
\end{aligned}
$$

故

$$
\lim_{n \to \infty} \frac{a_n}{b_n} = 0.
$$