Contents

1. Some results from geometry

▲ Surjectivity of Gauss map

Let $\Omega\left(\subset\mathbb{R}^N, N\geq 2\right)$ be smooth, simple-connected, bounded and open. Then

$$\{n(x);\ x\in\partial\Omega\}=S^{N-1}. \tag{1}$$

Proof. This is geometrically obvious. The analytical proof is as follows.

- $\{n(x);\ x\in\partial\Omega\}\subset S^{N-1}$ is OK, since $|n(x)|=1$.
- We proceed to show

$$S^{N-1}\subset\{n(x);\ x\in\partial\Omega\}\,.$$

To this end, choose a fixed ball $B\subset\Omega$. For $n\in S^{N-1}$, let H be the tangent space of ∂B such that $n\perp H$(see Figure 1). Consider

$$\{H+tn;\ 0\leq t<\infty\}\,,$$

since Ω is bounded, we have

$$0<t_0=\sup\left\{t>0;\ (H+tn)\cap\overline{\Omega}\neq\emptyset\right\}<\infty.$$

★ By continuity, $(H+t_0n)\cap\overline{\Omega}$ is a finite union of line segments (which may degenerate to points).

★ **Claim** $(H+t_0n)$ is the tangent space of $\partial\Omega$ at x_0 (thus, $n\perp(H+t_0n)=T_{x_0}\partial\Omega$, $n=n(x_0)$ as desired).
Indeed, for any $v\in T_{x_0}\partial\Omega$, let

$$\gamma:(-\varepsilon,\varepsilon)\to\partial\Omega,\ \gamma(0)=x_0,\ \dot{\gamma}(0)=v.$$

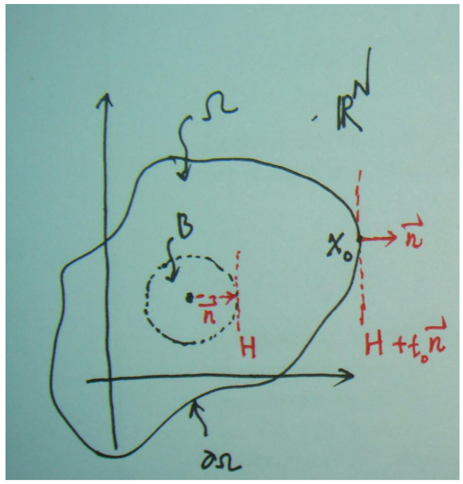

Figure 1: Surjectivity of Gauss map

Then

$$
\begin{aligned}
& (\gamma(h)-\gamma(0))\cdot n \leq 0 \\
\Rightarrow\ & \begin{cases} \frac{\gamma(h)-\gamma(0)}{h}\cdot n \leq 0, & h>0 \\ \frac{\gamma(h)-\gamma(0)}{h}\cdot n \geq 0, & h<0 \end{cases} \\
\Rightarrow\ & (\text{taking limits})\, 0 \leq v\cdot n \leq 0 \\
\Rightarrow\ & v\cdot n = 0 \\
\Rightarrow\ & v \in (H+t_0 n) \\
\Rightarrow\ & T_{x_0}\partial\Omega \subset (H+t_0 n) \\
\Rightarrow\ & T_{x_0}\partial\Omega = (H+t_0 n)\, (\text{by linear algebra})\, .
\end{aligned}
$$

Here we take $(H + t_0 n)$ as a linear space assuming x_0 to be 0.

□

▲ Relations to the second fundamental form.

Let $\Omega\left(\subset \mathbb{R}^N, N \geq 2\right)$ be a smooth, simple-connected, bounded and open. Assume that $u \in C^\infty(\overline{\Omega})$ satisfying $u \cdot n = 0$ on $\partial\Omega$, where n is the unit outward normal vector of $\partial\Omega$. Then

$$(u\cdot\nabla)\, n\cdot u = \sum_{i,j=1}^{N} u_j \partial_j n_i u_i = B(u,u) = \langle S\,(u)\,, u\rangle = \sum_{i=1}^{N} \lambda_i \left|u^i\right|^2 . \quad (2)$$

Here

- u^i is the components of u under some orthonormal principal directions e_i, to which the corresponding principle curvatures are λ_i.
- B is the second fundamental form of $\partial\Omega$.
- S is the shape operator.

Proof. Fixed a given point $x \in \partial\Omega$ we are calculating, let $\{e_i\}_{i=1}^N \subset T_x\partial\Omega$ be an orthonormal basis such that

$$S(e_i) = \lambda_i e_i,$$

where $S(v) = \nabla_v n,\ v \in T_x\partial\Omega$ is the shape operator.

Then one has

$$u = \sum_{i=1}^N u^i e_i,$$

and

$$\begin{aligned}
(u\cdot\nabla)\,n\cdot u &= \nabla_u n\cdot u \\
&= S(u)\cdot u\ \ (= B(u,u)) \\
&= S\left(\sum_{i=1}^N u^i e_i\right)\cdot\left(\sum_{j=1}^N u^j e_j\right) \\
&= \sum_{i=1}^N \langle S(e_i), e_i\rangle |u|^2 \\
&= \sum_{i=1}^N \lambda_i\cdot\left|u^i\right|^2.
\end{aligned}$$

□

Remark. *One is referred to [**?** **?**] for details of Gauss map, shape operator, principle curvatures (directions), and other fundamental concepts in Riemannian Geometry. Cheers!*

2. 6.4 The compactness result

- **Statement of the result** If $\left\{\begin{array}{l} q \geq 2 \\ q > \gamma \end{array}\right.$,then

$$\rho^n \to \rho \left\{\begin{array}{ll} \text{in } L^p, & \text{if (??) or periodic} \\ \text{in } L^p_{loc}, & \text{if } \mathbb{R}^N \end{array}\right. \quad \forall\ 1 \leq p < q.$$

Remark. ▲ *The reason for the convergence in local sense is twofold. The one comes from the compact imbedding $H^1_{local} \subset\subset L^{<\frac{2N}{N-2}}$, and the other is the utilization of weak-weak-convergence method.*

▲ *Recall that in (??), we have*

$$\rho^n \to \rho, \ in\ L^q\left(K_1 \times (0,T)\right),$$

where

$$K_1 = \left\{\begin{array}{ll} \Omega, & \textit{if periodic or } \mathbb{R}^N\,(N \geq 3)\,, \\ \subset\subset \Omega, & \textit{Dirichlet or } \mathbb{R}^2. \end{array}\right.$$

The local sense follows as

■ *Dirichlet: in order to apply the nonlocal operator $(-\triangle)^{-1}$ div , we need cut-off!*

■ *$\mathbb{R}^2$: no global control of L^p-norm on u^n.*

- **Proof of the compactness result**

We remark first that the proof here is similar in the spirit of that in Subsubsection **??**, however, we do not need to consider the nonlocal operator $(-\triangle)^{-1}$ div or invoke any L^p bounds of u^n, our result is global if the domain is bounded!

The proof is made up of the following four steps.

▲ The inequality satisfied by $\overline{(\varepsilon+\rho)^\theta}$, where $0 < \theta \ll 1$ to be specified later on.

Recall that $\overline{(\varepsilon+\rho)^\theta}$ is weak limit of $(\varepsilon+\rho^n)^\theta$ (in $L^{\frac{q}{\theta}}$ for example), and the $(\varepsilon > 0)$ is placed so that we are away from zones of vacuum.

Claim

$$\theta\alpha^n\left(\varepsilon+\rho^n\right)^\theta + \operatorname{div}\left\{\left(\varepsilon+\rho^n\right)^\theta u^n\right\} - \varepsilon^n\triangle\left(\varepsilon+\rho^n\right)^\theta \tag{3}$$

$$\geq \quad \theta\left[h^n + \varepsilon \mathrm{div}\, u^n + \alpha^n \varepsilon\theta\right](\varepsilon+\rho^n)^{\theta-1} + (1-\theta)(\varepsilon+\rho^n)^{\theta}\,\mathrm{div}\, u^n.$$

The formal proof of (3) is as follows. (**??**), together with the following observations:

- $$\begin{gathered}(\varepsilon+\rho^n)^\theta = (\rho^n)^\theta + \theta(\xi^n)^{\theta-1}\varepsilon \geq (\rho^n)^\theta + \theta(\varepsilon+\rho^n)^{\theta-1}\varepsilon \\ (\rho^n < \xi^n < \varepsilon+\rho^n) \\ \Rightarrow \quad \theta\alpha^n(\varepsilon+\rho^n)^\theta \geq \theta\alpha^n(\rho^n)^\theta + \theta\left[\alpha^n\varepsilon\theta\right](\varepsilon+\rho^n)^{\theta-1};\end{gathered}$$

- $$\begin{aligned} & \mathrm{div}\,\left\{(\varepsilon+\rho^n)^\theta u^n\right\} = (\varepsilon+\rho^n)^\theta\,\mathrm{div}\, u^n + \theta(\varepsilon+\rho^n)^{\theta-1} u^n\cdot\nabla\rho^n \\ = \quad & \theta(\varepsilon+\rho^n)^{\theta-1}\,\mathrm{div}\,(\rho^n u^n) + (\varepsilon+\rho^n)\,\mathrm{div}\, u^n \\ & -\theta(\varepsilon+\rho^n)^{\theta-1}\left[(\varepsilon+\rho^n)-\varepsilon\right]\mathrm{div}\, u^n \\ = \quad & \theta(\varepsilon+\rho^n)^{\theta-1}\,\mathrm{div}\,(\rho^n u^n) + (1-\theta)(\varepsilon+\rho^n)^\theta\,\mathrm{div}\, u^n \\ & +\theta\left[\varepsilon \mathrm{div}\, u^n\right](\varepsilon+\rho^n)^{\theta-1};\end{aligned}$$

- $$\begin{gathered}\partial_i(\varepsilon+\rho^n)^\theta = \theta(\varepsilon+\rho^n)^{\theta-1}\partial_i\rho^n,\ \forall\ 1\leq i\leq N, \\ \triangle(\varepsilon+\rho^n)^\theta = \theta(\theta-1)(\varepsilon+\rho^n)^{\theta-2}|\nabla\rho^n|^2 + \theta(\varepsilon+\rho^n)^{\theta-1}\triangle\rho^n,\end{gathered}$$
$$\begin{aligned} -\varepsilon^n\triangle(\varepsilon+\rho^n)^\theta \quad = \quad & \varepsilon^n\theta(1-\theta)(\varepsilon+\rho^n)^{\theta-2}|\nabla\rho^n|^2 \\ & -\varepsilon^n\theta(\varepsilon+\rho^n)^{\theta-1}\triangle\rho^n \\ \geq \quad & -\varepsilon^n\theta(\varepsilon+\rho^n)^{\theta-1}\triangle\rho^n;\end{aligned}$$

implies (3) (by multiplying (**??**) by $\theta(\varepsilon+\rho^n)^{\theta-1}$).
While the justifications invoking regularization needs only

$$\frac{\theta}{q} + \frac{1}{2} \leq 1 \Rightarrow \theta \leq \frac{q}{2}.$$

We now write (3) into a form suitable for weak limits as

$$\begin{aligned} & \theta\alpha^n(\varepsilon+\rho^n)^\theta + \mathrm{div}\,\left\{(\varepsilon+\rho^n)^\theta u^n\right\} - \varepsilon^n\triangle(\varepsilon+\rho^n) \\ \geq \quad & \theta\left[h^n + \alpha^n\varepsilon\theta\right](\varepsilon+\rho^n)^{\theta-1} + \theta(\varepsilon+\rho^n)^{\theta-1}\,\mathrm{div}\, u^n \\ & +(1-\theta)(\varepsilon+\rho^n)^\theta\left\{\mathrm{div}\, u^n - b(\rho^n)^\gamma\right\} \\ & +(1-\theta)\,b(\varepsilon+\rho^n)^\theta(\rho^n)^\gamma.\end{aligned}$$

Taking weak limits in the above inequality as $n\to\infty$, noticing

- $\alpha^n \to \alpha$;
- $u^n \to u$ in L^p or L^p_{loc} $1 \leq p < \frac{2N}{N-2}$, and $\frac{N-2}{2N} + \frac{\theta}{q} \leq 1 \rightsquigarrow \theta \leq 1$ is OK;
- $\varepsilon^n \to 0$;
- $h^n \to h$ in L^1;
- $\operatorname{div} u^n - b(\varepsilon+\rho^n)^\gamma \to \operatorname{div} u - b\overline{\rho^\gamma}$ in L^a or L^a_{loc} where $1 \leq a < \min\left\{2, \frac{q}{\gamma}\right\}$, and

$$\frac{\theta}{q} + \frac{1}{a} \leq 1 \Rightarrow 0 < \theta \leq q\left(1 - \frac{1}{a}\right) < \min\left\{\frac{q}{2}, q-\gamma\right\}; \tag{4}$$

we obtain

$$\begin{aligned} & \theta\alpha\overline{(\varepsilon+\rho)^\theta} + \operatorname{div}\left\{\overline{(\varepsilon+\rho)^\theta} u\right\} \\ \geq\ & \theta\left[h + \alpha\varepsilon\theta\right]\overline{(\varepsilon+\rho)^{\theta-1}} + \theta\varepsilon\overline{(\varepsilon+\rho)^{\theta-1}\operatorname{div} u} \\ & + (1-\theta)\overline{(\varepsilon+\rho)^\theta}\left\{\operatorname{div} u - b\overline{\rho^\gamma}\right\} + (1-\theta)\, b\overline{(\varepsilon+\rho)^\theta \rho^\gamma} \\ =\ & \theta\left[h + \alpha\varepsilon\theta\right]\overline{(\varepsilon+\rho)^{\theta-1}} + \theta\varepsilon\overline{(\varepsilon+\rho)^{\theta-1}\operatorname{div} u} \\ & + (1-\theta)\overline{(\varepsilon+\rho)^\theta}\operatorname{div} u + (1-\theta)\, b\left\{\overline{(\varepsilon+\rho)^\theta \rho^\gamma} - \overline{(\varepsilon+\rho)^\theta}\,\overline{\rho^\gamma}\right\}. \end{aligned} \tag{5}$$

▲ The inequality satisfied by $\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}}$.

Formally, multiplying (5) by $\frac{1}{\theta}\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1}$ yields

$$\begin{aligned} & \alpha\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}} + \operatorname{div}\left\{\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}} u\right\} \\ =\ & \frac{1}{\theta}\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1}\left[\theta\alpha\overline{(\varepsilon+\rho)^\theta} + \theta\overline{(\varepsilon+\rho)^\theta}\operatorname{div} u + u\cdot\nabla\overline{(\varepsilon+\rho)^\theta}\right] \\ =\ & \frac{1}{\theta}\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1}\left[\theta\alpha\overline{(\varepsilon+\rho)^\theta} + \operatorname{div}\left\{\overline{(\varepsilon+\rho)^\theta} u\right\} - (1-\theta)\overline{(\varepsilon+\rho)^\theta}\operatorname{div} u\right] \\ \geq\ & \left[h + \alpha\varepsilon\theta\right]\overline{(\varepsilon+\rho)^{\theta-1}}\cdot\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1} + \overline{\varepsilon(\varepsilon+\rho)^{\theta-1}\operatorname{div} u}\cdot\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1} \\ & + \frac{1-\theta}{\theta} b\left\{\overline{(\varepsilon+\rho)^\theta \rho^\gamma} - \overline{\rho^\gamma}\,\overline{(\varepsilon+\rho)^\theta}\right\}\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1}. \end{aligned} \tag{6}$$

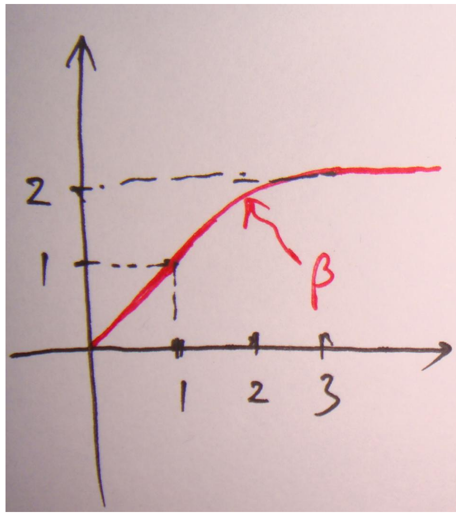

Figure 2: A concave, slowly increasing function

$$\equiv \quad I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon.$$

However, the justification is a bit delicate, in the same spirit as in Subsubsection ??. Multiplying (5) by $\frac{1}{\theta}\beta_R'\left(\overline{(\varepsilon+\rho)^\theta}\right)\beta_R\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}-1}$, where β has the graph as in Figure 2 , and $\beta_R(\cdot) = R\beta\left(\frac{\cdot}{R}\right)$. Noticing that

■

$$\begin{aligned} & \alpha\overline{(\varepsilon+\rho)^\theta}\beta_R'\left(\overline{(\varepsilon+\rho)^\theta}\right)\beta_R\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}-1} \\ = \quad & \alpha t\beta_R'(t)\beta_R(t)^{\frac{1}{\theta}-1}\left(t = \overline{(\varepsilon+\rho)^\theta}\right) \\ \leq \quad & \alpha\beta_R(t)^{\frac{1}{\theta}} = \alpha\beta_R\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}}. \end{aligned}$$

Indeed, we need only show

$$\begin{aligned} & t\beta_R'(t) \leq \beta_R(t) \Leftarrow t\beta'\left(\frac{t}{R}\right) \leq R\beta\left(\frac{t}{R}\right) \\ \Leftarrow \quad & x\beta'(x) \leq \beta(x)\left(x = \frac{t}{R}\right) \end{aligned}$$

$$\Leftarrow \quad 0 \leq [x\beta'(x) - \beta(x)]' = \beta''(x), \tag{7}$$

which is OK by Fig 2.

■

$$\begin{aligned}
& \text{div} \left\{ \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}} u \right\} \\
= & \frac{1}{\theta} \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} \beta_R' \left(\overline{(\varepsilon+\rho)^\theta} \right) u \cdot \nabla \overline{(\varepsilon+\rho)^\theta} + \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}} \text{div } u \\
= & \frac{1}{\theta} \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} \beta_R' \left(\overline{(\varepsilon+\rho)^\theta} \right) \text{div } \left\{ \overline{(\varepsilon+\rho)^\theta} u \right\} \\
& + \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} \left[\beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right) - \frac{1}{\theta} \beta_R' \left(\overline{(\varepsilon+\rho)^\theta} \right) \overline{(\varepsilon+\rho)^\theta} \right] \text{div } u.
\end{aligned}$$

We obtain

$$\begin{aligned}
& \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}} + \text{div } \left\{ \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}} u \right\} \\
\geq & [h + \alpha\varepsilon\theta] \, \overline{(\varepsilon+\rho)^{\theta-1}} \beta_R' \left(\overline{(\varepsilon+\rho)^\theta} \right) \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} \\
& + \overline{\varepsilon \, (\varepsilon+\rho)^{\theta-1} \text{ div } u} \cdot \beta_R' \left(\overline{(\varepsilon+\rho)^\theta} \right) \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} \\
& + \frac{1-\theta}{\theta} b \left\{ \overline{(\varepsilon+\rho)^\theta \rho^\gamma} - \overline{(\varepsilon+\rho)^\theta} \, \overline{\rho^\gamma} \right\} \beta_R' \left(\overline{(\varepsilon+\rho)^\theta} \right) \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} \\
& + \beta \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} \left[\beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right) - \beta_R' \left(\overline{(\varepsilon+\rho)^\theta} \right) \overline{(\varepsilon+\rho)^\theta} \right] \text{div } u \\
= & I_1^{\varepsilon,R} + I_2^{\varepsilon,R} + I_3^{\varepsilon,R} + I_4^{\varepsilon,R}.
\end{aligned}$$

Now it is the right time to take $R \to \infty$, observing that

■

$$\begin{aligned}
I_1^{\varepsilon,R} & \geq (h + \alpha\varepsilon\theta) \, \overline{(\varepsilon+\rho)^{\theta-1}} \beta_R \left(\overline{(\varepsilon+\rho)^\theta} \right)^{\frac{1}{\theta}-1} 1_{\overline{(\varepsilon+\rho)^\theta} \leq R} \\
& \geq (h + \alpha\varepsilon\theta) \, \overline{(\varepsilon+\rho)^{\theta-1}} \, \overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1} 1_{\overline{(\varepsilon+\rho)^\theta} \leq R} \\
& \geq 0,
\end{aligned}$$

thus by Fatou's lemma, $\liminf_{R\to\infty} I_1^{\varepsilon,R} \geq I_1^\varepsilon$ in $\mathcal{D}'$, that is,

$$\liminf_{R\to\infty} \int I_1^{\varepsilon,R} \phi dx \quad \geq \quad \int \lim_{R\to\infty} I_1^{\varepsilon,R} \phi dx$$

$$= \int I_1^\varepsilon \phi dx, \ \forall\, 0 \le \phi \in \mathcal{D}'.$$

-

$$\begin{aligned} \left|I_2^{\varepsilon,R}\right| &\le C\varepsilon \cdot \left(\varepsilon^{\theta-1}\left|\overline{\mathrm{div}\, u}\right|\right) \cdot \overline{(\varepsilon+\rho)}^{1-\theta} \\ &\le C\varepsilon^\theta \left|\overline{\mathrm{div}\, u}\right| \cdot \overline{(\varepsilon+\rho)}^{1-\theta}, \end{aligned}$$

  which is bounded in L^1; thus $\lim_{R\to\infty} I_2^{\varepsilon,R} = I_2^\varepsilon$, in L^1 by Lebesgue dominated convergence theorem;
- Due to (**??**),

$$\overline{(\varepsilon+\rho)^\theta \rho^\gamma} - \overline{(\varepsilon+\rho)^\theta}\,\overline{\rho^\gamma} \ge 0, \text{ a.e.},$$

$$I_3^{\varepsilon,R} \ge \tfrac{1-\theta}{\theta} b \left\{\overline{(\varepsilon+\rho)^\theta \rho^\gamma} - \overline{(\varepsilon+\rho)^\theta}\,\overline{\rho^\gamma}\right\} \overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}} 1_{\overline{(\varepsilon+\rho)^\theta} \le R} \ge 0,$$

  and again by Fatou's lemma, $\lim_{R\to\infty} I_3^{\varepsilon,R} \ge I_3^\varepsilon$ in $\mathcal{D}'$;
-

$$\begin{aligned} \left|I_4^{\varepsilon,R}\right| &\le \left|\beta_R(t)^{\frac{1}{\theta}} - \beta_R(t)^{\frac{1}{\theta}-1}\beta_R'(t)t\right| \cdot |\mathrm{div}\, u| \left(t = \overline{(\varepsilon+\rho)^\theta}\right) \text{ (by (7))} \\ &\le Ct^{\frac{1}{\theta}} |\mathrm{div}\, u|\, 1_{t\ge R} \\ &\to 0, \text{ in } L^1_{loc} \left(\mathrm{div}\, u \in L^2,\ \rho \in L^2\right); \end{aligned}$$

we find the desired inequality (6).

▲ Passage to limit $\varepsilon \to 0_+$.
In this circumstance, we shall invoke Lebesgue dominated convergence theorem, thus the following dominated functions are needed:

- We've already shown

$$\left|I_2^\varepsilon\right| \le C\left|\overline{\mathrm{div}\, u}\right| \cdot (1+\rho)^{1-\theta} \in L^1_{loc} \left(\frac{1}{2}, \frac{1}{2} + \frac{1-\theta}{2} \le 1\right);$$

- and for $\forall\, 0 \le \phi \in C_c^\infty$,

$$0$$

$$
\begin{aligned}
&\leq \int (I_2^\varepsilon + I_3^\varepsilon)\,\phi \\
&\leq \int \alpha\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}}\phi + \int \mathrm{div}\,\left\{\overline{(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}} u\right\}\phi - \int I_2^\varepsilon\phi \\
&\leq C\int \alpha\,(1+\rho)\,\phi + \int (1+\rho)\,|u\cdot\nabla\phi| + C\int \left|\overline{\mathrm{div}\,u}\right|(1+\rho)^{1-\theta}\,\phi,
\end{aligned}
$$

with

$$
\alpha\,(1+\rho)\,\phi, (1+\rho)\,|u\cdot\nabla\phi|\,, \left|\overline{\mathrm{div}\,u}\right|(1+\rho)^{1-\theta}\,\phi \in L^1.
$$

Thus letting $\varepsilon \to 0_+$ in (6), we find that

$$
\alpha\overline{\rho^\theta}^{\frac{1}{\theta}} + \mathrm{div}\,\overline{\rho^\theta}^{\frac{1}{\theta}} u \geq h + \frac{1-\theta}{\theta} b\left\{\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma}\,\overline{\rho^\theta}\right\}\overline{\rho^\theta}^{\frac{1}{\theta}-1}, \tag{8}
$$

where we use the fact

$$
\begin{aligned}
&h\overline{(\varepsilon+\rho)^{\theta-1}(\varepsilon+\rho)^\theta}^{\frac{1}{\theta}-1} = h\overline{t^{1-\frac{1}{\theta}}}\cdot\bar{t}^{\frac{1}{\theta}-1}\left(t = (\varepsilon+\rho)^\theta\right) \\
\geq\ & h\bar{t}^{1-\frac{1}{\theta}}\cdot\bar{t}^{\frac{1}{\theta}-1}\ \text{(by convexity)} \\
=\ & h.
\end{aligned}
$$

▲ Invoking convexity to conclude the proof.
Taking $\varepsilon \to 0_+$ in (**??**), we have

$$
\alpha\rho + \mathrm{div}\,\{\rho u\} = h, \tag{9}
$$

thus

$$
\begin{aligned}
& (9) - (8) \\
\Rightarrow\ & \alpha s + \mathrm{div}\,\{su\} \leq -\frac{1-\theta}{\theta} b\left\{\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma}\,\overline{\rho^\theta}\right\}\overline{\rho^\theta}^{\frac{1}{\theta}-1} \\
& \left(s = \rho - \overline{\rho^\theta}^{\frac{1}{\theta}} \in [0,\rho]\right) \\
\left(\int\right)\ \Rightarrow\ & 0 \leq \overline{\rho^\gamma}\,\overline{\rho^\theta} - \overline{\rho^{\gamma+\theta}},\ \text{a.e. on}\ \left\{\overline{\rho^\theta} > 0\right\} \qquad (10) \\
\Rightarrow\ & 0 = \overline{\rho^\gamma}\,\overline{\rho^\theta} - \overline{\rho^{\gamma+\theta}},\ \text{a.e. on}\ \left\{\overline{\rho^\theta} > 0\right\} \\
\Rightarrow\ & \overline{\rho^\theta} = \overline{\rho^{\gamma+\theta}}^{\frac{\theta}{\gamma+\theta}}, \overline{\rho^\gamma} = \overline{\rho^{\gamma+\theta}}^{\frac{\gamma}{\gamma+\theta}},\ \text{a.e. on}\ \left\{\overline{\rho^\theta} > 0\right\}
\end{aligned}
$$

$$\begin{aligned}
&\Rightarrow \quad \overline{\rho^{\gamma+\theta}} = \overline{\rho^\theta}^{\frac{\gamma+\theta}{\theta}}, \text{ a.e. on } \left\{\overline{\rho^\theta} > 0\right\} \\
&\Rightarrow \quad (\rho^n)^\theta \to \rho^\theta, \text{ in } L^2_{loc} \\
&\qquad \left((\rho^n)^\theta \to \rho^\theta \text{ in } L^1_{loc}\left(\left\{\overline{\rho^\theta} = 0\right\}\right)\right) \\
&\Rightarrow \quad \rho^n \to \rho \text{ in } L^{2\theta<1}_{loc} \\
&\Rightarrow \quad \rho^n \to \rho \text{ in } L^p_{loc}, \ \forall\ 1 \leq p < q.
\end{aligned}$$

Remark. ■ *Here, the local convergence really mean the convergence in* $\Omega \cap B_R$, $\forall\ 0 < R < \infty$. *Thus the global convergence for bounded domains.*

■ *The integration over* Ω *in* (10) *needs justification in different settings.*

★ *Periodic case. In this case,* Ω *is a smooth compact manifold without boundary (closed manifold), thus divergence theorem tells us* $\int_\Omega div\ (su) = 0$.

★ (**??**) *case. In this case,*

$$\int div\ (su) = \int_{\partial\Omega} su \cdot n = 0.$$

★ $\mathbb{R}^N$ *case. Cut-off function technique is needed, for a* ϕ *as in Figure* **??**, *we have*

$$\begin{aligned}
RHS\ of\ (10) \quad &\geq \quad \int div\ \{su\}\,\phi_R = -\int su \cdot \nabla\phi_R \\
&\geq \quad -\frac{C}{R}\int \sqrt{\rho}\cdot(\sqrt{\rho}\,|u|) \geq -\frac{C}{R}\,\|\rho\|_2\,\left\|\sqrt{\rho}u\right\|_2 \\
&\qquad (\textit{see the Claim in Subsubsection } \mathbf{??}) \\
&\to \quad 0,\ as\ R \to \infty.
\end{aligned}$$

3. Acknowledgements

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Thanks are also due to Professor H.X. Zhao for his patience and constant encouragements.