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AN EXISTENCE THEOREM FOR STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we show the existence of a solution to the stationary compressible Navier-Stokes equations under Dirichlet boundary conditions. This is [1, Page 121], and is delivered on Dec. 4th, 2010.

Theorem 1. *(Existence/Dirichlet BVP). Let* $\gamma = 5/3$, $N = 3$, $p \in (1,2)$. *Then* $\exists$ *a continuum* $C\left(\subset L^q \cap W^{1,q}, 1 \leq q < 2\right)$ *of solutions of*

$$\begin{cases} div\ (\rho u) = 0, \\ div\ (\rho u \otimes u) - \mu \Delta u - \xi \nabla div\ u + a \nabla \rho^\gamma = \rho f + g, \end{cases} \quad in\ \Omega \tag{1}$$

such that

1. $C \cap \{(\rho, u, M)\,;\ 0 \leq M \leq R\}$ *is bounded in* $L^2 \times H_0^1$, $\forall\ R > 0$;
2. $(0, u_0) \in C$ *where* u_0 *satisfies*

$$\begin{cases} -\mu \Delta u_0 - \xi \nabla div\ u_0 = g, & in\ \Omega, \\ u_0 = 0, & on\ \partial\Omega; \end{cases} \tag{2}$$

3. $\forall\ M > 0$, $\exists\ (\rho, u) \in C$ *such that* $\int_\Omega \rho^p = M$.

Proof. **Step I: Bounds for solution of the approximate problems:**

$$\begin{cases} \alpha \rho^p + \mathbf{div}\ (\rho u) = \alpha \frac{M}{|\Omega|}, \\ \alpha \rho^p u + \mathbf{div}\ (\rho u \otimes u) - \mu \Delta u - \xi \nabla \mathbf{div}\ u + a \nabla \rho^\gamma = \rho f + g, \end{cases} \quad \mathbf{in}\ \Omega. \tag{3}$$

1. $\int_\Omega \rho^p = M$;

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2. $\|u\|_{H^1} \leq C\left(1+\|\rho\|_{6/5}\right),\ \|\rho\|_\gamma \leq C\left(1+\|u\|_{H^1}^{3/2}\right)$; which follows form the energy identity:

$$\int_\Omega \left\{\alpha \frac{M}{|\Omega|}\frac{|u|^2}{2} + \alpha\rho^p \frac{|u|^2}{2} + \frac{a\alpha\gamma}{\gamma-1}\left(\rho^\gamma - h\rho^{\gamma-1}\right) \right.$$
$$\left. +\mu\,|Du|^2 + \xi\,|\mathrm{div}\ u|^2 - \rho u\cdot f - u\cdot g\right\} = 0.$$

3. $\|\rho\|_2 \leq C,\ \|u\|_{H^1} \leq C.$

Direct computations show

$$\begin{aligned}
\|\rho^\gamma\|_r &\leq \left\|\rho^\gamma - \fint_\Omega \rho^\gamma\right\|_r + |\Omega|^{1/r}\fint_\Omega \rho^\gamma \\
&\leq C\,\|\nabla\rho^\gamma\|_{W^{-1,r}} + C + C\,\|u\|_{H^1}^{5/2} \\
&\leq C + C\left\|\rho\,|u|^2\right\|_r + C\,\|u\|_{H^1}^{5/2} \\
&\leq C + C\,\|\rho\|_{\gamma r}\left\||u|^2\right\|_{\frac{\gamma}{\gamma-1}r} + C\,\|u\|_{H^1}^{5/2} \\
&\leq C + C\,\|\rho\|_{\gamma r}\,\|u\|_6^2 + C\,\|u\|_{H^1}^{5/2}\ \ (\text{if } \gamma r = 3(\gamma-1) = 2)\,.
\end{aligned}$$

Thus

$$\|\rho\|_2^\gamma \leq C\left(1 + \|\rho\|_2\,\|\nabla u\|_2^2 + \|u\|_{H^1}^{5/2}\right),$$

$$\|\rho\|_2^{1/3} \leq C\left(1+\|\rho\|_{6/5}\right).$$

To proceed further, we split into two cases.

(a) When $6/5 \leq p < 2$, $\|\rho\|_{6/5} \leq |\Omega|^{1/p-5/6}\,\|\rho\|_p \leq C$.

(b) In case $1 < p < 6/5$, $\|\rho\|_{6/5} \leq \|\rho\|_p^{1-\vartheta}\,\|\rho\|_2^\vartheta$ with

$$\frac{5}{6} = \frac{1-\vartheta}{p} + \frac{\vartheta}{2} \Rightarrow \vartheta = \frac{6-5p}{3(2-p)} < \frac{1}{3}.$$

Step II: The second approximation scheme and continuum.

We approximate (3) further by

$$\left.\begin{array}{l}
\alpha\rho^p + \mathrm{div}\ (\rho u) - \varepsilon\Delta\rho = \frac{\alpha M}{|\Omega|}, \\
\frac{\alpha M}{|\Omega|}\frac{u}{2} + \frac{1}{2}\rho u\cdot\nabla u + \alpha\rho^p\frac{u}{2} + \frac{1}{2}\mathrm{div}\ (\rho u\otimes u) \\
\qquad -\mu\Delta u - \xi\nabla\mathrm{div}\ u + a\nabla\rho^\gamma + \delta\nabla\rho^2 = \rho f + g,
\end{array}\right\}\ \text{in } \Omega,$$
$$\frac{\partial\rho}{\partial n} = 0, \quad u = 0, \qquad\qquad \text{on } \partial\Omega, \qquad (4)$$

where $\varepsilon, \delta \in (0, 1]$. Here we add **viscosity and artificial pressure**.

We shall next establish the existence of a continuum (parameterized by M) of solutions of (4), and by taking $\varepsilon \to 0_+$, then $\delta \to 0_+$, then $\alpha \to 0_+$, in the next step, to conclude the proof of Theorem 1.

Before invoking Leray-Schauder's fixed point theorem to show such a solution continuum, we first establish some a priori estimates, which shall be useful later on.

1. $\int_\Omega \rho^p = M$.
2. Energy identity:

$$\int_\Omega \left\{ \frac{\alpha}{2} h |u|^2 + \frac{1}{2}\alpha\rho^p |u|^2 + \mu |Du|^2 + \xi |\mathrm{div}\, u|^2 + \varepsilon a\gamma\rho^{\gamma-2} |\nabla\rho|^2 + 2\varepsilon\delta |\nabla\rho|^2 \right.$$
$$\left. \frac{a\alpha\gamma}{\gamma-1}\left(\rho^{\gamma+p-1} - h\rho^{\gamma-1}\right) + 2\delta\alpha\left(\rho^{p+1} - h\rho\right) \right\} = \int_\Omega \{\rho u \cdot f + u \cdot g\}.$$

3. $\|\rho\|_3 \le C$, $\|u\|_{H^1} \le C$, independent of $\varepsilon \in (0, 1]$.

   Notice that the improved regularity of ρ comes from the artificial pressure:

$$\frac{5}{3} \to 2, \quad 2 \to 3.$$

We now show the existence of a solution continuum $C_\alpha^{\delta,\varepsilon}$ to (4) by invoking the following

Theorem 2. *(Leray-Schauder). Let X be a Banach space, and $T : X \times [0, 1] \to X$ be compact. Assume*

1. *$T(x, 0) = x$, $\forall\, x \in X$;*
2. *$\exists\, M > 0$, s.t. $x = T(x, \sigma)$, $\sigma \in [0, 1] \Rightarrow \|x\| \le M$.*

Then $T(\cdot, 1)$ has a fixed point.

The Banach space we live is chosen to be $X = W^{1,\infty} \times \left(W^{1,\infty}\right)^N$; and $[0, 1]$ is rescaled to be $[0, M]$. The compact mapping is defined as

$$T(M, \varphi, v) = (\rho, u) - (0, u_0),$$

where (ρ, u) satisfy

$$\left.\begin{array}{l} \alpha\rho^p + \text{div }(\rho v) - \varepsilon\Delta\rho = \frac{\alpha M}{|\Omega|}, \\ \frac{\alpha M}{|\Omega|}\frac{u}{2} + \rho v\cdot\nabla v + \frac{1}{2}\varepsilon\Delta\rho v - \mu\Delta u - \xi\nabla\text{div } u + a\nabla\rho^\gamma + \delta\nabla\rho^2 = \rho f + g, \end{array}\right\} \text{ in } \Omega,$$

$$\frac{\partial\rho}{\partial n} = 0, \quad u = 0, \qquad \text{on } \partial\Omega.$$

Notice that the compactness follows from the fact that $\cap_{1\leq q<\infty}W^{2,q} \hookrightarrow\hookrightarrow W^{1,\infty}$, and the uniform bounds in Condition 2 of Theorem 2 follows readily from the classical elliptic estimates in $W^{2,q}$, $1 \leq q < \infty$ and a bootstrap argument.

Step III: Passage to limits.

Before passing to limit $\varepsilon \to 0_+$, then $\delta \to 0_+$, then $\alpha \to 0_+$, we recall

Lemma 3. ([1, Appendix D]). *Let (E, d) be a complete metric space and $\{C_n\}$ be a sequence of continua (closed, connected subsets) in $E \times [0, \infty)$ with*

(A1) C_n is unbounded in $E \times \boldsymbol{R}$;

(A2) $\exists\ x_0 \in E,\ s.t.\ (x_0, 0) \in C_n$;

(A3) $C_n \cap (E \times [0, R]) \subset K_R$, K_R compact in $E \times \boldsymbol{R}$, $\forall\ R > 0$; or equivalently

(A3′) $C_n \cap (E \cap [0, R])$ is compact:

$$(x_n, t_n) \in C_n,\ t_n \text{ bounded} \Rightarrow x_n \text{ relatively compact in } E.$$

Then the limit continuum

$$C = \{(x, t) \in E \times [0, \infty);\ \exists\ \{n_k\},\ \exists\ x_{n_k} \to x,\ \exists\ t_{n_k} \to t,\ (x_{n_k}, t_{n_k}) \in C_{n_k}\}$$

satisfies

(C1) C is unbounded in $E \times \boldsymbol{R}$:

$$\forall\ t \geq 0,\ \exists\ x \in E,\ s.t.\ (x, t) \in C;$$

(C2) $(x_0, 0) \in C$;

(C3) $C \cap (E \times [0, R]) \subset K_{R'}$, $\forall\ R' > R \geq 0$.

We now commence our passage to limits, $\varepsilon \to 0_+$, then $\delta \to 0_+$, then $\alpha \to 0_+$, by invoking Lemma 3 to construct

$$C_\alpha^{\delta,\varepsilon} \to_\varepsilon C_\alpha^\delta \to_\delta C_\alpha \to_\alpha C \ \left(\text{this } C \text{ being what we pursue}\right).$$

1. $\varepsilon \to 0_+$**, for** $\alpha, \delta \in (0,1]$ **fixed.**

The underlying $E = L^{q_1} \times \left(W^{1,q_2}\right)^N, \ 1 \le q_1 < 3, \ 1 \le q_2 < 2$.

(A1) holds since $\int_\Omega \rho^p = M$.

(A2) holds since $(0, u_0) \in C_\alpha^{\delta,\varepsilon}$.

(A3′) Let $0 < \varepsilon_n \to 0$, $0 \le M_n \to M$, $(\rho_n, u_n) \in C_\alpha^{\delta,\varepsilon_n}$. We show the compactness of (ρ_n, u_n) in E as

$\rho_n \rightharpoonup \rho \ge 0$ in L^3; $u_n \rightharpoonup u$ in H^1, $u_n \to u$ in $L^p (1 \le p < 6)$, $u_n \to u$ a.e.;

$$\nabla\left\{\mathrm{div}\ u_n - \tfrac{a}{\mu+\xi}\rho_n^{5/3} - \tfrac{\delta}{\mu+\xi}\rho_n^2\right\} + \tfrac{\mu}{\mu+\xi}\mathrm{curl\ curl}\ u$$
$$= (\rho u \cdot \nabla)\, u + \cdots \text{ bounded in } \left(L^3 \cdot L^6\right) \cdot L^2 \subset \mathcal{H}^1$$
$$\Rightarrow \nabla\left\{\mathrm{div}\ u_n - \frac{a}{\mu+\xi}\rho_n^{5/3} - \frac{\delta}{\mu+\xi}\rho_n^2\right\}, \nabla \mathrm{curl}\ u_n \text{ bounded in } \mathcal{H}^1$$
$$\Rightarrow \mathrm{div}\ u_n - \frac{a}{\mu+\xi}\rho_n^{5/3} - \frac{\delta}{\mu+\xi}\rho_n^2 \text{ compact in } L^s\left(1 \le s < \frac{3}{2}\right); \mathrm{curl}\ u_n \text{ compact in } L^r (1 \le r < 2)$$
$$\Rightarrow \rho_n \to \rho \text{ in } L^{q_1} (1 \le q_1 < 3)$$
$$\Rightarrow \mathrm{div}\ u_n, \mathrm{curl}\ u_n, \text{ and thus } Du_n \ \to \ \mathrm{div}\ u, \mathrm{curl}\ u, Du \text{ in } L^{q_2}(1 \le q_2 < 2), \text{ respectively.}$$

Thus we have a continuum C_α^δ of solutions of

$$\left.\begin{aligned} &\alpha\rho^p + \mathrm{div}\ (\rho u) = \tfrac{\alpha M}{|\Omega|}, \\ &\alpha\rho^p u + \mathrm{div}\ (\rho u \otimes u) - \mu\Delta u - \xi\nabla \mathrm{div}\ u + a\nabla\rho^{5/3} + \delta\nabla\rho^2 = \rho f + g \end{aligned}\right\} \text{in } \Omega$$

satisfying (C1), (C2), (C3) in Lemma 3 and

$$C_\alpha^\delta \cap \{(\rho, u, M)\, ;\ 0 \le M \le R\} \text{ is bounded in } L^3 \times H_0^1 \times \boldsymbol{R}, \ \forall\, R > 0,$$

and the energy inequality

$$\int_\Omega \left\{\frac{\alpha}{2} h\,|u|^2 + \frac{1}{2}\alpha\rho^p\,|u|^2 + \mu\,|Du|^2 + \xi\,|\mathrm{div}\ u|^2 + \frac{a\alpha\gamma}{\gamma-1}\left(\rho^{\gamma+p-1} - h\rho^{\gamma-1}\right) + 2\delta\alpha\left(\rho^{p+1} - h\rho\right)\right\}$$
$$\le \int_\Omega \{\rho u \cdot f + u \cdot g\}, \ \forall\ (\rho, u, M) \in C_\alpha^\delta \left(h = \frac{\alpha M}{|\Omega|}\right).$$

2. $\delta \to 0_+$**, for** $\alpha \in (0,1]$ **fixed.**

The space we live now is $E = L^q \times \left(W^{1,q}\right)^N$, $1 \leq q < 2$. And the crucial key point is the compact assertion (A3′), which is proved as

$$\begin{aligned} &\nabla\left\{\text{div } u_n - \tfrac{a}{\mu+\xi}\rho_n^{5/3}\right\} + \tfrac{\mu}{\mu+\xi}\text{curl curl } u_n \\ &\quad = (\rho_n u_n \cdot \nabla) u_n + \cdots \text{ bounded in } \left(L^2 \cdot L^6\right) \cdot L^2 \subset \mathcal{H}^{6/7} \text{ (by Step I)} \end{aligned}$$

$$\Rightarrow \begin{cases} \text{div } u_n - \frac{a}{\mu+\xi}\rho_n^{5/3} \text{ compact in } L^s\left(1 \leq s < \frac{6}{5}\right) \\ \text{curl } u_n \text{ compact in } L^r(1 \leq r < 2) \end{cases} \left(-1 + \frac{3}{6/7} = \frac{3}{6/5}\right)$$

$$\Rightarrow \quad \rho_n \to \rho \text{ in } L^q(1 \leq q < 2)$$

$$\Rightarrow \quad \text{div } u_n, \text{curl } u_n, \text{ and thus } Du_n \text{ compact in } L^q(1 \leq q < 2).$$

Thus we find a continuum of solutions of (3) satisfying (C1), (C2), (C3) and

$$C_\alpha \cap \{(\rho, u, M)\,;\ 0 \leq M \leq R\}$$

$$\text{is bounded in } L^{\max\{2,p+2/3\}} \times H_0^1 \times \boldsymbol{R},\ \forall\, R > 0,$$

and the energy inequality

$$\begin{aligned} &\int_\Omega \left\{\frac{\alpha}{2} h\,|u|^2 + \frac{\alpha}{2}\rho^p\,|u|^2 + \mu\,|Du|^2 + \xi\,|\text{div } u|^2 + \frac{a\alpha\gamma}{\gamma-1}\left(\rho^{\gamma+p-1} + h\rho^{\gamma-1}\right)\right\} \\ &\quad \leq \int_\Omega \{\rho u \cdot f + u \cdot g\},\ \forall\ (\rho, u, M) \in C_\alpha \left(h = \frac{\alpha M}{|\Omega|}\right). \end{aligned}$$

3. $\alpha \to 0_+$ **finally.**

The space we work in now is $E = L^p \times \left(W^{1,p}\right)^N$, $1 \leq p < 2$. The details being exactly the same as the passage to limit $\delta \to 0_+$. And we conclude the existence of such a continuum C of solutions of (1) stated in Theorem 1.

□

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