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EXTRINSIC VIEW OF HARMONIC MAPS

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ABSTRACT. We see harmonic maps in an extrinsic point of view.
This is [1, 1.3].

By the isometric embedding theorem of Nash [2], we may assume

$$\left(N^l, h\right) \hookrightarrow \boldsymbol{R}^L,$$

for some $L \geq 1$. Then

$$C^2(M;N) = \left\{u = \left(u^1, \cdot, u^L\right) \in C^2(M;\boldsymbol{R}^L);\ u(M) \subset N\right\},$$

and

$$e(u) = \frac{1}{2} g^{\alpha\beta} u^i_\alpha u^i_\beta.$$

Since N is a closed submanifold of $\boldsymbol{R}^L$, we can construct the **nearest point projection map**

$$\Pi_N:\ N_\delta \to N$$

1. where

$$N_\delta = \left\{y \in \boldsymbol{R}^L;\ d(y,N) \equiv \inf_{z\in n} |y-z| < \delta\right\};$$

2. for $y \in N_\delta$, $\Pi_N(y) \in N$ is such that

$$|y - \Pi_N(y)| = d(y,N);$$

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3. and Π_N is smooth, the gradient of which,

$$P(y) = \nabla\Pi_N(y) : \ \boldsymbol{R}^L \to T_y N$$

is an orthogonal projection; the Hessian of which, induces the second fundamental form of $N \subset \boldsymbol{R}^L$:

$$\begin{array}{rccccl} A(y) \equiv \nabla P(y) : & T_y N & \times & T_y N & \to & \left(T_y N\right)^{\perp} \\ & (v & , & w) & \mapsto & \sum_{i=l+1}^{L} Hess\Pi_N(v,w)\nu_i(y), \end{array}$$

where $\{\nu_i(y)\}_{i=l+1}^{L}$ is a local orthonormal frame of the normal bundle $\left(T_y N\right)^{\perp}$.

Now, we have

Proposition 1. *$u \in C^2(M;N)$ is a* **harmonic map** *iff u satisfies*

$$\Delta_g u \perp T_u N.$$

Proof. For $\varphi \in C_0^2(M;\boldsymbol{R}^L)$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} \int_M |\nabla\Pi_N(u+t\varphi)|^2 \, \mathrm{d}v_g \\ &= 2\int_M \langle \nabla u, \nabla\left(P(u)\varphi\right)\rangle \, \mathrm{d}v_g \\ &= -2\int_M \left\langle \Delta_g u, P(u)\varphi \right\rangle \mathrm{d}v_g \\ &= -2\int_M \left\langle P(u)\left(\Delta_g u\right), \varphi \right\rangle \mathrm{d}v_g. \end{aligned}$$

□

Remark 2. *Notice that*

$$\Delta_g u \perp T_u N$$

is equivalent to the PDE :

$$\Delta_g u + A(u)\left(\nabla u, \nabla u\right) = 0, \ \textit{in } M.$$

In fact, we may write

$$\Delta_g u = \sum_{i=l+1}^{L} \lambda_i(x) \nu_i(u),$$

with

$$\begin{aligned} \lambda_i &= \left\langle \Delta_g u, \nu_i(u) \right\rangle \\ &= div_g (\nabla u \cdot \nu_i(u)) - \nabla u \cdot \nabla (\nu_i(u)) \\ &= -(\nabla \nu_i)(u)(\nabla u, \nabla u) \\ &= -A(u)(\nabla u, \nabla u). \end{aligned}$$

Example 3. *Let*

(a) $M = T^n$ be the n-dimensional flat torus;

(b) and $N = S^k \subset \boldsymbol{R}^{k+1}$ be the unit sphere.

Then $u \in C^2(T^n, S^k)$ is a harmonic map iff

$$\Delta u + |\nabla u|^2 u = 0, \text{ in } T^n.$$

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