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Contact graphs of disk packings as a model of spatial planar networks

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Abstract. Spatially constrained planar networks are frequently encountered in real-life systems. In this paper, based on a space-filling disk packing we propose a minimal model for spatial maximal planar networks, which is similar to but different from the model for Apollonian networks (Andrade *et al* 2005 *Phys. Rev. Lett.* **94** 018702). We present an exhaustive analysis of various properties of our model, and obtain the analytic solutions for most of the features, including degree distribution, clustering coefficient, average path length and degree correlations. The model recovers some striking generic characteristics observed in most real networks. To address the robustness of the relevant network properties, we compare the structural features between the investigated network and the Apollonian networks. We show that topological properties of the two networks are encoded in the way of disk packing. We argue that spatial constraints of nodes are relevant to the structure of the networks.

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Contents

1. Introduction

The past decade has witnessed a great deal of activity devoted to complex networks by the scientific community, since many systems in the real world can be described and characterized by complex networks [1, 2]. Prompted by the computerization of data acquisition and the increased computing power of computers, researchers have done a lot of empirical studies on diverse real networked systems, unveiling the presence of some generic properties of various natural and manmade networks: power-law degree distribution $P(k) \sim k^{-\gamma}$ with characteristic exponent γ in the range between 2 and 3 [3], small-world effect including large clustering coefficient and small average path length (APL) [4] and degree correlations [5]. These findings are important for our understanding of the real-life systems, since they strongly affect almost all aspects of various dynamical processes taking place on networks [6]–[8].

In order to reproduce or explain the above-mentioned striking common features of real-life systems, there has been a concerted effort in the last few years within the physical circle and elsewhere to develop network models to uncover and understand the complexity of real systems [1, 2]. In addition to the seminal Watts–Strogatz's (WS) small-world network model [4] and Barabási–Albert's (BA) scale-free network (SFN) model [3], a huge variety of models and mechanisms have been proposed to mimic real-world systems, including initial attractiveness [9], aging and cost [10], fitness model [11], duplication [12], weight- or traffic-driven evolution [13, 14], accelerating growth [15, 16], coevolution [17], visibility graph [18], to name but a few. For reviews, see [1, 2, 6, 7]. Although significant progress has been made in the field of network modeling and has led to a significant improvement in our understanding of complex systems, it is still a fundamental task and of current interest to construct models mimicking real networks and reproducing their generic properties from different angles.

Most previous network models concentrated on topological aspects and ignored the geographical effects. In these works, the node position has no special signification, which is reasonable for some networks that can be considered as lying in an abstract space, such as scientific collaboration network [19], metabolic network [20] and so forth. However, a plethora of real networks have well-defined node positions, including the Internet [21], power grid [22], airline networks [23], to name only a few. This class of networks is often referred to as geographical or spatial networks, which is a promising kind of network, since its geography has a pivotal influence on the dynamics running on the networks, such as robustness [24],

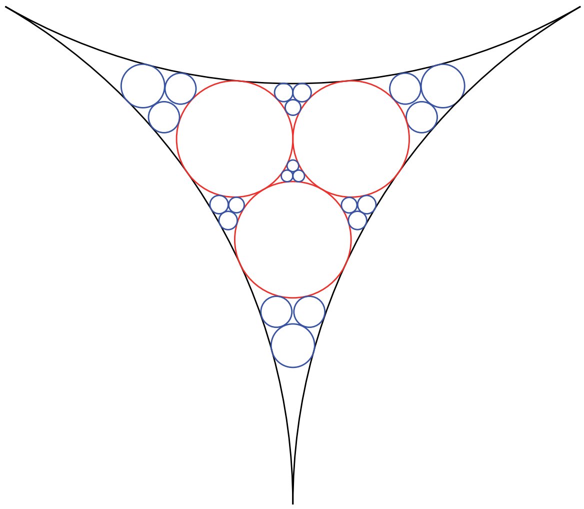

Figure 1. The first two generations for the construction of the disk packing.

cascading breakdown [25], synchronization [26], disease spreading [27], among others. In addition to the geography, some real-world geographical networks are also planar. Typical examples are street networks [28], electronic circuit networks [29], ant-trail networks [30] and neural networks [31]. Recently, several models [27], [32]–[37] have been presented that take the geographical position into consideration. However, models for spatial planar networks are much fewer, with the exception of space-filling networks—Apollonian networks [38] that have received a great amount of attention [39]–[43].

In this paper, inspired by the disk packing and the construction method of Apollonian networks [38], we propose a spatial planar network, where nodes (located at the center of the disks) correspond to the disks and edges represent contact relation. The suggested network belongs to a deterministically growing class of self-similar graphs. On the basis of the recursive construction and self-similar structure of the network, we determine analytically many relevant topological properties, such as degree distribution, clustering coefficient, APL as well as degree correlations. The obtained results show that the network is simultaneously scale-free, small world and disassortative, as observed in a variety of real networks. Except for the fact that all nodes of the network have separate spatial positions that encode the network structure, we also show that the network is not only planar but also maximally planar. Note that although Apollonian networks are also derived from a recursive disk packing, a main subject of the current paper is to compare the topological features of the presented network and Apollonian networks, with an aim to address the effect of the way of disk packings (thus the spatial constraints of nodes) on the network structure, which is important to the theory of complex networks.

2. Construction of the network

Before defining the network, we first introduce a disk packing [44, 45], which is a variation of the celebrated Apollonian packing [46]. The introduced disk packing, shown in figure 1, is constructed as follows. We start with three mutually touching disks, the interstice of which is a curvilinear triangle, and we denote this initial configuration by generation $t = 0$. Then in the first generation $t = 1$, three smaller mutually touching disks are added to fill the interstice of

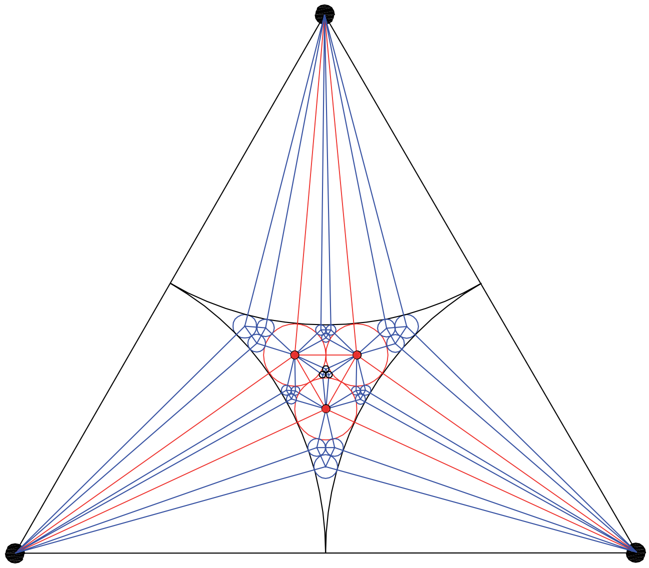

Figure 2. Illustration of the network corresponding to the disk packing shown in figure 1.

the initial configuration, each of which touches two adjacent sides of the initial curvilinear triangle, giving rise to seven new smaller curvilinear triangles. For subsequent generations we indefinitely repeat the packing process for all the new curvilinear triangles. In the limit of infinite t generations, we obtain a disk packing.

The above-mentioned disk packing can be used as a basis for a network, which is the research object of the current paper. The translation from the disk packing to network generation is quite straightforward. Let the nodes (vertices) of the network correspond to the disks and make two nodes connected if the corresponding disks are in contact. Alternatively, one can also connect the centers of the touching disks by lines to obtain the network. Figure 2 shows the network corresponding to the disk packing in figure 1.

In the construction process of the disk packing, for each interstice at arbitrary generation, once we add three disks to fill it, seven new interstices are created that will be filled in the next iteration. When building the network, it is equivalent to say that for each group of three new nodes added, seven new triangles are generated in the network, into each of which a group of three nodes will be inserted in the next iteration. According to this, we can introduce an iterative algorithm to create the network, denoted by F_t after t generation evolutions.

The iterative algorithm for the network is as follows: for $t = 0$, F_0 consists of three nodes forming a triangle. Then, we add three nodes into the original triangle. These three new nodes are linked to each other shaping a new triangle, and both ends of each edge of the new triangle are connected to a node of the original triangle. Thus we get F_1, see figure 3. For $t \geqslant 1$, F_t is obtained from F_{t-1}. For each of the existing triangles of F_{t-1} that has never generated nodes before, we call it an active triangle. We replace each of the existing active triangles of F_{t-1} by the connected cluster on the right-hand side of figure 3 to obtain F_t. The growing process is repeated until the network reaches a desired order (node number of network). Figure 2 shows the network growing process for the first two steps.

Next, we compute the order and size (number of all edges) of the network F_t. Let $L_v(t)$, $L_e(t)$ and $L_\Delta(t)$ be the number of nodes, edges and active triangles created at step t, respectively.

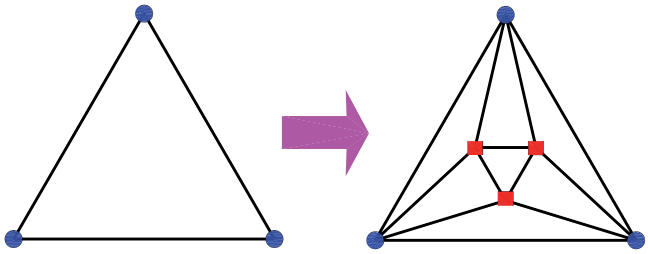

Figure 3. Iterative construction method for the network.

By construction (see also figure 3), each active triangle in F_{t-1} will be replaced by seven active triangles in F_t. Thus, it is not difficult to find the following relation: $L_\Delta(t) = 7\,L_\Delta(t-1)$. Since $L_\Delta(0) = 1$, we have $L_\Delta(t) = 7^t$.

Note that each active triangle in F_{t-1} will lead to an addition of three new nodes and nine new edges at step t, then one can easily obtain the following relations: $L_v(t) = 3\,L_\Delta(t-1) = 3 \times 7^{t-1}$ and $L_e(t) = 9\,L_\Delta(t-1) = 9 \times 7^{t-1}$ for arbitrary $t > 0$. From these results, we can compute the order and size of the network. The total number of vertices N_t and edges E_t present at step t is

$$N_t = \sum_{t_i=0}^{t} L_v(t_i) = \frac{7^t + 5}{2} \tag{1}$$

and

$$E_t = \sum_{t_i=0}^{t} L_e(t_i) = \frac{3 \times 7^t + 3}{2}, \tag{2}$$

respectively. So for large t, the average degree $\bar{k}_t = \frac{2E_t}{N_t}$ is approximately 6, which shows the network is sparse as most real systems.

From equations (1) and (2), we have $E_t = 3N_t - 6$. In addition, by the very construction of the network, it is obvious that two arbitrary edges in the network never cross each other. Thus our network is a maximal planar network (or graph) [47].

3. Structural properties of the network

In this section, we study the statistical properties of the network, in terms of degree distribution, clustering coefficient, APL and degree correlations.

3.1. Degree distribution

When a new node i is added to the graph at step t_i ($t_i \geqslant 1$), it has a degree of 4. Let $L_\Delta(i, t)$ be the number of active triangles at step t that will create new nodes connected to node i at step $t+1$. Then at step t_i, $L_\Delta(i, t_i) = 4$. From the iterative generation process of the network, one can see that at any step each two new neighbors of i generate three new active triangles involving i, and one of its existing active triangles is deactivated simultaneously. We define $k_i(t)$ as the degree of node i at time t, then the relation between $k_i(t)$ and $L_\Delta(i, t)$ satisfies:

$$L_\Delta(i, t) = k_i(t). \tag{3}$$

Now we compute $L_\Delta(i,t)$. By construction, $L_\Delta(i,t) = 3\,L_\Delta(i,t-1)$. Considering the initial condition $L_\Delta(i,t_i)=4$, we can derive $L_\Delta(i,t) = 4\times 3^{t-t_i}$. Then at time t, the degree of vertex i becomes

$$k_i(t) = 4\times 3^{t-t_i}. \tag{4}$$

It should be mentioned that the initial three vertices created at step 0 have a little different evolution process from the other ones. We can easily obtain $L_\Delta(0,t)=3^t$ and $k_i(t)=3^t+1$. Thus, at step t, the degree of the initial three nodes is less than that of those three nodes born at step 1 but larger than that for those nodes emerging at other steps.

Equation (4) shows that the degree spectrum of the network is discrete. It follows that the cumulative degree distribution [6] is given by

$$P_{\mathrm{cum}}(k) = \sum_{\tau\leqslant t_i} \frac{L_v(\tau)}{N_t} = \frac{7^{t_i}+5}{7^t+5}, \tag{5}$$

which is valid for all $t_i \geqslant 2$.

Substituting for t_i ($t_i \geqslant 2$) in this expression using $t_i = t - \frac{\ln(k/4)}{\ln 3}$ gives

$$P_{\mathrm{cum}}(k) = \frac{7^t\times(k/4)^{-(\ln 7/\ln 3)}+5}{7^t+5}. \tag{6}$$

When t is large enough, one can obtain

$$P_{\mathrm{cum}}(k) = \left(\frac{k}{4}\right)^{-(\ln 7/\ln 3)}. \tag{7}$$

So the degree distribution follows a power law form with the exponent $\gamma = 1+\frac{\ln 7}{\ln 3}$.

3.2. Clustering coefficient

The clustering coefficient [4] C_i of node i is defined as the ratio between the number of edges e_i that actually exist among the k_i neighbors of node i and its maximum possible value, $k_i(k_i-1)/2$, i.e. $C_i = 2e_i/[k_i(k_i-1)]$. The clustering coefficient of the whole network is the average of C_i's over all nodes in the network.

For our network, the analytical expression of clustering coefficient $C(k)$ for a single node with degree k can be derived exactly. When a node enters the system, both k_i and e_i are 4. In the following iterations, each of its active triangles increases both k_i and e_i by 2 and 3, respectively. Thus, e_i is equal to $4+\frac{3}{2}(k_i-4)$ for all nodes at all steps. So one can see that there exists a one-to-one correspondence between the degree of a node and its clustering. For a node of degree k, we have

$$C(k) = \frac{2\,e}{k(k-1)} = \frac{2\left[4+\frac{3}{2}(k-4)\right]}{k(k-1)} = \frac{4}{k} - \frac{1}{k-1}. \tag{8}$$

In the limit of large k, $C(k)$ is inversely proportional to degree k. The same scaling of $C(k)\sim k^{-1}$ has also been observed in several real-life networks [48].

Using equation (8), we can obtain the clustering C_t of the network at step t:

$$C_t = \sum_{r=0}^{t}\left[\frac{L_v(r)}{N_t}\left(\frac{4}{D_r} - \frac{1}{D_r-1}\right)\right], \tag{9}$$

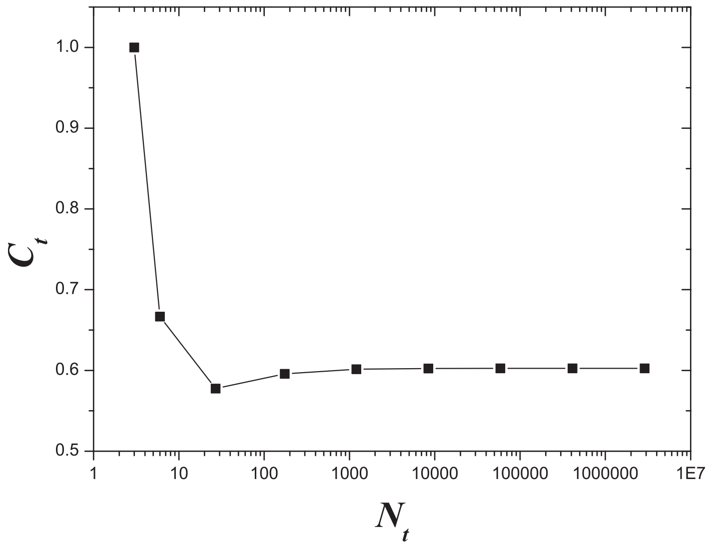

Figure 4. Semilogarithmic plot of average clustering coefficient C_t versus network order N_t.

where the sum runs over all the nodes and D_r is the degree of the nodes created at step r, which is given by equation (4). In the infinite network order limit ($N_t \to \infty$), equation (9) converges to a nonzero value $C = 0.603$, as shown in figure 4. Therefore, the average clustering coefficient of the network is very high.

3.3. APL

We represent all the shortest path lengths of network F_t as a matrix in which the entry d_{ij} is the distance between node i and j, that is the length of a shortest path joining i and j. A measure of the typical separation between two nodes in F_t is given by the APL (also called average distance) d_t defined as the mean of distances over all pairs of nodes:

$$d_t = \frac{D_t}{N_t(N_t-1)/2}, \tag{10}$$

where

$$D_t = \sum_{i\in F_t,\, j\in F_t,\, i\neq j} d_{ij} \tag{11}$$

denotes the sum of the distances between two nodes over all couples. Note that in equation (11), for a pair of nodes i and j ($i \neq j$), we only count d_{ij} or d_{ji}, not both.

3.3.1. Recursive equation for total distances. We continue by exhibiting the procedure for determining the total distance and present the recurrence formula, which allows us to obtain D_{t+1} of the $t+1$ generation from D_t of the t generation. The network F_t under consideration has a self-similar structure that allows one to calculate D_t analytically [49]–[51]. As shown in figure 5, network F_{t+1} may be constructed by joining at six edge nodes (i.e. A, B, C, X, Y and Z) seven copies of F_t that are labeled as $F_t^{(1)}, F_t^{(2)}, \ldots, F_t^{(7)}$.

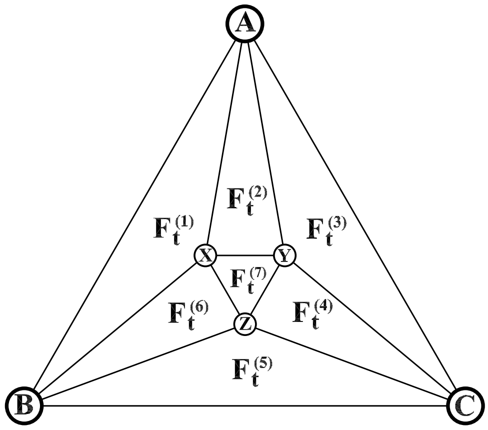

Figure 5. Schematic illustration of second construction means of the network. F_{t+1} may be obtained by joining seven copies of F_t denoted as $F_t^{(\eta)}(\eta = 1, 2, \ldots, 7)$, which are connected to one another at the six edge nodes, i.e. A, B, C, X, Y and Z.

According to the second construction method, the total distance D_{t+1} satisfies the recursion relation

$$D_{t+1} = 7\,D_t + \Delta_t - 9, \tag{12}$$

where Δ_t is the sum over all shortest path lengths whose endpoints are not in the same $F_t^{(\eta)}$ branch. The last term -9 on the right-hand side of equation (12) compensates for the overcounting of certain paths: the shortest path d_{AX} between A and X, with length 1, is included in both $F_t^{(1)}$ and $F_t^{(2)}$; similarly, the shortest paths d_{AY}, d_{BX}, d_{BZ}, d_{CY}, d_{CZ}, d_{XY}, d_{XZ} and d_{YZ} are all computed twice. To determine D_t, all that is left is to calculate Δ_t.

3.3.2. Definition of crossing distance. In order to compute Δ_t, we classify the nodes in F_{t+1} into two categories: the six edge nodes (such as A, B, C, X, Y and Z in figure 5) are called connecting nodes, while the other nodes are named non-connecting nodes. Thus Δ_t, named the crossing distance, can be obtained by summing the following path lengths that are not included in the distance of node pairs in $F_t^{(\eta)}$: length of the shortest paths between non-connecting nodes, length of the shortest paths between connecting and non-connecting nodes, and length of the shortest paths between connecting nodes (i.e. d_{AZ}, d_{BY} and d_{CZ}).

Denote $\Delta_t^{\alpha,\beta}$ as the sum of all shortest paths between non-connecting nodes, whose endpoints are in $F_t^{(\alpha)}$ and $F_t^{(\beta)}$, respectively. That is to say, $\Delta_t^{\alpha,\beta}$ rules out the paths with end-point at the connecting nodes belonging to $F_t^{(\alpha)}$ or $F_t^{(\beta)}$. For example, each path contributed to $\Delta_t^{1,2}$ does not end at node A, B, X or Y. On the other hand, let Ω_t^{η} be the set of non-connecting nodes in F_{t+1}, which belong to $F_t^{(\eta)}$. Then the total sum Δ_t is given by

$$\Delta_t = \sum_{\beta=\alpha+1}^{7}\sum_{\alpha=1}^{6}\Delta_t^{\alpha,\beta} + \sum_{\substack{j\in\Omega_t^{\eta}\\ \eta\in\{4,5,6,7\}}} d_{\mathrm{A}j} + \sum_{\substack{j\in\Omega_t^{\eta}\\ \eta\in\{2,3,4,7\}}} d_{\mathrm{B}j} + \sum_{\substack{j\in\Omega_t^{\eta}\\ \eta\in\{1,2,6,7\}}} d_{\mathrm{C}j} + \sum_{\substack{j\in\Omega_t^{\eta}\\ \eta\in\{3,4,5\}}} d_{\mathrm{X}j} + \sum_{\substack{j\in\Omega_t^{\eta}\\ \eta\in\{1,5,6\}}} d_{\mathrm{Y}j}$$
$$+ \sum_{\substack{j\in\Omega_t^{\eta}\\ \eta\in\{1,2,3\}}} d_{\mathrm{Z}j} + d_{\mathrm{AZ}} + d_{\mathrm{BY}} + d_{\mathrm{CX}}. \tag{13}$$

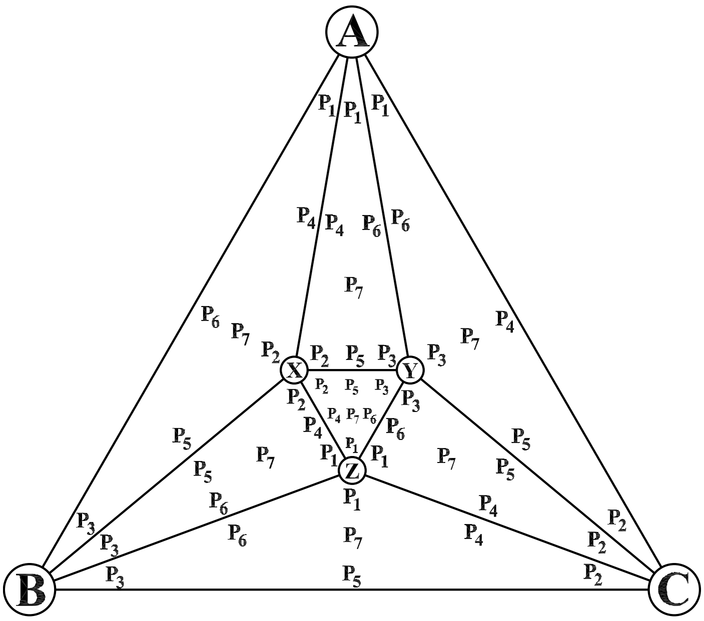

Figure 6. Illustration of the classification of interior nodes in $F_t^{(\eta)}(\eta = 1, 2, \ldots, 7)$, from which we can derive recursively the classification of interior nodes in network F_{t+1}.

By symmetry, equation (13) can be simplified as

$$\Delta_t = 9\Delta_t^{1,2} + 9\Delta_t^{1,3} + 3\Delta_t^{1,4} + 21 \sum_{j \in \Omega_t^4} d_{Aj} + 6. \tag{14}$$

Having Δ_t in terms of the quantities of $\Delta_t^{1,2}$, $\Delta_t^{1,3}$, $\Delta_t^{1,4}$ and $\sum_{j\in\Omega_t^4} d_{Aj}$, the next step is to explicitly determine these quantities.

3.3.3. Classification of interior nodes. To calculate the crossing distances $\Delta_t^{1,2}$, $\Delta_t^{1,3}$, $\Delta_t^{1,4}$ and $\sum_{j\in\Omega_t^4} d_{Aj}$, we classify interior nodes in network F_{t+1} into seven different parts according to their shortest path lengths to each of the three peripheral nodes (i.e. A, B and C). Notice that nodes A, B and C themselves are not partitioned into any of the seven parts represented as P_1, P_2, P_3, P_4, P_5, P_6 and P_7, respectively. The classification of nodes is shown in figure 6. For any interior node v, we denote the shortest path lengths from v to A, B and C as a, b and c, respectively. By construction, a, b, c can differ by at most 1 since vertices A, B and C are adjacent. Then the classification function $class(v)$ of node v is defined to be

$$class(v) = \begin{cases} P_1, & \text{for } a < b = c, \\ P_2, & \text{for } b < a = c, \\ P_3, & \text{for } c < a = b, \\ P_4, & \text{for } a = c < b, \\ P_5, & \text{for } a = b < c, \\ P_6, & \text{for } b = c < a, \\ P_7, & \text{for } a = b = c. \end{cases} \tag{15}$$

It should be mentioned that the definition of node classification is recursive. For instance, classes P_1 and P_4 in $F_t^{(1)}$ belong to class P_1 in F_{t+1}, classes P_3 and P_5 in $F_t^{(1)}$ belong to class

P_2 in F_{t+1}, classes P_2, P_6 and P_7 in $F_t^{(1)}$ belong to class P_5 in F_{t+1}. Since the three nodes A, B and C are symmetrical, in the network we have the following equivalent relations from the viewpoint of class cardinality: classes P_1, P_2 and P_3 are equivalent to one another, and it is the same with classes P_4, P_5 and P_6. We denote the number of nodes in network F_t that belong to class P_1 as N_{t,P_1}, the number of nodes in class P_2 as N_{t,P_2} and so on. By symmetry, we have $N_{t,P_1} = N_{t,P_2} = N_{t,P_3}$ and $N_{t,P_4} = N_{t,P_5} = N_{t,P_6}$. Therefore in the following computation we will only consider N_{t,P_1}, N_{t,P_4} and N_{t,P_7}. It is easy to conclude that

$$\begin{aligned} N_t &= N_{t,P_1} + N_{t,P_2} + N_{t,P_3} + N_{t,P_4} + N_{t,P_5} + N_{t,P_6} + N_{t,P_7} + 3 \\ &= 3\,N_{t,P_1} + 3\,N_{t,P_4} + N_{t,P_7} + 3. \end{aligned} \tag{16}$$

Considering the self-similar structure of the network, we can easily know that at time $t+1$, the quantities N_{t+1,P_1}, N_{t+1,P_4} and N_{t+1,P_7} evolve according to the following recursive equations

$$\begin{cases} N_{t+1,P_1} = 3\,N_{t,P_1} + 4\,N_{t,P_4} + N_{t,P_7}, \\ N_{t+1,P_4} = 4\,N_{t,P_1} + N_{t,P_4} + N_{t,P_7} + 1, \\ N_{t+1,P_7} = 6\,N_{t,P_4} + N_{t,P_7}, \end{cases} \tag{17}$$

where we have used the equivalent relations $N_{t,P_1} = N_{t,P_2} = N_{t,P_3}$ and $N_{t,P_4} = N_{t,P_5} = N_{t,P_6}$. With the initial condition $N_{2,P_1} = 4$, $N_{2,P_4} = 2$ and $N_{2,P_7} = 6$, we can solve the recursive equation (17) to obtain

$$\begin{cases} N_{t,P_1} = \frac{1}{124}[-62 + 10 \times 7^t + 26(-1-\sqrt{2})^t + 26(-1+\sqrt{2})^t + 11\sqrt{2}(-1+\sqrt{2})^t \\ \qquad -11\sqrt{2}(-1-\sqrt{2})^t], \\ N_{t,P_4} = \frac{1}{124}[8 \times 7^t - 4(-1-\sqrt{2})^t - 4(-1+\sqrt{2})^t + 15\sqrt{2}(-1+\sqrt{2})^t \\ \qquad -15\sqrt{2}(-1-\sqrt{2})^t], \\ N_{t,P_7} = \frac{1}{62}[62 + 4 \times 7^t - 33(-1-\sqrt{2})^t - 33(-1+\sqrt{2})^t + 39\sqrt{2}(-1-\sqrt{2})^t \\ \qquad -39\sqrt{2}(-1+\sqrt{2})^t]. \end{cases} \tag{18}$$

For a node v in network F_{t+1}, we are also interested in the smallest value of the shortest path length from v to any of the three peripheral nodes A, B and C. We denote the shortest distance as f_v, which can be defined to be

$$f_v = \min(a, b, c). \tag{19}$$

Let d_{t,P_1} denote the sum of f_v of all nodes belonging to class P_1 in network F_t. Analogously, we can also define the quantities $d_{t,P_2}, d_{t,P_3}, \ldots, d_{t,P_7}$. Again by symmetry, we have $d_{t,P_1} = d_{t,P_2} = d_{t,P_3}$, $d_{t,P_4} = d_{t,P_5} = d_{t,P_6}$ and d_{t,P_1}, d_{t,P_4}, d_{t,P_7} can be written recursively as follows:

$$\begin{cases} d_{t+1,P_1} = 3\,d_{t,P_1} + 4\,d_{t,P_4} + d_{t,P_7}, \\ d_{t+1,P_4} = 4\,d_{t,P_1} + d_{t,P_4} + d_{t,P_7} + 4\,N_{t,P_1} + 1, \\ d_{t+1,P_7} = 6\,(d_{t,P_4} + N_{t,P_4}) + (d_{t,P_7} + N_{t,P_7}). \end{cases} \tag{20}$$

Substituting equation (18) into equation (20), and considering the initial condition $d_{2,\mathrm{P}_1} = 4$, $d_{2,\mathrm{P}_4} = 2$ and $d_{2,\mathrm{P}_7} = 12$, equation (20) is solved inductively

$$\begin{cases} d_{t,\mathrm{P}_1} = \frac{1}{5004\,888}\Big[7(-3(-1+\sqrt{2})^t(20\,082+18\,235\sqrt{2}) \\ \quad +(-1-\sqrt{2})^t(-60\,246+54\,705\sqrt{2})+4(29\,791+3327^t)) \\ \quad +93(1760\times 7^t-7(-1+\sqrt{2})^t(1256+1149\sqrt{2}) \\ \quad +(-1-\sqrt{2})^t(-8792+8043\sqrt{2}))t\Big], \\ d_{t,\mathrm{P}_4} = \frac{1}{5004\,888}\Big[14(-59\,582+13\,700\times 7^t+(22\,941-56\,145\sqrt{2})(-1-\sqrt{2})^t \\ \quad +3(-1+\sqrt{2})^t(7647+18\,715\sqrt{2})) \\ \quad +93(1408\times 7^t-7(-1+\sqrt{2})^t(1042+107\sqrt{2}) \\ \quad +(-1-\sqrt{2})^t(-7294+749\sqrt{2}))t\Big], \\ d_{t,\mathrm{P}_7} = \frac{1}{5004\,888}\Big[7(9(-1-\sqrt{2})^t(-16\,540+33\,393\sqrt{2}) \\ \quad -9(-1+\sqrt{2})^t(16\,540+33\,393\sqrt{2})+8(29\,791+7424\times 7^t)) \\ \quad +186(704\times 7^t-21(-1-\sqrt{2})^t(-1149+628\sqrt{2}) \\ \quad +21(-1+\sqrt{2})^t(1149+628\sqrt{2}))t\Big]. \end{cases} \tag{21}$$

3.3.4. Calculation of crossing distances. Having obtained the quantities N_{t,P_i} and d_{t,P_i} ($i = 1, 2, \cdots, 7$), we now begin to determine the crossing distance $\Delta_t^{1,2}$, $\Delta_t^{1,3}$, $\Delta_t^{1,4}$ and $\sum_{j\in\Omega_t^4} d_{\mathrm{A}j}$ expressed as a function of N_{t,P_i} and d_{t,P_i}. Here, we only give the computation details of $\Delta_t^{1,2}$, while the computing processes of $\Delta_t^{1,3}$, $\Delta_t^{1,4}$ and $\sum_{j\in\Omega_t^4} d_{\mathrm{A}j}$ are similar. For convenience of computation, we use $\Gamma_t^{\eta,i}$ to denote the set of interior nodes belonging to class P_i in $F_t^{(\eta)}$. Then $\Delta_t^{1,2}$ can be written as

$$\Delta_t^{1,2} = \sum_{\substack{u\in\Gamma_t^{1,i},\, i\in\{1,2,3,4,5,6,7\} \\ v\in F_t^{(2)},\, v\neq A,X,Y}} d_{uv}. \tag{22}$$

The seven terms on the right-hand side of equation (22) are represented consecutively as δ_t^i ($i = 1, 2, \cdots, 7$). Next, we will calculate the quantities δ_t^i. By symmetry, $\delta_t^1 = \delta_t^2$, $\delta_t^5 = \delta_t^6$. Therefore, we need only compute δ_t^1, δ_t^3, δ_t^4, δ_t^5 and δ_t^7. First, we evaluate δ_t^1. By definition,

$$\begin{aligned} \delta_t^1 &= \sum_{\substack{u\in\Gamma_t^{1,1},\, v\in F_t^{(2)} \\ v\neq A,X,Y}} d_{uv} \\ &= \sum_{\substack{u\in\Gamma_t^{1,1}, v\in\Gamma_t^{2,i} \\ i\in\{1,4,6,7\}}} (d_{u\mathrm{A}}+d_{\mathrm{A}v}) + \sum_{\substack{u\in\Gamma_t^{1,1} \\ v\in\Gamma_t^{2,3}}} (d_{u\mathrm{A}}+d_{\mathrm{AY}}+d_{\mathrm{Y}v}) + \sum_{\substack{u\in\Gamma_t^{1,1} \\ v\in\Gamma_t^{2,2}\bigcup\Gamma_t^{2,5}}} (d_{u\mathrm{A}}+d_{\mathrm{AX}}+d_{\mathrm{X}v}) \\ &= N_{t,\mathrm{P}_1}(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7}+2N_{t,\mathrm{P}_1}+N_{t,\mathrm{P}_4})+d_{t,\mathrm{P}_1}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7}). \end{aligned} \tag{23}$$

Proceeding similarly, we obtain

$$\delta_t^3 = N_{t,\mathrm{P}_1}(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7}+4N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})+d_{t,\mathrm{P}_1}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7}), \tag{24}$$

$$\delta_t^4 = N_{t,\mathrm{P}_2}(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7}+N_{t,\mathrm{P}_1})+d_{t,\mathrm{P}_2}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7}), \tag{25}$$

$$\delta_t^5 = N_{t,\mathrm{P}_2}(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7}+2N_{t,\mathrm{P}_1}+N_{t,\mathrm{P}_4})+d_{t,\mathrm{P}_2}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7}), \tag{26}$$

and

$$\delta_t^7 = N_{t,\mathrm{P}_3}(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7}+N_{t,\mathrm{P}_1})+d_{t,\mathrm{P}_7}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7}). \tag{27}$$

With the obtained results for δ_t^i, we have

$$\Delta_t^{1,2} = 2(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7})(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})+N_{t,\mathrm{P}_1}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7}) \\ +2(N_{t,\mathrm{P}_1}+N_{t,\mathrm{P}_4})(2N_{t,\mathrm{P}_1}+N_{t,\mathrm{P}_4})+N_{t,\mathrm{P}_1}(N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})+(N_{t,\mathrm{P}_1})^2. \tag{28}$$

Analogously, we find

$$\Delta_t^{1,3} = 2(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7})(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})+2(N_{t,\mathrm{P}_1})^2 \\ +2N_{t,\mathrm{P}_1}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})+N_{t,\mathrm{P}_4}(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7}) \\ +(N_{t,\mathrm{P}_1}+2N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})(2N_{t,\mathrm{P}_1}+N_{t,\mathrm{P}_4}), \tag{29}$$

$$\Delta_t^{1,4} = 2(3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7})(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})+(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})^2+3(N_{t,\mathrm{P}_1})^2 \tag{30}$$

and

$$\sum_{j\in\Omega_t^4} d_{\mathrm{A}j} = (3d_{t,\mathrm{P}_1}+3d_{t,\mathrm{P}_4}+d_{t,\mathrm{P}_7})+(3N_{t,\mathrm{P}_1}+3N_{t,\mathrm{P}_4}+N_{t,\mathrm{P}_7})+N_{t,\mathrm{P}_1}. \tag{31}$$

Substituting equations (28)–(31) into equation (14), we obtain the final expression for cross distances Δ_t,

$$\Delta_t = \frac{1}{3844}\Big[5766+2728\times 7^t+30\,290\times 49^t-10\,974(-1+\sqrt{2})^t-9300\sqrt{2}(-1+\sqrt{2})^t \\ +3114(-1+\sqrt{2})^{2t}+1017\sqrt{2}(-1+\sqrt{2})^{2t}+186(-1-\sqrt{2})^t(-59+50\sqrt{2}) \\ -9(-1-\sqrt{2})^{2t}(-346+113\sqrt{2})+16\,368t\times 49^t\Big]. \tag{32}$$

3.3.5. Exact result for APL. With the above-obtained results and recursion relations, we now readily calculate the sum of the shortest path lengths between all pairs of nodes. Inserting equation (32) into equation (12) and using the initial condition $D_2 = 717$, equation (12) is solved inductively,

$$D_t = \frac{1}{161\,448}\Big[201\,810+215\,270\times 7^t+11\,194\times 7^{2t}+72\,072(-1-\sqrt{2})^t-57\,834\sqrt{2}(-1-\sqrt{2})^t \\ -44\,037(-1-\sqrt{2})^{2t}-11\,340\sqrt{2}(-1-\sqrt{2})^{2t}+72\,072(-1+\sqrt{2})^t \\ +57\,834\sqrt{2}(-1+\sqrt{2})^t-44\,037(-1+\sqrt{2})^{2t}+11\,340\sqrt{2}(-1+\sqrt{2})^{2t} \\ +16\,368t\times 7^t+16\,368t\times 7^{2t}\Big]. \tag{33}$$

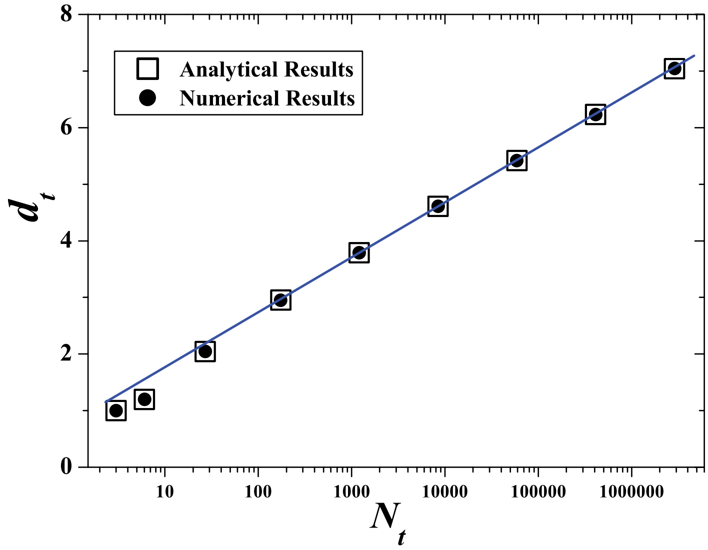

Figure 7. Average distance d_t versus network order N_t on a semilogarithmic scale. The solid line serves as guide to the eye.

Substituting equation (33) into equation (10) yields the exactly analytic expression for APL

$$\begin{aligned} d_t = \frac{1}{20\,181(15+8\times 7^t+7^{2t})}\Big[&201\,810+215\,270\times 7^t+11\,194\times 7^{2t}+72\,072(-1-\sqrt{2})^t \\ &-57\,834\sqrt{2}(-1-\sqrt{2})^t-44\,037(-1-\sqrt{2})^{2t}-11\,340\sqrt{2}(-1-\sqrt{2})^{2t} \\ &+72\,072(-1+\sqrt{2})^t+57\,834\sqrt{2}(-1+\sqrt{2})^t-44\,037(-1+\sqrt{2})^{2t} \\ &+11\,340\sqrt{2}(-1+\sqrt{2})^{2t}+16\,368t\times 7^t+16\,368t\times 7^{2t}\Big]. \end{aligned} \quad (34)$$

In the large t limit, $d_t \sim t$, while the network order $N_t \sim 7^t$, which is obvious from equation (1). Thus, the APL grows logarithmically with increasing order of the network. We have checked our analytic result provided by equation (34) against numerical calculations for different network order up to $t = 8$, which corresponds to $N_8 = 1\,007\,772$. In all cases we obtain complete agreement between our theoretical formula and the results of numerical investigation, see figure 7.

Recently, it has been suggested that for random uncorrelated SFNs with degree exponent $\gamma < 3$ and network order N, their average distance $d(N)$ behaves as a double logarithmic scaling with N: $d(N) \sim \ln\ln N$ [52, 53]. However, for the deterministic network considered here, in despite of the fact that its degree exponent $\gamma = 1 + \frac{\ln 7}{\ln 3} < 3$, its APL scales as a logarithmic scaling with network order, showing an obvious difference from that of the stochastic scale-free counterparts. The logarithmic scaling of d_t with N_t for our network as well as the Apollonian networks [41] shows that the previous relation between APL and the network order obtained for uncorrelated SFNs [52, 53] is not valid for disassortative SFNs [52, 53], at least for some spatial networks, e.g. the Apollonian networks and the network considered here. This leads us to the conclusion that the degree exponent itself does not suffice to characterize the APL of SFNs.

3.4. Degree correlations

An interesting quantity related to degree correlations is the average degree of the nearest neighbors for nodes with degree k, denoted as $k_{\rm nn}(k)$, which is a function of node degree k [54, 55]. When $k_{\rm nn}(k)$ increases with k, it means that nodes have a tendency to connect to nodes with a similar or larger degree. In this case the network is defined as assortative [5]. In contrast, if $k_{\rm nn}(k)$ is decreasing with k, which implies that nodes of large degree are likely to have near neighbors with small degree, then the network is said to be disassortative. If correlations are absent, $k_{\rm nn}(k) = \text{const}$.

We can exactly calculate $k_{\rm nn}(k)$ for network F_t using equations (3) and (4) to work out how many links are made at a particular step to nodes with a particular degree. By construction, we have the following expression [56, 57]

$$k_{\rm nn}(k)=\frac{1}{L_v(t_i)k(t_i,t)}\left[\sum_{t_i'=0}^{t_i'=t_i-1}2L_v(t_i')L_\Delta(t_i',t_i-1)k(t_i',t)+\sum_{t_i'=t_i+1}^{t_i'=t}2L_v(t_i)L_\Delta(t_i,t_i'-1)k(t_i',t)\right]+2 \tag{35}$$

for $k = 4\times 3^{t-t_i}$ $(t_i \geqslant 1)$, where $k(t_i, t)$ is the degree of a node i at time t that was born at step t_i. Here, the first sum on the right-hand side accounts for the links made to nodes with larger degree (i.e. $t_i' < t_i$) when the node was generated at t_i. The second sum describes the links made to the current smallest degree nodes at each step $t_i' > t_i$. The last term 2 accounts for the two links connected to two simultaneously emerging nodes. After some algebraic manipulations, we can rewrite equation (35) in terms of k to obtain

$$k_{\rm nn}(k)=\frac{(3^{2t+1}+3^{t-1})\,(4/k)^{2-(\ln 7/\ln 3)}}{2\times 7^{t-1}}+\frac{8\ln(k/4)}{3\ln 3}-10. \tag{36}$$

For $k = 3^t + 1$ $(t_i = 0)$, we have

$$\begin{aligned}k_{\rm nn}(k)&=\frac{1}{k(t_i,t)}\left[\sum_{t_i'=t_i+1}^{t_i'=t}2L_\Delta(t_i,t_i'-1)k(t_i',t)\right]+2\\&=\frac{8t\times 3^t}{3^{t+1}+3}+2.\end{aligned} \tag{37}$$

Therefore, for large t and k, $k_{\rm nn}(k)$ is approximately a power-law function of k as $k_{\rm nn}(k)\sim k^{-\omega}$ with $\omega = 2-\frac{\ln 7}{\ln 3}\simeq 0.229$, which shows that the network is disassortative. Note that $k_{\rm nn}(k)$ of the Internet exhibits a similar power-law scaling with exponent $\omega = 0.5$ [54].

4. Conclusion

In summary, motivated by the disk packing and Apollonian networks, we have presented a model for spatial planar networks introducing the influence of geography encoded in the disk packing. According to the construction, we have studied analytically the main structural features of the network. We have shown that the network has a power-law distribution with exponent $\gamma = 1+\frac{\ln 7}{\ln 3}$, it has a large clustering coefficient 0.603, its APL scales logarithmically with the number of network nodes and it is disassortative with the average degree of the nearest neighbors for nodes having degree k being roughly a power-law function of k with exponent -0.229.

Note that although both the network considered here and the Apollonian network [38, 56] are translated from disk packings, and both networks have qualitatively similar topologies, their structural characteristics are quantitatively different. For example, the exponent of degree distribution is $1+\frac{\ln 7}{\ln 3}$ for our network, while for Apollonian network it is $1+\frac{\ln 3}{\ln 2}$; the average clustering coefficients for our network and Apollonian network are 0.603 and 0.828, respectively. In addition, the APL [41] and the degree correlations [56] for both networks are also quantitatively different. These disparities of the two networks show that the ways of disk packing lead to different spatial constraints of network nodes, which in turn have a significant impact on network properties and thus dynamics running on networks, such as cascading failing [25], random walks [58] and so on. Thus, we can conclude that for spatial networks the positions, where nodes are geographically located, matter greatly and should be incorporated when modeling such networks. Ignoring the geography will lead to missing some important attributes and properties of the systems.

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