华南理工大学2005数分

To my parents

1. 设 $x_1 > a > 0$, $x_{n+1} = \sqrt{x_n^2 - 2ax_n + 2a^2}$. 求极限 $\lim\limits_{n\to\infty} x_n$.

解答. 由

$$x_{n+1} = \sqrt{(x_n - a)^2 + a^2} \geq a > 0, \tag{1}$$

知

$$\frac{x_{n+1}}{x_n} = \sqrt{\left(1 - \frac{a}{x_n}\right)^2 + \left(\frac{a}{x_n}\right)^2} \leq \sqrt{\left[\left(1 - \frac{a}{x_n}\right) + \frac{a}{x_n}\right]^2} = 1,$$

而 $\{x_n\}$ 单减有下界, $\lim\limits_{n\to\infty} x_n$ 存在. 设 $\lim\limits_{n\to\infty} x_n = c$, 则于 (1) 中令 $n \to \infty$, 有 $c = \sqrt{(c-a)^2 + a^2}$, 而 $c = a$. ■

2. 求积分 $\oint_C \dfrac{x^3dx - y^3dy}{x^4 + y^4}$, 其中 C 是单位圆周: $x^2 + y^2 = 1$, 逆时针为正向.

解答.

$$\begin{aligned}\oint_C \frac{x^3dx - y^3dy}{x^4 + y^4} &= \int_0^{2\pi} \frac{\cos^3\vartheta(-\sin\vartheta) - \sin^3\vartheta\cos\vartheta}{\cos^4\vartheta + \sin^4\vartheta} d\vartheta \\ &= -\int_0^{2\pi} \frac{\sin\vartheta\cos\vartheta}{1 - 2\cos^2\vartheta\sin^2\vartheta} d\vartheta = -\int_{-\pi}^{\pi} \frac{\sin\vartheta\cos\vartheta}{1 - 2\cos^2\vartheta\sin^2\vartheta} d\vartheta = 0.\end{aligned}$$

■

3. 讨论函数序列 $f_n(t) = \dfrac{\sin nt}{n\sqrt{t}}$ 在 $(0, \infty)$ 上的一致收敛性.

解答. 由

$$0 \leq \left|\frac{\sin nt}{n\sqrt{t}}\right| \leq \frac{\sqrt{|\sin nt|}}{n\sqrt{t}} \leq \frac{\sqrt{nt}}{n\sqrt{t}} = \frac{1}{\sqrt{n}}$$

知 $f_n \rightrightarrows 0$, 于 $(0, \infty)$. ■

4. 设 $z = z(x, y)$ 由方程 $F\left(x + \frac{z}{y}, y + \frac{z}{x}\right) = 0$ 所确定. 证明:

$$xz_x + yz_y = z - xy.$$

解答. 设 $F = F(p, q)$, 则

$$\begin{cases} F_p \frac{y+z_x}{y} + F_q \frac{xz_x - z}{x^2} = 0, \\ F_p \frac{yz_y - z}{y^2} + F_q \frac{x+z_y}{x} = 0. \end{cases}$$

于是

$$\frac{y + z_x}{y} \cdot \frac{x + z_y}{x} = \frac{yz_y - z}{y^2} \cdot \frac{xz_x - z}{x^2},$$

$$xy\left(xy + xz_x + yz_y + z_x z_y\right) = xyz_x z_y - z\left(xz_x + yz_y\right) + z^2,$$

$$(z + xy)\left(xz_x + yz_y\right) = z^2 - x^2y^2,$$

$$xz_x + yz_y = z - xy.$$

■

5. 设 $f(x)$ 是偶函数, 在 $x = 0$ 的某个领域中有连续的二阶导数 $f(0) = 1$. 试证明: $\sum_{n=1}^{\infty}\left[f\left(\frac{1}{n}\right) - 1\right]$ 绝对收敛.

证明. 由 f 是偶函数知 f' 是奇函数, 而 $f'(0) = 0$. 故

$$\begin{aligned} \left|f\left(\frac{1}{n}\right) - 1\right| &= \left|f\left(\frac{1}{n}\right) - f(0) - f'(0)\frac{1}{n}\right| \\ &= \frac{1}{2}\left|f''(\xi_n)\right|\frac{1}{n^2} \quad \left(0 < \xi_n < \frac{1}{n}\right) \\ &\leq \frac{M}{n^2} \quad \left(M = \frac{1}{2}\max_{U(0,\delta)}\left|f''\right|\right). \end{aligned}$$

于是 $\sum_{n=1}^{\infty}\left[f\left(\frac{1}{n}\right) - 1\right]$ 绝对收敛. ■

6. 设曲线 $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ 由方程组 $\begin{cases} x + y + 2t(1-t) = 1 \\ te^y + 2x - y = 2 \end{cases}$ 确定. 求该曲线在 $t = 0$ 处动切线方程和法线方程.

解答. *(a)*

$$\begin{cases} x(0) + y(0) = 1 \\ 2x(0) - y(0) = 2 \end{cases} \Rightarrow \begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$$

(b)

$$\begin{cases} \dot{x}(t) + \dot{y}(t) + 2 - 4t = 0 \\ e^{y(t)} + te^{y(t)}\dot{y}(t) + 2\dot{x}(t) - \dot{y}(t) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}(0) + \dot{y}(0) = -2 \\ 2\dot{x}(0) - \dot{y}(0) = -1 \end{cases} \Rightarrow \begin{cases} \dot{x}(0) = -1 \\ \dot{y}(0) = -1 \end{cases} \Rightarrow \frac{dy}{dx} = 1$$

(c) 曲线在 $t = 0$ 处的切线方程为 $y = x - 1$, 法线方程为 $y = -x + 1$.

■

7. 求幂级数 $\sum_{n=0}^{\infty}(-1)^n(n^2+n+1)x^n$ 的收敛域, 并求该级数的和.

解答. *(a)* 由 $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\left[(n+1)^2+(n+1)+1\right]}{(-1)^n(n^2+n+1)}\right| = 1$, 易知原幂级数收敛域为 $(-1, 1)$.

(b) 由

$$\begin{aligned} \sum_{n=0}^{\infty}(n^2+n+1)t^n &= t\sum_{n=1}^{\infty}(n+1)nt^{n-1} + \sum_{n=0}^{\infty}t^n \\ &= t\left(\sum_{n=0}^{\infty}t^n\right)'' + \sum_{n=0}^{\infty}t^n = t\left(\frac{1}{1-t} - 1 - t\right)'' + \frac{1}{1-t} = \frac{1+t^2}{(1-t)^3} \end{aligned}$$

知

$$\sum_{n=0}^{\infty}(-1)^n(n^2+n+1)x^n = \frac{1+x^2}{(1+x)^3}.$$

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8. 求 $\iint_S xdydz - ydxdz + zdxdy$, S 为椭球面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 的上半部分, 其定向为下侧.

解答.

$$\iint_S xdydz - ydxdz + zdxdy = -\iint_{S^+} dxdy + ydzdx + zdxdy$$
$$= -\iiint_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\\ z\ge 0}} 3dxdydz = -3abc\iiint_{\substack{x^2+y^2+z^2\le 1\\ z\ge 0}} dxdydz$$
$$= -3abc\cdot\frac{2\pi}{3} = -2\pi abc.$$

■

9. (a) 设 $a_0 > 0$. 证明积分 $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ 关于 $|a| \ge a_0$ 一致收敛.

证明. $\forall\,\varepsilon > 0$, $\exists\, X = \frac{1}{\sqrt[3]{3\varepsilon}} > 0$, $s.t.$

$$X_2 > X_1 \ge X \Rightarrow \int_{X_1}^{X_2} \frac{dx}{(x^2+a^2)^2} \le \int_{\frac{1}{\sqrt[3]{3\varepsilon}}}^{\infty} \frac{dx}{x^4} = \varepsilon.$$

■

(b) 设 $a > 0$. 计算积分 $\int_0^\infty \frac{dx}{x^2+a^2}dx$ 和 $\int_0^\infty \frac{dx}{x^2+a^{2^3}}dx$.

解答.

$$\int_0^\infty \frac{dx}{x^2+a^2}dx = \int_0^{\frac{\pi}{2}} \frac{a\sec^2\vartheta}{(a\tan\vartheta)^2+a^2}dx = \frac{\pi}{2a};$$

$$\int_0^\infty \frac{dx}{x^2+a^{2^3}}dx = \int_0^{\frac{\pi}{2}} \frac{a\sec^2\vartheta}{\left[(a\tan\vartheta)^2+a^2\right]^3}dx = \frac{1}{a^5}\int_0^{\frac{\pi}{2}} \cos^4\vartheta d\vartheta$$
$$= \frac{1}{a^5}\cdot\left[\frac{\vartheta+\frac{1}{2}\sin 2\vartheta}{2} - \frac{\vartheta-\frac{1}{4}\sin 4\vartheta}{8}\right]_0^{\frac{\pi}{2}}$$

$$\left(\cos^4\vartheta = \cos^2\vartheta\left(1-\sin^2\vartheta\right) = \cos^2\vartheta - \frac{1}{4}\sin^2\vartheta\right)$$
$$= \frac{1}{a^5}\cdot\left(\frac{1}{2}-\frac{1}{8}\right)\cdot\frac{\pi}{2} = \frac{3\pi}{16a^5}.$$

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10. 设 $f(x)$ 在 $[0,\infty)$ 上二阶连续可微, $|f(x)|\leq A$, $|f''(x)|\leq B$. 试证明: $|f'(x)|\leq\sqrt{2AB}$.

证明. 首先, $B\neq 0$, 因为

$$B=0\Rightarrow f(x)=ax+b\Rightarrow f \text{ 无界}.$$

由 *Taylor* 展式,

$$f(x+h)=f(x)+f'(x)h+\frac{f''(\xi)}{2}h^2,$$

知

$$\begin{aligned}|f'(x)| &= \left|\frac{f(x+h)-f(x)}{h}-\frac{f''(\xi)}{2}h\right|\\ &\leq \frac{A}{|h|}+\frac{B}{2}|h|\\ &= 2\cdot\frac{B}{2}\sqrt{\frac{2A}{B}}\quad\left(h=\sqrt{\frac{2A}{B}}\right)\\ &= \sqrt{2AB}.\end{aligned}$$

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11. 设 $f(x)\geq 0$ 在 $(-\infty,\infty)$ 上一致连续, $\int_{-\infty}^{\infty} f(x)dx$ 收敛. 试证明 $\lim\limits_{|x|\to\infty} f(x)=0$.

证明. 用反证法. 若 $\lim\limits_{|x|\to\infty} f(x)\neq 0$, 则

$$\exists\, \varepsilon_0>0,\ |x_n|\to\infty,\ |x_n-x_{n-1}|>1,\ s.t.\ f(x_n)\geq\varepsilon_0.$$

而

$$\begin{aligned}
& f\ \text{一致连续}\\
\Rightarrow\ & \exists\, 0<\delta_0<1,\ s.t.\ |f(x)-f(x')|<\frac{\varepsilon_0}{2},\ \forall\ |x-x'|<\delta_0\\
\Rightarrow\ & f(x)\geq f(x_n)-|f(x)-f(x_n)|\geq\varepsilon_0-\frac{\varepsilon_0}{2}=\frac{\varepsilon_0}{2},\ \forall\ |x-x_n|<\delta_0\\
\Rightarrow\ & \int_{-\infty}^{\infty} f(x)dx\geq\sum_{n=1}^{\infty}\int_{x_n}^{x_n+\delta_0} f(t)dt\geq\sum_{n=1}^{\infty}\delta\cdot\frac{\varepsilon_0}{2}=\infty,
\end{aligned}$$

矛盾. 故有 $\lim\limits_{|x|\to\infty} f(x)=0$. ■

12. 设 $f(x)$ 在 $[0,\pi]$ 上二阶连续可微, $f(0)=f(\pi)=0$.

$$a_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nxdx,\ n=1,2,\cdots.$$

试证明 $\sum\limits_{n=1}^{\infty} n^2a_n^2$ 收敛.

证明. 由

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin nxdx=\frac{2}{n\pi}\int_0^\pi f'(x)\cos nxdx\\
&= -\frac{2}{n^2\pi}\int_0^\pi f''(x)\sin nxdx
\end{aligned}$$

知

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2a_n^2 &= \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[\int_0^\pi f''(x)\sin nxdx\right]^2\\
&\leq \frac{4}{\pi^2}\left[\int_0^\pi |f''(x)|\,dx\right]^2\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty.
\end{aligned}$$

■