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DIRICHLET PRINCIPLE OF HARMONIC MAPS

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ABSTRACT. We define harmonic maps as the critical point of Dirichlet energy functional. This is [1, 1.1].

Let

1. (M^n, g) be a Riemanninan manifold with or without boundary;
2. (N^l, h) be a compact Riemanninan manifold without boundary (closed).

The we may define the **Dirichlet energy functional**

$$E(u) = \int_M e(u) \mathrm{d}v_g,$$

where $e(u)$ is the **Dirichlet energy density function**, the expression of which in local coordinates (U, x^α), (V, u^i) is

$$e(u) \equiv \frac{1}{2} |\nabla u|_g^2 = \frac{1}{2} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Definition 1. *A map $u \in C^2(M, N)$ is a* **harmonic map** *if it is a critical point of the Dirichlet energy functional E.*

Proposition 2. *A map $u \in C^2(M; N)$ is a* **harmonic map** *iff u satisfies*

$$\Delta_g u^i + g^{\alpha\beta} \Gamma^i_{jk}(u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0, \ in\ M,\ 1 \le i \le l.$$

Here

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1. *Δ_g is the* **Laplace-Beltrami operator** *on (M, g) given by*

$$\Delta_g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right);$$

2. *and Γ^i_{jk} is the* **Christoffel symbol** *of the metric h on N given by*

$$\Gamma^i_{jk} = \frac{1}{2} h^{il} \left(h_{li,k} + h_{kl,j} - h_{jk,l} \right).$$

Proof. 1. Let $U \subset M$ be any coordinate chart and $\varphi \in C^2_0(U; \boldsymbol{R}^l)$. Then we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \left[\frac{1}{2} \int_M g^{\alpha\beta} h_{ij} (u + t\varphi) \left(u^i_\alpha + t\varphi^i_\alpha \right) \left(u^j_\beta + t\varphi^j_\beta \right) \sqrt{g} \right] \mathrm{d}x \\ &= \frac{1}{2} \int_M g^{\alpha\beta} h_{ij,k}(u) \varphi_k u^i_\alpha u^j_\beta \sqrt{g} \mathrm{d}x + \int_M g^{\alpha\beta} h_{ij}(u) u^i_\alpha \varphi^j_\beta \sqrt{g} \mathrm{d}x. \end{aligned}$$

2. Direct computations show

$$\begin{aligned} \int_M \Delta_g u^i h_{ij}(u) \varphi^j \mathrm{d}v_g &= \int_M \frac{\partial}{\partial x^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial u^i}{\partial x^\beta} \right) h_{ij}(u) \varphi^i \mathrm{d}x \\ &= -\int_M \sqrt{g} g^{\alpha\beta} u^i_\beta \left(h_{ij,k}(u) u^k_\alpha \varphi^j + h_{ij}(u) \varphi^j_\alpha \right) \mathrm{d}x \\ &= -\frac{1}{2} \int_M \sqrt{g} g^{\alpha\beta} u^i_\alpha u^j_\beta \left(h_{ik,j} + h_{kj,i} - h_{ij,k} \right) (u) \varphi^k \mathrm{d}x \\ &= -\int_M g^{\alpha\beta} \Gamma^l_{ij}(u) h_{lk}(u) u^i_\alpha u^j_\beta \varphi^k \mathrm{d}v_g. \end{aligned}$$

This implies that

$$\Delta u^i + g^{\alpha\beta} \Gamma^i_{kl}(u) u^k_\alpha u^l_\beta = 0, \ \forall\ 1 \leq i \leq l.$$

□

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