

DIRICHLET PRINCIPLE OF HARMONIC MAPS

ZUJIN ZHANG

ABSTRACT. We define harmonic maps as the critical point of Dirichlet energy functional. This is [1, 1.1].

Let

1. (M^n, g) be a Riemannian manifold with or without boundary;
2. (N^l, h) be a compact Riemannian manifold without boundary (closed).

The we may define the **Dirichlet energy functional**

$$E(u) = \int_M e(u) dv_g,$$

where $e(u)$ is the **Dirichlet energy density function**, the expression of which in local coordinates $(U, x^\alpha), (V, u^i)$ is

$$e(u) \equiv |\nabla u|_g^2 = \frac{1}{2} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Definition 1. A map $u \in C^2(M, N)$ is a **harmonic map** if it is a critical point of the Dirichlet energy functional E .

Proposition 2. A map $u \in C^2(M; N)$ is a **harmonic map** iff u satisfies

$$\Delta_g u^i + g^{\alpha\beta} \Gamma_{jk}^i(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0, \text{ in } M, 1 \leq i \leq l.$$

Here

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1. Δ_g is the **Laplace-Beltrami operator** on (M, g) given by

$$\Delta_g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right);$$

2. and Γ_{jk}^i is the **Christoffel symbol** of the metric h on N given by

$$\Gamma_{jk}^i = \frac{1}{2} h^{il} (h_{li,k} + h_{kl,j} - h_{jk,l}).$$

Proof. 1. Let $U \subset M$ be any coordinate chart and $\varphi \in C_0^2(U; \mathbf{R}^l)$.

Then we have

$$\begin{aligned} 0 &= \frac{d}{dt} \Big|_{t=0} \left[\frac{1}{2} \int_M g^{\alpha\beta} h_{ij}(u + t\varphi) (u_\alpha^i + t\varphi_\alpha^i) (u_\beta^j + t\varphi_\beta^j) \sqrt{g} \right] dx \\ &= \frac{1}{2} \int_M g^{\alpha\beta} h_{ij,k}(u) \varphi_k u_\alpha^i u_\beta^j \sqrt{g} dx + \int_M g^{\alpha\beta} h_{ij}(u) u_\alpha^i \varphi_\beta^j \sqrt{g} dx. \end{aligned}$$

2. Direct computations show

$$\begin{aligned} \int_M \Delta_g u^i h_{ij}(u) \varphi^j dv_g &= \int_M \frac{\partial}{\partial x^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial u^i}{\partial x^\beta} \right) h_{ij}(u) \varphi^j dx \\ &= - \int_M \sqrt{g} g^{\alpha\beta} u_\beta^i (h_{ij,k}(u) u_\alpha^k \varphi^j + h_{ij}(u) \varphi_\alpha^j) dx \\ &= - \frac{1}{2} \int_M \sqrt{g} g^{\alpha\beta} u_\alpha^i u_\beta^j (h_{ik,j} + h_{kj,i} - h_{ij,k}) (u) \varphi^k dx \\ &= - \int_M g^{\alpha\beta} \Gamma_{ij}^l(u) h_{lk}(u) u_\alpha^i u_\beta^j \varphi^k dv_g. \end{aligned}$$

This implies that

$$\Delta u^i + g^{\alpha\beta} \Gamma_{kl}^i(u) u_\alpha^k u_\beta^l = 0, \quad \forall 1 \leq i \leq l.$$

□

REFERENCES

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DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU,
510275, P.R. CHINA

E-mail address: uia.china@gmail.com