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EQUIPARTITION OF ENERGY

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ABSTRACT. In this paper, we show the equipartition of energy for the 1D wave equation [2], and suggest a challenging open problem.

Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ u = g,\ u_t = h, & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

Suppose g, h have compact support. The **kinetic energy** is

$$k(t) \equiv \frac{1}{2} \int_{\mathbb{R}} u_t^2 dx,$$

and the **potential energy** is

$$p(t) \equiv \frac{1}{2} \int_{\mathbb{R}} u_x^2 dx.$$

Prove

1. $k(t) + p(t)$ is constant in t;
2. $k(t) = p(t)$ for all large enough times t.

Proof. 1. Since

$$\begin{aligned} \frac{d}{dt}[k(t) + p(t)] &= \int_{-\infty}^{\infty} [u_t u_{tt} + u_x u_{xt}]\, dx = \int_{-\infty}^{\infty} [u_t u_{xx} + u_x u_{xt}]\, dx \\ &= \int_{-\infty}^{\infty} [u_t u_x]_x\, dx = 0, \end{aligned}$$

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we see

$$k(t) + p(t) = k(0) + p(0) = \frac{1}{2}\int_{-\infty}^{\infty}\left[g'^2 + h^2\right]dx.$$

2. In view of **d'Alembert's formula**,

$$u(x,t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy,$$

and thus

$$u_t(x,t) = \frac{g'(x+t) - g'(x-t)}{2} + \frac{h(x+t) + h(x-t)}{2},$$

$$u_x(x,t) = \frac{g'(x+t) + g'(x-t)}{2} + \frac{h(x+t) - h(x-t)}{2}.$$

Consequently,

$$\begin{aligned} u_t^2 - u_x^2 &= [u_t + u_x]\cdot[u_t - u_x] \\ &= \left[g'(x+t) + h(x+t)\right]\cdot\left[-g'(x-t) + h'(x-t)\right] \\ &= -g'(x+t)g'(x-t) + g'(x+t)h(x-t) \\ &\quad -h(x+t)g'(x-t) + h(x+t)h(x-t) \\ &= 0, \text{ for all large } t, \end{aligned}$$

the last equality holding since both g and h have compact support:

$$\text{supp } (g,h) \subset [a,b]$$
$$\Rightarrow \text{either } x+t \text{ or } x-t \text{ leaves away } [a,b],\ \forall\, t > \frac{b-a}{2},\ x \in \mathbb{R}.$$

We obtain finally that

$$k(t) - p(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left[u_t^2 - u_x^2\right]dx = 0,$$

for all large t.

□

Remark 1. *1. This result can be extended to the wave equation in general odd space dimensions. However, it involves Fourier analysis, mainly the Paley-Wiener theorem* [1].

2. *To the authors' best knowledge, this equipartition of energy was first introduced by Einstein in* 1901*s. Since then, many mathematicians have been devoted to studying this problem.*
3. *Just in May* 2010*, some experiments established in Texas showed that equipartition of energy was valid for Brownian motion.* **This would give a challenging and interesting open problem whether we can give a mathematical proof of it**.

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