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ON SOLUTION FORMULAE OF IBVP FOR THE HEAT EQUATION

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ABSTRACT. In this paper, we give a solution formula of the initial/boundary-value problem for the heat equation via reflection method. This problem is 2.5.13 in [1].

Given a smooth $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, we have the solution formula

$$u(x, t) = \frac{x}{4\pi} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds$$

for the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } \mathbb{R}_+ \times (0, \infty), \\ u = 0, & \text{on } \mathbb{R}_+ \times \{t = 0\}, \\ u = g, & \text{on } \{x = 0\} \times [0, \infty). \end{cases}$$

Proof. Setting $v(x, t) \equiv u(x, t) - g(t)$, due to the fact that

$$v = 0, \quad \text{on } \{x = 0\} \times [0, \infty),$$

we may odd reflect v. Still denoting the resulting function by v yields

$$\begin{cases} v_t - v_{xx} = \begin{cases} g_t, & x < 0 \\ -g_t, & x > 0 \end{cases}, & \text{in } \mathbb{R} \times (0, \infty), \\ v = 0, & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Solution formula for the heat equation in one dimension then gives

$$u(x, t) - g(t)$$

Key words and phrases. heat equation, solution formula, reflection method.

$$
\begin{aligned}
&= v(x,t) \\
&= \int_0^t g_s(s)ds\left[\int_{-\infty}^0 \Phi(x-y,t-s)dy - \int_0^\infty \Phi(x-y,t-s)dy\right] \\
&= \int_0^t g_s(s)\left[-\int_{-x}^x \Phi(y,t-s)dy\right]ds \\
&= -\int_0^t g(s)\left[\int_{-x}^x \Phi_t(y,t-s)dy\right]ds - g(t)\lim_{s\to t_-}\int_{-x}^x \Phi(y,t-s)dy \\
&= -\int_0^t g(s)\left[\int_{-x}^x \Phi_{yy}(y,t-s)dy\right]ds - 2g(t)\lim_{s\to t_-}\int_0^x \frac{1}{[4\pi(t-s)]^{1/2}}e^{-\frac{|y|^2}{4(t-s)}}dy \\
&= -\int_0^t g(s)\left[\Phi_y(t,y-s) = \frac{1}{[4\pi(t-s)]^{1/2}}\cdot\frac{-2y}{4(t-s)}e^{-\frac{|y|^2}{4(t-s)}}\Bigg|_{-x}^x\right]ds \\
&\quad -\frac{2g(t)}{\pi^{1/2}}\lim_{s\to t_-}\int_0^{\frac{x}{[4(t-s)]^{1/2}}} e^{-z^2}dz \\
&= \frac{x}{(4\pi)^{1/2}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{-\frac{|x|^2}{4(t-s)}}g(s)ds - g(t)\left(\int_0^\infty e^{-z^2}dz = \frac{\pi^{1/2}}{2}\right).
\end{aligned}
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□

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