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TOWARD A HARMONIOUS UNIFYING HYBRID MODEL FOR ANY EVOLVING COMPLEX NETWORKS

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The current interest in complex networks is a part of a broader movement towards research on complex systems. Motivation of this work raises the two challenging questions: (i) Are real networks fundamentally random preferential attached without any deterministic attachment for both un-weighted and weighted networks? (ii) Is there a coherent physical idea and model for unifying the study of the formation mechanism of complex networks? To answer these questions, we propose a harmonious unifying hybrid preferential model (HUHPM) to a certain class of complex networks, which is controlled by a hybrid ratio, d/r, and study their behavior both numerically and analytically. As typical examples, we apply the concepts and method of the HUHPM to un-weighted scale-free networks proposed by Barabasi and Albert (BA), weighted evolving networks proposed by Barras, Bartholomew and Vespignani (BBV), and the traffic driven evolution (TDE) networks proposed by Wang *et al.*, to get the so-called HUHPM-BA, HUHPM-BBV and HUHPM-TDE networks. All the findings of topological properties in the above three typical HUHPM networks give certain universal meaningful results which reveal some essential hybrid mechanisms for the formation of nontrivial scale-free and small-world networks.

Keywords: Network science; harmonious unifying hybrid preferential model; hybrid ratio d/r; un-weighted and weighted networks; universal properties; small-world; effects; scale-free property; sensitivity of exponent to hybrid ratio.

1. Introduction

The interdisciplinary research into complex networks has exploded across the academic spectrum as has the great potential for applications. Conferences on these fields from natural to social sciences have become a very hot activity all over the world since the small world (SW) effect and the scale-free (SF) property were discovered by Watts and Strogatz (the so-called WS model) in 1998 [1] and by Barabasi and Albert (the so-called BA model) in 1999 [2], respectively. These two discoveries and substantial advances on a number of previously intractable problems has shown that complex network research has broken free of the imprisonment of random graph analysis that began in the 1960s and has since made unprecedented progress in the following forty years. A variety of theoretical models of networks have been proposed and investigated [1–3, 8–24]. Most real-world networks have shown both the SF and the SW effects, which include small average path length (APL), large average clustering coefficient (ACC), and other topological and dynamical properties. As pointed out by Buchanan [6] and Watts [4, 5], these results have been called the new science of networks. As a pioneer in the field of networking as a unified science and author of "Linked: The New Science of Networks", Barabasi was awarded a medal by the Hungarian von Neumann Computer Society for outstanding achievement in computer-related science and technology (as reported from University of Notre Dame on August 24, 2006; see http://newsinfo.nd.edu).

One of the significant problems at present is that the majority of research concentrates on the BA network and its varieties [3–20], which introduces the random preferential attachment (RPA) mechanism to mimic un-weighted growing networks. Many current models are not completely consistent with those ubiquitous properties in real world networks although they have been useful and successful at reproducing features to approach real world networks. As pointed out by American scientist, Wilson [7]: "The greatest challenge today, not just in cell biology and ecology but in all of science, is the accurate and complete description of complex systems. Scientists have broken down many kinds of systems. They think they know most of the elements and forces. The next task is to reassemble them. At least in mathematical models that capture the key properties of the entire ensembles." That is the motivation for this work.

Indeed, in most of the models either the random factor or the deterministic factor is ignored. Based on basic observations for a unified world, we cannot ignore either of them since the interactions in the real world are neither completely regular nor completely random but lie somewhere between the extremes of order and randomness. The world should be a harmoniously unified one. To do so, we propose a harmonious unifying hybrid preferential model (HUHPM) for a certain class of evolving complex networks, which is controlled only by a hybrid ratio, d/r. Through numerical simulation and theoretical analysis, we find that the HUHPM has a series of universal properties, and possesses both the SF and the SW effects, which allows HUHPM networks to be closer to real-world networks and suitable for any evolving complex networks, including un-weighted and weighted networks.

The un-weighted network models may reflect the most topological characteristics and dynamical behavior between network nodes and connectivity, but they cannot describe the strength of interaction and difference of connected edges. On the other hand, weighted networks can carefully portray the node connections and mutual interactions that not only reflect the topology of real networks, but also reveal physical and dynamic characteristics of real-world networks. Recently, several weighted networks have been proposed in the literature [21–25]. Among them, Barrat, Barthelemy and Vespignani proposed the weighted evolving network model (the so-called BBV model) [21, 22]. The BBV model yields scale-free properties of the degree, weight and strength distributions, but its weight dynamical evolution is triggered only by newly added vertices, resulting in few satisfying interpretations of collaboration networks and transport networks (e.g. airline systems). Improving on this, Wang *et al.* presented the weighted traffic-driven evolution (TDE) model for technological networks [24]. They considered the dynamics of weight taking place along the links and introduced two coupled mechanisms: topological growth and the strengths' dynamics. Their model is a better one for technological networks. However, both the BBV model and the TDE model considered *only* RPA, but did not consider deterministic preferential attachment (DPA). This is in contradiction to a fundamental observation in which one can see that both RPA and DPA, or the deterministic and the random factors, exist extensively in our unified real world.

In short, the many existing models mentioned above belong to the class of generalized random network models (GRNMs). They are not completely consistent with the ubiquitous properties in real-world networks although many models have been useful and successful at reproducing some features that approach real-world networks. The reason for this is that RPA is only considered in GRNMs and utilized to generate networks with the SF and the SW effects. However, based on the observations mentioned above, we cannot ignore RPA and DPA, thus both in various complex network models should be investigated. Motivated by this and the two questions mentioned in the abstract in this work, the HUHPM for a certain class of complex networks is investigated both numerically and analytically. We *only* define a hybrid ratio d/r as DPA/RPA, which is a key parameter to control topological and dynamical properties of evolving complex networks. When applying the HUHPM method to the BA, BBV and TDE models, they are called the HUHPM-BA network, the HUHPM-BBV network and the HUHPM-TDE network, respectively. Through numerical simulation and theoretical analysis, we find that the HUHPM has a series of universal topological properties, which include the exponents of the three power laws (node degree, node strength, and edged weight), that are sensitive to the d/r or depend on the d/r strongly. A threshold of the exponent exists at $d/r = 1/1$; beyond it, the exponent for the HUHPM-BA, HUHPM-BBV and HUHPM-TDE increases rapidly until it approaches a large value, and may tend to infinity. The HUHPM models possess both the SF and SW effects. As the hybrid d/r increases, their APL and assortative coefficient decrease, while their ACC increases. All the results in this article show that HUHPM networks can give satisfactory answers to the questions mentioned above, and are suitable for un-weighted and weighted evolving networks.

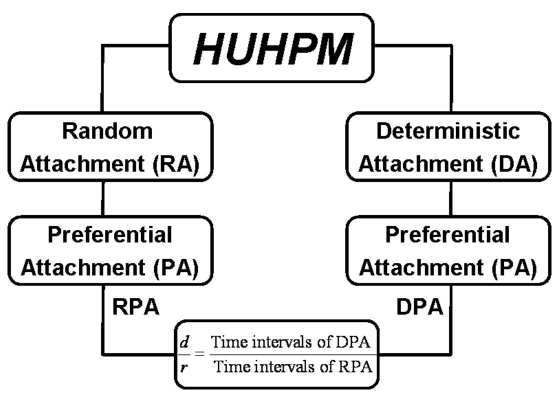
Discoveries of the SW and the SF in complex networks have had potential applications in many fields, such as the Internet, the WWW, power networks, transport networks, social networks, and so on. For this work, the following three aspects for applications may principally be considered. Firstly, because the exponents of the three power laws for HUHPM networks have high sensitivity to the hybrid ratio d/r change, this may yield a corresponding encryption method that can be applied to the cryptology and privacy communication domain. Its basic principles may be similar to chaotic communication, based on chaotic sensitivity to initial conditions.

Secondly, due to the fact that the APL and the ACC can be adjusted by the hybrid ratio d/r, in practice one may design the most required network architecture to satisfy different special requirements. Thirdly, the above discoveries could be helpful for understanding certain phenomena of complex networks that occur in social and biological nervous networks.

The paper is organized as follows. In Sec. 2, the basic concept and method of HUHPM are described. In Sec. 3, sensitivity of the exponent of three power laws to the hybrid ratio d/r is illustrated through numerical simulations. In Sec. 4, complex relationships between the power-law exponent and hybrid ratio are analyzed theoretically for three typical networks and compared with simulation results. The effects of hybrid ratio on small world properties are shown with a comparison between theory and simulation in Sec. 5. The second SW effect and comparisons with others are shown in Sec. 6. Conclusions and a summary are given in the final section.

2. Basic Concept and Method for HUHPM

The basic concept and method for the HUHPM can be expressed as the following:



This implies that any generalized random preferential models can be changed into the HUHPM only by adding DPA in it. This implementation combines the random connection with the determination connection by using the hybrid ratio to request growth scale size of the networks. Hence, the unified hybrid ratio can be defined as:

$$\frac{d}{r} = \frac{\text{Time intervals } d \text{ of deterministic preferential attachment (DPA)}}{\text{Time intervals } r \text{ of random preferential attachment (RPA)}}, \tag{1}$$

where d is a number of time intervals (step) for DPA, and r is a number for RPA. In the process of network evolution, the hybrid ratio must maintain the same value by combining RPA and DPA. In fact, the rank of RPA and DPA can be flexible. This means that one can use different orders to make the two hybrids grow the network in turn, until the required scale size is achieved. The main mechanism and principle for implementation of hybrid network growth are as follows:

(i) **The growth way.** First, use each growing rule of the model to realize the growth. For example, the un-weighted BA model starts growth from less isolated nodes, m_0, increasing at each time-interval to reach a new node with m ($\leq m_0$) edges, and connects this new node to m different existing nodes. For the weighted BBV model, new edges can be produced among the old nodes, while for the TDE model, a new node is grown at each time step. The new node connects to the old nodes with m new edges, and then the network continues to grow.

(ii) **The growth connection way.** Each step adopting this kind of connection mechanism must accord with the hybrid ratio d/r. Keeping down the same hybrid ratio d/r, there are three kinds of hybrid connection orders:

HPAS-1: First conduct RPA, then arrange the rank of the degree of nodes from the biggest to the smallest, selecting m biggest degree nodes to conduct DPA.

HPAS-2: First DPA then RPA.

HPAS-3: RPA or DPA is conducted randomly.

(iii) **The DPA way.** After each attachment, the rank of the degree of nodes is reordered again from the biggest to the smallest: $k_1 > k_2 > \cdots > k_m > \cdots > k_n$, then m nodes are attached preferentially. This is a general way for DPA, which is quite natural.

(iv) **RPA way.** The above ideas and method are applied to some current typical models, and we follow the above steps to give the rules of the three present models BA, BBV and TDE, or as mentioned above, the so-called HUHPM-BA, HUHPM-BBV and HUHPM-TDE network. The concrete constructions are followed by their respective preferential attachment ways [2, 21, 22, 24, 25].

For the HUHPM-BA network, the RPA is firstly connected with the nodes which have massive connections, the probability of preferential attachment for the new node n with old node i is proportional to the degree of node k:

$$P_{n\to i}^{\mathrm{BA}} = \frac{k_i}{\sum_j k_j}. \tag{2}$$

For the HUHPM-BBV network, new preferential node n attaches to old node i according to the intensity coupling probability:

$$\Pi_{n\to i}^{\mathrm{BBV}} = \frac{s_i}{\sum_j s_j} = \frac{s_n s_i}{\sum_j s_n s_j}, \tag{3}$$

where s_i represents the intensity corresponding node (i, j). This model considers the weight of edges; it is chosen with probability $\Pi_{n\to i}^{\mathrm{BBV}}$ to be connected to n. Thus, its connectivity increases by 1, and its strength by $1 + \delta$, where strength weight parameter $\delta = \text{const}$. This means that it allows the production of new connection edges among the old nodes, at the same time, also considering already existing transportation flows along that connection edges unceasingly renewing with the growth of the network.

For the HUHPM-TDE network, we maintain the topological growth rule: at each time step, the network grows a new node connected to m new edges with the old nodes, the connective probability coinciding with Eq. (3), namely, preferential attachment is based on the intensity of nodes. Each weight of the new edge is assumed to be $w_0 = 1$. Among the old nodes, preferential attachment (PA) obeys the intensity coupling dynamic rule: at each time step, all possible (existing or not yet existing) connected edges are subject to the probability of preferential attachment, according to the intensity coupling renewal mechanism:

$$w_{ij} \to \begin{cases} w_{ij} + 1, & \text{with probability } wp_{ij}, \\ w_{ij}, & \text{with probability } 1 - wp_{ij}, \end{cases} \tag{4}$$

where

$$p_{ij} = \frac{s_i s_j}{\sum_{a<b} s_a s_b}, \tag{5}$$

and the intensity coupling arrangement between node i and node j, which determines the increase of weight w_{ij}. If the node i and node j are not connected, then $w_{ij} = 0$. The total weight of the edges in the statistical sense is modified by following amount:

$$\left\langle \sum_{i<j} \Delta w_{ij} \right\rangle = w. \tag{6}$$

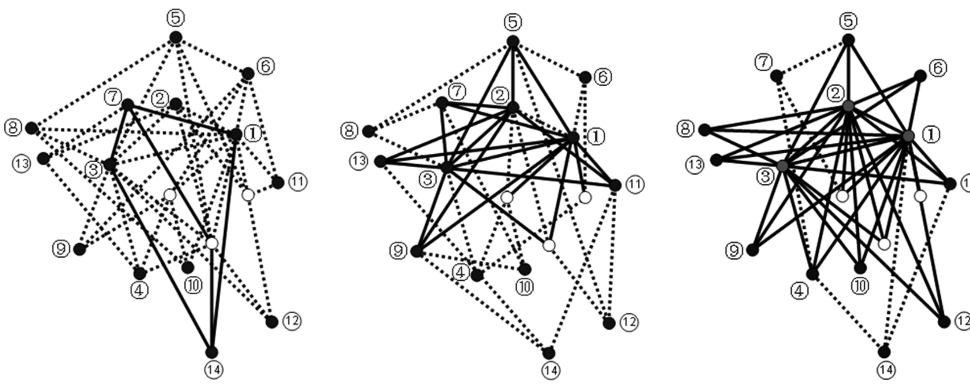
This model simply assumes that w is a constant, which is used approximately to reflect the growth speed for the total transportation load in the entire network. $w > 1$ is taken as in Ref. 24. However, we will take $w < 1$ in this paper since it is more suitable for practical situations.

After the procedures above for each model, the rank of the node degrees is then rearranged from the largest to the smallest as $k_1 > k_2 \cdots > k_n$. The DPA is conducted for d time steps for the HUHPM network according to the new rank of the vertex degrees above when choosing the nodes connected to the new node. This procedure creates a network with $N = r + d + m_0$ nodes.

Then, steps (iii) and (iv) of the HUHPM algorithm are repeated again. In this HUHPM algorithm, two kinds of preferential attachments are applied in turn under a certain hybrid ratio d/r but may be different HPAS ways, until the desired size of network is reached.

Figure 1 illustrates the evolution of HUHPM growing networks with $N = 14$ and $m = m_0 = 3$ under different hybrid ratios $d/r = 1/6$, $1/1$ and $6/1$.

We use the solid line to express deterministic connection in Fig. 1, while the dashed line expresses random connection; the numbers in circles represent the order



(a) (b) (c)

Fig. 1. Evolution of HUHPM growing networks with $N = 14$ and $m = m_0 = 3$: (a) $d/r = 1/6$, (b) $d/r = 1/1$ and (c) $d/r = 6/1$. The solid line denotes a deterministic preferential link; the dashed line is a random preferential link. $m = m_0 =$ is the number of initial nodes; m is the number of nodes to be added at each step.

of newly joined nodes. Figure 1(a) for $d/r = 1/6$ corresponds to the RPA dominating; the network topology is similar to a random network. Moreover, d/r is smaller, and r is bigger. The network should be more similar to the completely random network. Figure 1(b) for $d/r = 1/1$ demonstrates the clustering effect brought by DPA. Figure 1(c) for $d/r = 6/1$ belongs to the case where the DPA plays the leading role. Therefore, the network appears to have high clustering nodes. Moreover, they will increase while d/r increases until condensation occurs on certain nodes, where the condensation degree only depends on the hybrid ratio d/r. In addition, for the same hybrid ratio d/r, the three different HPAS orders of the connected network show no differences [see Fig. 2(a)]. The simulation below further confirms that their evolution is indeed independent of the order of the connections.

3. Sensitivity of the Exponent to the Hybrid Ratio d/r

In fact, the most important topological characteristic for the BA model and its varieties of generalized random network is that their degree distribution obeys the negative exponent of power law. Some real networks also have a similar property, but some power-law curves often show deviation from the heavy tail in the end or top part of the curves, and only the middle part shows the proper logarithm linearity. This phenomenon has various explanations: for the GRNM, the negative power law distribution is the result of the RPA mechanism; the heavy tail is also created by the random connection. Besides this, the statistical data are not sufficient and the rather little rigor of the theory does not coincide with real networks, which can also cost this deviation.

Figure 2 shows the negative power exponent of γ verses the logarithm of the hybrid ratio, $\log(d/r)$, for the HUHPM with three typical models, where the

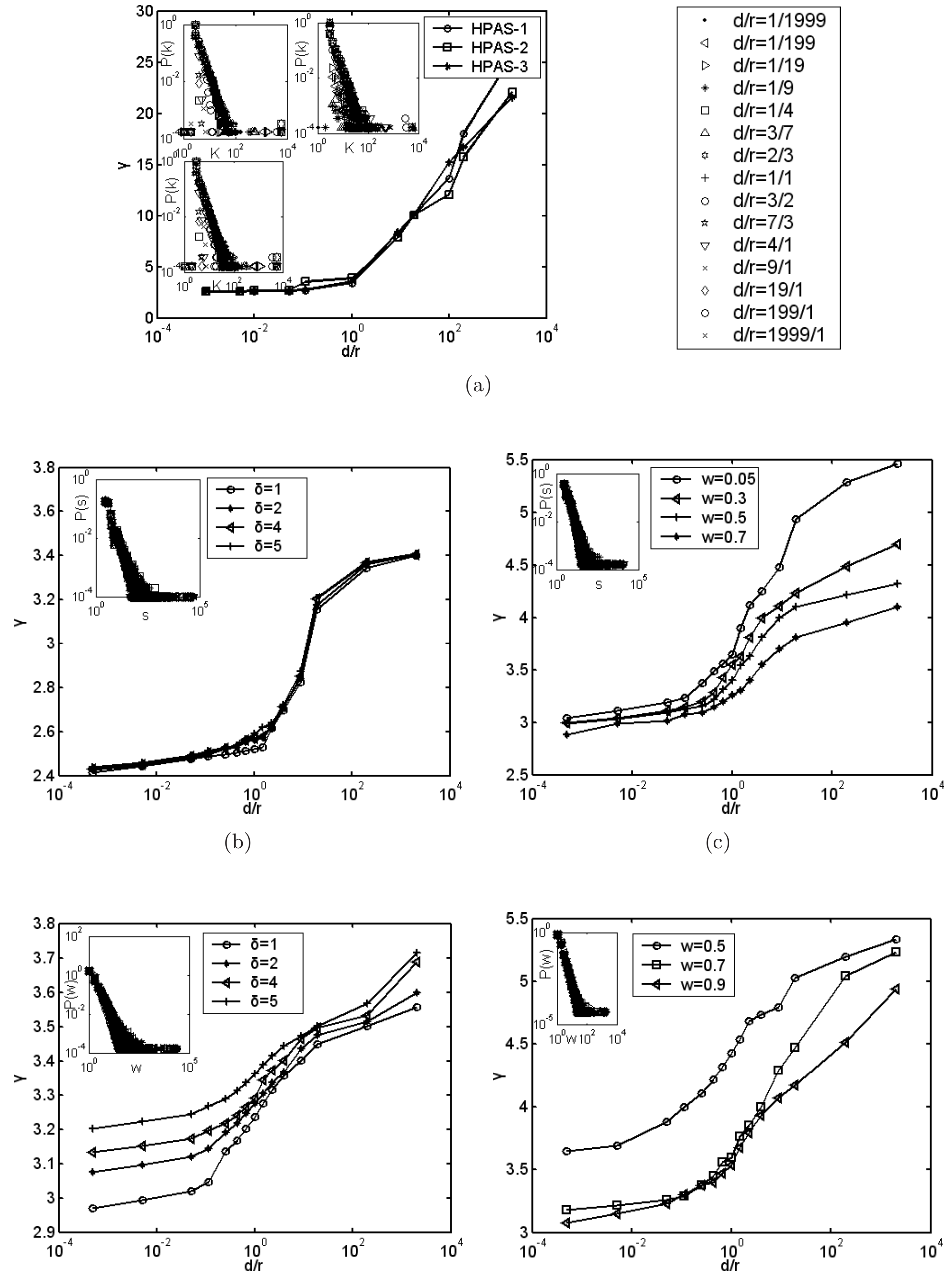


Fig. 2. The exponent γ of power law versus $\log(d/r)$. The insets are power-law distributions for node degree, strength and edge weight, respectively. Here k is the node degree, s denotes the node strength, w is the weight of edges. Here $N = 6{,}000$ and $m = m_0 = 3$. (a) HUHPM-BA with three orders of connections (HPAS-1, HPAS-2, HPAS-3). (b) and (c) HUHPM-BBV and HUHPM-TDE for the node strength. (d) and (e) HUHPM-BBV and HUHPM-TDE for the edge weight.

insets are power-law distributions for the node degree, strength and weight, respectively: (a) HUHPM-BA, comparing three orders of connections (HPAS-1, HPAS-2, HPAS-3); (b) and (c) HUHPM-BBV and HUHPM-TDE for the node strength; (d) and (e) HUHPM-BBV and HUHPM-TDE for the edge weight, where $N = 6{,}000$ and $m = m_0 = 3$.

From Fig. 2(a), one can see that the three different PHAS orders of connected networks show no difference in their power-law distributions. For the three link cases, there is a common universal power law, and when the determinism holds to a certain degree, all the heavy tails are restricted or eliminated. When determinism dominates, the degree distribution is convergent to some highly clustering nodes. Furthermore, the power-law exponents are very sensitive to the change of hybrid ratio d/r, as a new topological characteristic. Here, $d/r = 1/1$ is a threshold value. If $d/r \leq 1$, $\gamma \leq 3$, this is consistent with the RPA playing the leading role. If $d/r > 1/1$, in the HUHPM-BA and HUHPM-BBV models, γ increases as $\log(d/r)$ increases, although δ is different. For the HUHPM-TDE model, when $w < 1$, which is reasonable in the actual situation (because actual relative disturbances generally cannot be bigger than 1), γ increases as $\log(d/r)$ increases, until a large number is approached. Even if $w > 1$, γ still depends on d/r. Once RPA approaches zero ($r = 0$, d/r), γ is very large or even approaches infinity. The power law then vanishes, and often concentrates on several highly clustering nodes.

As shown in Figs. 2(b)–(e), it is noted that the original power law of node strength and edge weight in the BBV and TDE still exist in the corresponding HUHPM-BBV and HUHPM-TDE networks. Moreover, their node strength and edge weight have similar characteristics, i.e. power-law sensitivity to the hybrid ratio d/r. For Fig. 2(b), HUHPM-BBV at $\delta > 1$ and Fig. 2(c), HUHPM-TDE at $w < 1$, both node strength power exponents γ of $P(s)$ are still very sensitive to the d/r change in the range $1/1 < d/r < 10/1$, beyond which the γ tend to linearly increase with $\log(d/r)$. For Fig. 2(d), HUHPM-BBV at $\delta > 1$, and Fig. 2(e), HUHPM-TDE at $w < 1$, both edge weight γ of $P(w)$ are also sensitive to the d/r and increase linearly with $\log(d/r)$ after $d/r > 100/1$. These results reveal that the HUHPM networks have some important new common characteristics and differences, which are worth paying attention to.

4. Theoretical Analysis for the Relation of γ and d/r

In this section, we theoretically analyze the HUHPM-BA, HUHPM-BBV and the HUHPM-TDE network models. According to Ref. 12, a degree rate equation for $k(t)$ can be given by

$$\frac{\partial k_i}{\partial t} = m \frac{k_i}{\sum_{j=1}^{N-1} k_j} = \frac{\beta k_i}{t}, \tag{7}$$

since $\sum_{j=1}^{N-1} k_j = \frac{m}{\beta} t$ may affirm that, under HUHPM, parameter β must be a kind of function of d/r, which only has different parameters for different models. Through preliminary analysis and careful observation, we obtain that β is a kind of exponential function combination:

$$\beta = \frac{\exp\left[\left(\frac{d/r}{A_2}\right)^{A_3}\right]}{A_1 + A_4 \exp\left[\left(\frac{d/r}{A_2}\right)^{A_3}\right]}, \tag{8}$$

where A_1, A_2, A_3, A_4 are relevant parameters. Keeping k_i proportion to β with increasing t, substituting Eq. (8) in Eq. (7), and considering the initial condition [12], we obtain

$$k_i = m\left(\frac{t}{t_i}\right)^{\beta}. \tag{9}$$

Thus, from the degree distribution of the network $P(k)$:

$$P(k) \propto 2m^{\frac{1}{\beta}} k^{-\left(\frac{1}{\beta}+1\right)}. \tag{10}$$

For the HUHPM-BA network, we get the function relationship of power index γ with d/r as:

$$\gamma_{\mathrm{BA}}^{\mathrm{HUHPM}} = \frac{1}{\beta} + 1 = \frac{A_1}{\exp\left[\left(\frac{d/r}{A_2}\right)^{A_3}\right]} + A_4 \quad \text{(formula 1)}. \tag{11}$$

Adjusting relevant parameters, one can make β another function of d/r:

$$\beta = \frac{1}{\gamma_0 + A_1\left(1 - \exp\left(-\frac{\frac{d}{r}}{A_2}\right)\right) + A_3\left(1 - \exp\left(-\frac{\frac{d}{r}}{A_4}\right)\right)}. \tag{12}$$

Then, from Eqs. (9)–(12) we can get another function of γ and d/r:

$$\gamma_{\mathrm{BA}}^{\mathrm{HUHPM}} = \gamma_0 + A_1 * \left(1 - \exp\left(-\frac{\frac{d}{r}}{A_2}\right)\right) + A_3 * \left(1 - \exp\left(-\frac{\frac{d}{r}}{A_4}\right)\right) \quad \text{(formula 2)}, \tag{13}$$

where $\gamma_0 = 3$. The parameters for Eqs. (11) and (13) (formulas 1 and 2) are listed in Appendix A (see Table 1). As shown in Fig. 3(a), the theoretical results coincide well with the numerical curves for corresponding different parameters.

In the same way, we can get the γ and d/r relation for HUHPM-BBV and HUHPM-TDE by considering the change of δ and w, respectively. The power exponent γ of the HUHPM-BBV model is given by

$$\gamma_{\mathrm{BBV}}^{\mathrm{HUHPM}} = \frac{4\delta + \dfrac{A_1}{\exp\left[\left(\dfrac{d/r}{A_2}\right)^{A_3}\right]} + A_4}{2\delta + 1} \quad \text{(formula 3)}, \tag{14}$$

and another is

$$\gamma_{\mathrm{BBV}}^{\mathrm{HUHPM}} = \frac{4\delta + \gamma_0 + A_1\left(1 - \exp\left(-\dfrac{d/r}{t_1}\right)\right) + A_2\left(1 - \exp\left(-\dfrac{d/r}{t_2}\right)\right)}{2\delta + 1} \quad \text{(formula 4)}, \tag{15}$$

where parameters for different δ are also given in Appendix 1 (see Table 1); while the HUHPM-TDE γ is given by

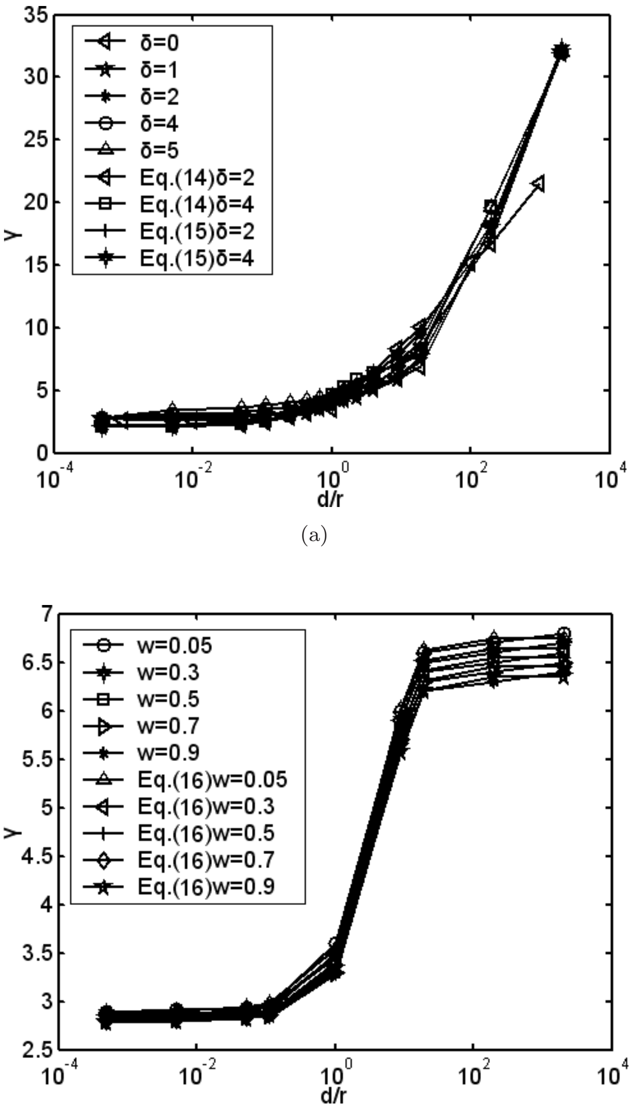
$$\gamma_{\mathrm{TDE}}^{\mathrm{HUHPM}} = 1 + \chi\left\{\left[1 + \left\{\frac{A_1}{\exp\left[\left(\dfrac{d/r}{A_2}\right)^{\alpha_1}\right]} + (A_4 - 2)\right\}\frac{m}{2w + m}\right]\right\} \quad \text{(formula 5)}, \tag{16}$$

and

$$\gamma_{\mathrm{TDE}}^{\mathrm{HUHPM}} = 1+\chi\left[1+\frac{\left\{A_1\left(1-\exp\left(-\dfrac{d/r}{t_1}\right)\right)+A_2\left(1-\exp\left(-\dfrac{d/r}{t_2}\right)\right)+(\gamma_0-1)\right\}m}{2w+m}\right] \quad \text{(formula 6)}, \tag{17}$$

where the strength of node $s \propto k^{\chi}$ and $\chi = 1$ for HUHPM-BBV and χ is related to w for HUHPM-TDE [24]. Relevant parameters A_1, A_2, A_3, A_4 are also given in Appendix A (Table 1).

Figure 3 shows the power-law exponent γ versus the $\log(d/r)$ by comparing the numerical simulation result with the theoretical results of Eqs. (11), (13) and (16). We see that for two typical networks, (a) HUHPM-BBV and (b) HUHPM-TDE, under different parameters (see Table 1), the simulation results are consistent with the theoretical results. Moreover, for all three HUHPM models, their power-law



(a)

(b)

Fig. 3. The power-law exponent γ versus the $\log(d/r)$, comparing simulation results with theoretical results: (a) the HUHPM-BBV and (b) the HUHPM-TDE, where $N = 6{,}000$ and $m = m_0 = 3$.

exponent has quite a complicated relation with the hybrid ratio d/r, which is beyond the simple relations in the corresponding original models. This reflects either mutual competition or the harmonious unification between DPA and RPA. Furthermore, the HUHPM-BBV and HUHPM-TDE models are related to δ or w, respectively, which is closely connected to the architecture producing the network. This makes the relationship more complicated.

5. Small World Effect for HUHPM — Short APL

The SW effect discovered by Watts and Strogatz [1, 4, 5] has both short APL and large ACC. In fact, this SW phenomenon is similar to a famous poem of the Tang dynasty in ancient China: "It is as neighbor as his hand and as remote as a star", which might demonstrate a wonderful and surprising prediction regarding the SW phenomenon in society and philosophy. It is very interesting that the previously proposed HUHPM network also merits these two SW characteristics, very short APL and very big ACC, except that their power exponent distributions are sensitive to the hybrid ratio. Below, we discuss these two characteristics using the computed results and comparison with other models.

5.1. *The relation of APL with the hybrid ratio d/r*

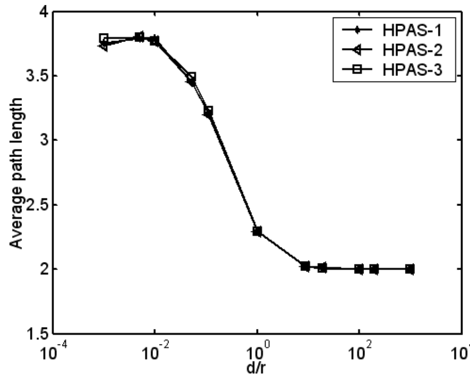
Figure 4(a) shows the relation between the shortest APL, L, and the hybrid ratio d/r for the three kinds of different preferential attachment orders (HPAS-1 $\sim$ 3). One can see that L is not influenced by the orders. For the HUHPM-BA model, the change of L with the ratio d/r has three stages: for the first stage, $d/r < 1/100$, randomness dominates: L slowly drops, and is almost invariable near 3.7. In the second stage, $1/100 < d/r < 1/1$, the value of L rapidly drops from 3.7 to 2.3, randomness gradually approaches the same proportion as that of determination, and d/r approaches the threshold value of $d/r = 1/1$. Therefore, the shortest APL change is extreme. This is the very natural tendency. In the third stage, $1/1 \leq d/r < \infty$, determination starts to dominate. The L value drops extremely slowly: from 2.3 it drops to 2 until it becomes invariable and arrives at the shortest APL, which is the shortest distance between the two most closely connected neighbor nodes. The SW effect is prominently evident.

For the HUHPM networks, by numerical simulation, the relation between L and d/r is obtained from

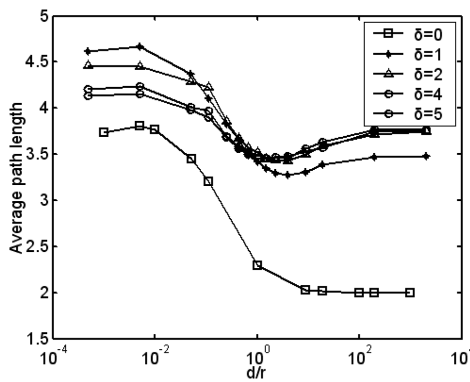
$$L = l_2 + \frac{l_1 - l_2}{\left(1 + \dfrac{d/r}{a}\right)^{\alpha}}, \tag{18}$$

where $l_1 = 3.79754$, $l_2 = 1.99859$, $a = 0.21263$, $\alpha = 1.07902$ for HUHPM-BA. Equation (18) shows that the shortest APL basically decreases with the power function of the hybrid ratio d/r and comes close to 2 after $d/r \geq 10/1$. This demonstrates that in the complete determination connection, the shortest APL is two nodes apart.

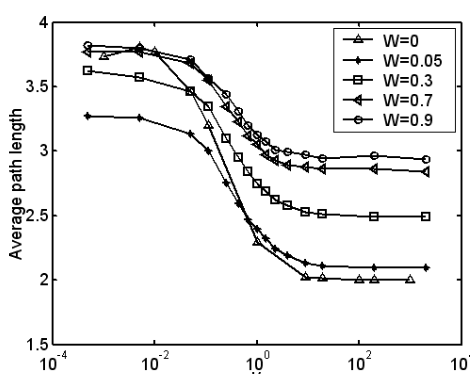
Figures 4(b) and (c), respectively, show for the HUHPM-BBV model (for $\delta \geq 0$) and the HUHPM-TDE model (for $w < 1$) the relation between L and d/r, where each network size is $N = 6{,}000$ and $m = m_0 = 3$. From Figs. 4(b) and (c), one can see that the HUHPM-BBV and HUHPM-TDE networks have similar relations



(a)



(b)



(c)

Fig. 4. The relationship between the shortest APL and d/r for (a) the HUHPM-BA network under three different HPAS orders, (b) the HUHPM-BBV model ($\delta > 1$), and (c) the HUHPM-TDE model ($w < 1$), where $N = 6{,}000$ and $m = m_0 = 3$.

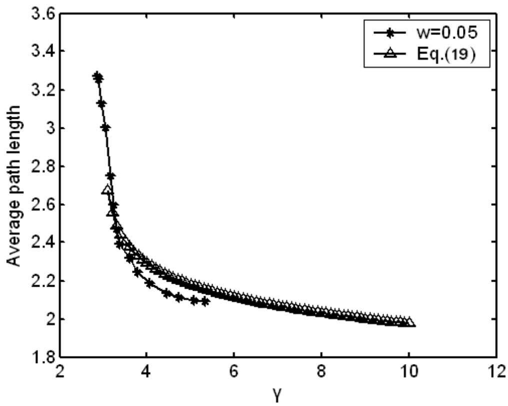


Fig. 5. The relation between L and γ for HUHPM-TDE by comparing of the numerical curve with Eq. (19).

between L and d/r to the HUHPM-BA network. When $\delta = 0$ and $w = 0$, they are both reduced to HUHPM-BA model completely. However, they also show some new characteristics. For the HUHPM-BBV model, when $\delta > 1$, the curve appears as the small V glyph, which means that the existing minimum L, corresponding to d/r nearly at 10, is when δ is bigger, and L is smaller. For the HUHPM-TDE model $w < 1$, when d/r increases; w becomes bigger, and L becomes smaller. This fully explains why HUHPM-TDE has the SW effect and demonstrates that HUHPM-TDE is closer to the actual network than the original TDE model.

5.2. *The relation of APL with N*

Here, we study numerically the relation of the APL with the network scale size N in detail. Theoretically, we get the interesting conclusion that L decreases as γ (or d/r) increases. This is because the present shortest L is a function of γ, and γ is also the function of d/r; therefore, L depends on d/r. If γ rises as d/r increases, then L drops along as γ increases. In fact, from the common random graph APL formula [26], considering the determination factor possibly having the influence, if we choose $N = 6{,}000$ and $m = 3$ for the HUHPM-TDE, we have

$$l_{\mathrm{TDE}}^{\mathrm{HUHM}}(\gamma) = \frac{\ln\left[\left(\frac{9}{\gamma - 3}\right)\right] - \frac{2}{9(\gamma - 1)} + 8.6192}{4.788} + \frac{1}{2}. \tag{19}$$

This is represented by Fig. 5, which shows the relationship between L and γ for the HUHPM-TDE network. The result is quite consistent with the numerical curve. For HUHPM-BA and HUHPM-BBV, similar results are obtained.

In addition, we get some fit unification formulas representing the relations of note strength and edged weight with weight parameters for the HUHPM-BBV and HUHPM-TDE networks, respectively, as follows:

$$\gamma^{\mathrm{HUHPM}} = A_1 + \frac{A_2 - A_1}{\delta + \left(\dfrac{d/r}{A_3}\right)^{A_4}} \quad \text{and} \quad \gamma^{\mathrm{HUHPM}} = A_1 + \frac{A_2 - A_1}{w + \left(\dfrac{d/r}{A_3}\right)^{A_4}},$$

where A_1, A_2, A_3, A_4 are some constants (see Appendix C; Table 3). These formulas imply again that the power laws of the node strength and edged weight distributions strongly depend on the hybrid ratio and there is quite a complex relation between the hybrid ratio and their weight parameters.

6. The Second SW Effect and Comparisons with Others

The second SW effect is the big average clustering coefficient (ACC) C. This describes a local organized topological quantity for complex networks [1, 4, 5]. The big ACC means that the local network partial group degree is higher. In the HUHPM models, the ACC depends not only on the network architecture parameter, but also on the hybrid ratio. Theoretically, for the HUHPM models, we find a ACC formula:

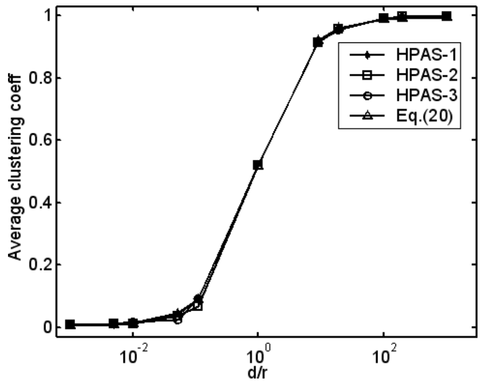
$$C = c_2 + (c_1 - c_2)/\{1 + [(d/r)/c_3)]^q\}, \tag{20}$$

where the parameters c_1, c_2, c_3, q for three typical networks, HUHPM-BA, HUHPM-BBV and HUHPM-TDE, are given in Appendix B (see Table 2), respectively.

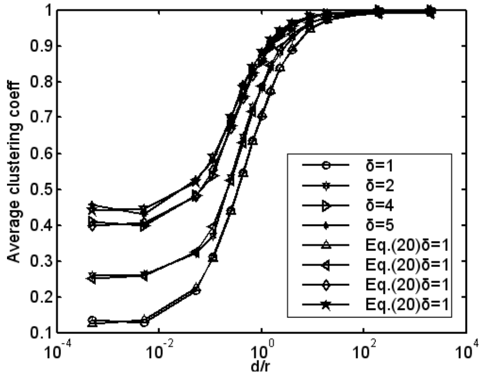
6.1. *The relation between C and d/r*

Figure 6 shows the relationship between ACC and d/r for the three types of HUHPM network. From Fig. 6, one can see that three stages for the three cases coexist. Interestingly, they are opposite to the previous curve of L and d/r, and show a mirror image relation, which is the anti-symmetric curve. For Fig. 6(a), the HUHPM-BA model, the first stage is at $d/r < 1/10$, corresponding to randomness dominating; then $C < 0.1$ is a very small constant. For the second stage, $1/10 \le d/r < 10/1$, C rapidly rises from 0.2 to 0.9, and $d/r = 1/1$; C arrives at 0.5. For $10/1 < d/r < 100/1$, C rises to 0.98. During this period, determination plays a leading role. In the third stage, $100/1 < d/r < \infty$, C approaches 1 from 0.98 and remains constant.

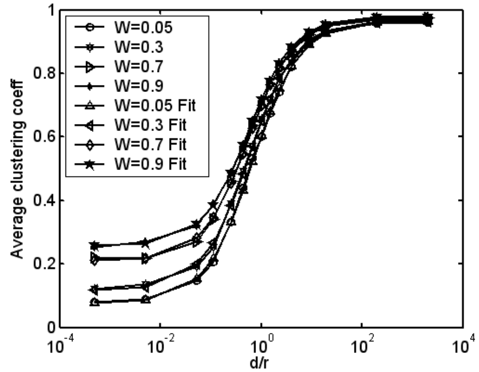
For Fig. 6(b), the behavior of the HUHPM-BBV model is like the HUHPM-BA model: with increasing δ, C approaches 0.16 and then has the same behavior in the next two stages as HUHPM-BA. For Fig. 6(c), the HUHPM-TDE model, the behavior is similar to the HUHPM-BBV model, except when $w > 1$ increases, C is bigger than HUHPM-BBV, about $C > 0.4$. But in the third stage, C is smaller than



(a)



(b)



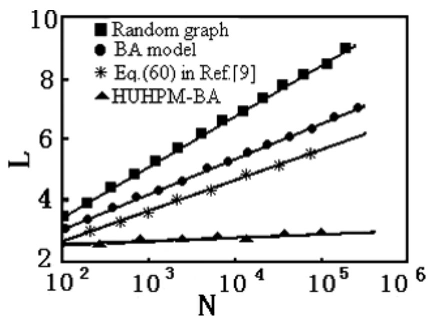
(c)

Fig. 6. The relation of ACC with d/r and compared simulation results with theoretical results. (a) the HUHPM-BA network, (b) the HUHPM-BBV network, (c) the HUHPM-TDE network for $w < 1$.

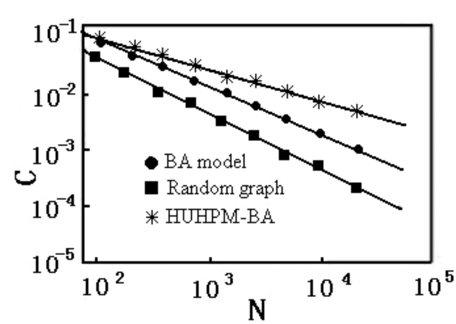
the HUHPM-BA and HUHPM-BBV models, about 0.8, and continuously decreases as w increases.

6.2. *Comparing the small-world effects of the HUHPM with other models*

By comparing the APL with network size N for the three kinds of models (the HUHPM, the original BA model and the random graph model) Fig. 7(a) shows the APL of HUHPM to be the shortest. The block is the result for the random graph model; its L value is the biggest. The dot is the result for the original BA model; its L value is the second biggest. The asterisk is the result for Eq. (60) in Ref. 12, whose L value is the middle. The triangle is the HUHPM-BA result; the L value is the smallest. This explains once more that HUHPM can reduce the APL for the same size N, achieving the smallest APL compared with other networks.



(a)



(b)

Fig. 7. Comparing small-world effects of the HUHPM with other models. (a) illustrates that when $d/r = 1$ and $\langle k \rangle = 4$, using the HUHPM-BA method to obtain APL is shortest. The brown (triangle number) is the HUHPM-BA result; the asterisk is the result of Eq. (60) in Ref. 12; the circle is the result of the original BA model; the block is the result of random graph. (b) is a similar comparison between ACC and N for the HUHPM-BA, random graph and original BA models.

Figure 7(b) compares the ACC with the network size N for the three kinds of models (HUHPM, the original BA model and the random graph model). One can see that the ACC of HUHPM-BA network is biggest. The block is the result for the random graph model; its C value is the smallest. The dot is the result for the original BA model; its C value is the second biggest. The asterisk is the result of HUHPM-BA, whose C value is the biggest. This explains once more that the HUHPM can really enhance the ACC for the same level size N, achieving the biggest ACC compared with other networks.

7. Effect of Hybrid Ratio on Entropy Characteristics of HUHPM

Entropy is another important characteristic quantity for statistical mechanics. What role does entropy play in complex networks? What is the effect of the hybrid ratio on the entropy characteristics of HUHPM? We will use the entropy concept proposed by Boltzmann and Gibbs: the entropy of a system is defined by the so-called Boltzmann–Gibbs entropy (BGS):

$$\mathrm{BGS} = -k \sum_{i=1}^{\infty} P(i) \ln P(i) \tag{21}$$

with the normalization condition $\sum_{i=1}^{\infty} P(i) = 1$, where $P(i)$ is the probability of the system being in the ith microstate, and k is the Boltzmann constant. Without loss of generality, one can also arbitrarily assume $k = 1$. We can use the BGS to measure the relationship between entropy and d/r, as well as the power exponent γ. We will study the effects of hybrid entropy on un-weighted HUHPM-BA and weighted HUHPM-BBV, respectively.

7.1. *For the HUHPM-BA*

To compute the entropy, based on Eqs. (10) and (11), we consider the approximation

$$P(k) = 2m^{\frac{1}{\beta}} k^{-\left(\frac{1}{\beta}+1\right)}. \tag{22}$$

Using formulas (21), (11) and (22), we have the theoretical curves of BGS versus d/r and BGS versus γ, as shown in Figs. 8(a) and (b). The corresponding results from numerical simulation are shown in Figs. 8(c) and (d). Comparing the results implies that the theoretical results are consistent with the simulation results.

7.2. *For the weighted HUHPM-BBV*

Similarly, for the weighted HUHPM-BBV model, to compute the entropy, we have the approximation

$$P(k) = \frac{1}{m^{1-\frac{1}{\beta}} \beta k^{1+\frac{1}{\beta}}}. \tag{23}$$

Using formulas (21), (16) and (23), we obtain the curves for BGS versus d/r, and BGS versus γ, as shown in Figs. 9(a) and (b). The corresponding results from numerical simulation are shown in Figs. 9(c) and (d).

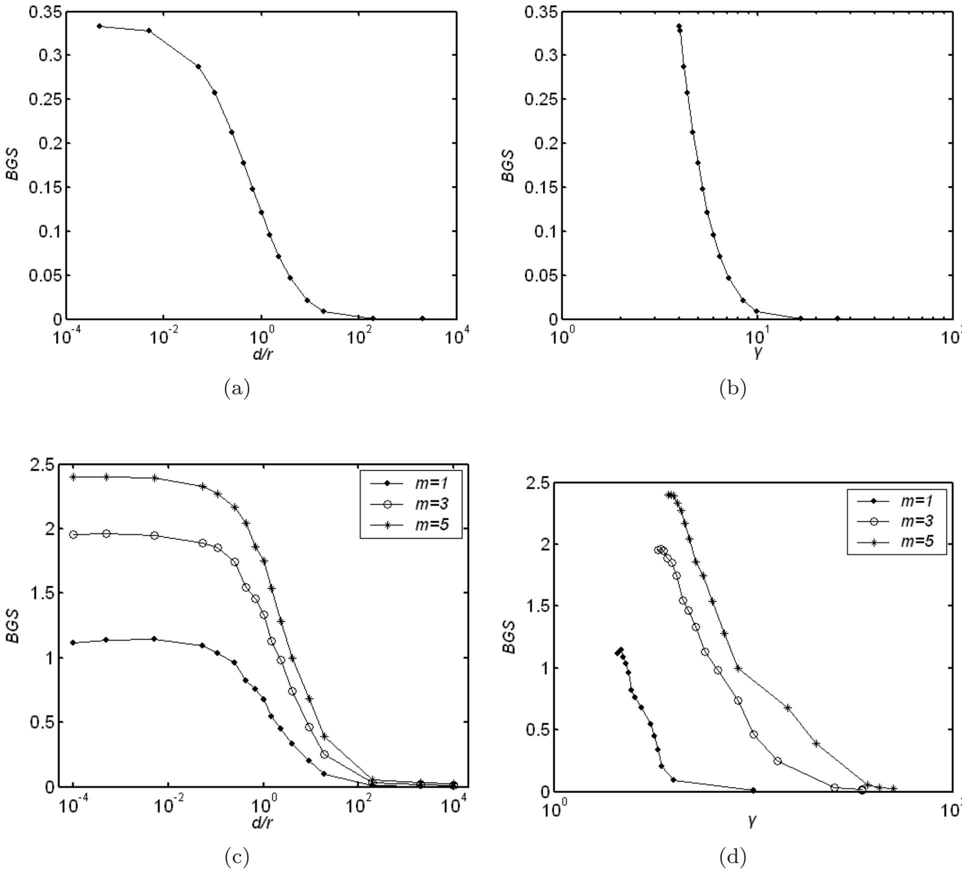


Fig. 8. BGS versus d/r and γ with network size $N = 2{,}000$, $m = 1, 3, 5$. (a) and (b) Theoretical results for BGS. (c) and (d) Numerical simulation results.

It is found from Figs. 8 and 9 that BGS decreases as d/r increases and the current of the BGS along with hybrid ratio d/r or exponent γ of the power-law is consistent. These results can provide a better understanding of the evolution characteristics in growing HUHPM complex networks.

8. Assortativity Coefficient and Synchronizability

The effects of hybrid ratio on assortative mixing and synchronizability in complex networks are also interesting problems [25–29]. The amount of assortative mixing can be quantified by the assortativity coefficient r_{ac}, which measures the tendency of nodes to be connected to other nodes. Our numerical simulations were shown in Ref. 30, where there were three stages in the HUHPM, which is similar to the APL versus d/r, but the values of r_{ac} are negative. For the same value of d/r,

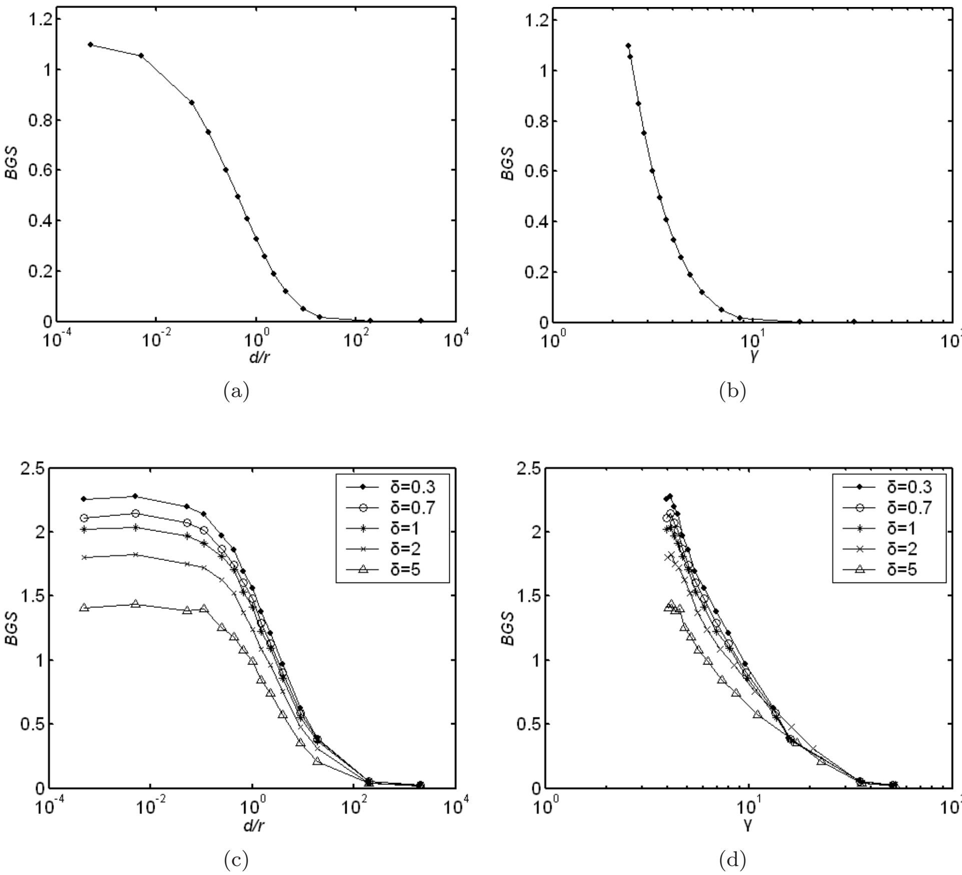


Fig. 9. BGS versus d/r and γ with $N = 2{,}000$, $m = 5$. (a) and (b) Theoretical results for BGS. (c) and (d) Numerical simulation results.

the r_{ac} decreases with the increase of the value of the strength parameter δ for the HUHPM-BBV networks. However, for HUHPM-TDE, r_{ac} decreases as the weight parameter w increases for $d/r < 1/1$, and r_{ac} increases as w increases for $d/r > 1/1$. Theoretically, we obtain the following complex relationship between r_{ac} and d/r:

$$r_{ac} = r_0 + A_{11}\left(\exp\left(-\frac{d/r}{A_{12}}\right)\right) + A_{21}\left(\exp\left(-\frac{d/r}{A_{22}}\right)\right) + A_{31}\left(\exp\left(-\frac{d/r}{A_{32}}\right)\right). \tag{24}$$

For HUHPM-BA, the parameters are $r_0 = -0.9977$, $A_{11} = 0.06264$, $A_{12} = 51.53277$, $A_{21} = 0.23651$, $A_{22} = 0.56142$, $A_{31} = 0.62374$, $A_{32} = 0.99036$. Similar relationships for the HUHPM-BBV and the HUHPM-TDE models are also obtained with different parameter values. Our study demonstrates that the theoretical results are quite consistent with numerical simulation results. The above three types of HUHPM network have a bigger assortativity coefficient, which is strongly affected by the hybrid ratio d/r.

We have also investigated the effect of the hybrid ratio d/r on dynamical synchronizability for the HUHPM networks [30]. The main conclusion is that the synchronizability of the HUHP-BBV network is enhanced if the hybrid ratio increases and the HUHPM network satisfies the so-called type-I synchronization condition, while the synchronizability of the HUHP-BBV network is weakened if it satisfies the type-II synchronization condition. Therefore, the synchronizability of the HUHPM networks is affected by the hybrid ratio and the synchronization condition. The results above are useful for comprehensively understanding the relationship between the topology properties and the dynamical synchronization behaviors for un-weighted and weighted complex networks.

9. Conclusions

The above analysis and simulation results have shown that the proposed HUHPM is suitable for many types of un-weighted and weighted networks, and theoretical results are in good agreement with numerical results. We have discovered some universal laws, such as: the SF power exponents have high sensitivity to or dependence on the hybrid ratio d/r; the HUHPM models have the shortest APL; and the properties of the biggest ACC and the bigger negative r_{ac}. The APL, ACC and r_{ac} are also greatly affected by the hybrid ratio d/r. One may also adjust APL and ACC by changing d/r. This situation is easily understood since when the hybrid ratio $d/r \geq 1$, the determinism in the network dominates, which results in more orders to grow the evolution network.

We have shown that all the characteristics of HUHPM models are much closer to those of real networks. The results demonstrate that, although the RPA is the main mechanism to produce the distributed power law function, the DPA also plays a vital role that can suppress and eliminate the heavy tail. The decrease of BGS entropy of the HUHPM models as the hybrid ratio increases enhances the self-organization of the HUHPM networks. The HUHPM models can cause randomness and determinism to arrive at harmonious unification, which shows that they are general and succinct models. We believe that HUHPM networks reveal more essential mechanisms for producing both the SF and SW properties in many actual networks. Hence, further applications for more widespread types of network are possible as mentioned in introduction.

Theoretical analysis of the HUHPM networks is still an open problem, and there are some extremely challenging problems for further study.

Acknowledgments

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Appendix A

Table 1. The parameters of exponent formulas for HUHPM-BA, HUHPM-BBV and HUHPM-TDE models for different weight parameters.

		Control variable/parameter				Parameter			
Model	Equation	d/r	χ	δ	w	A_1	A_2	A_3	A_4
HUHPM-BA	(11) Fomula 1	Yes				107.62317	74594.7464	−0.12849	$A_4 = \gamma_0 = 3$
	(13) Fomula 2	Yes				6.13617	6.26677	13.60908	323.67565
HUHPM-BBV	(14) Fomula 3	Yes		2		3972.21249	2813168942.9169	−0.08381	
				4		5551.413	1540244321.0424	−0.08145	
	(15) Formula 4	Yes		2		134.04128	302.69868	15.14441	1.02978
				4		35.96354	1.06106	233.1824	267.94085
HUHPM-TDE	(16) Fomula 5	Yes	2		0.05	0.07738	0.3114	−1.39247	1.03961
					0.3	0.07548	0.33715	−1.45829	1.03419
					0.5	0.06249	0.34483	−1.53529	1.0262
					0.7	0.06277	0.34907	−1.53086	1.02447
					0.9	0.06115	0.35477	−1.52504	1.02457
	(17) Fomula 6	Yes	0.5		0.05	7.92309	5.39626	0.89938	0.00014
					0.3	9.07618	5.53752	1.26503	0.0001
					0.5	9.8145	5.81671	1.65413	0.0001
					0.7	10.70325	5.88584	1.83152	0.0001
					0.9	11.40359	5.93872	2.12625	0.0001

Appendix B

Table 2. The parameters in three models ACC calculation, Eq. (20).

Model	Control variable and parameters			Parameters			
	d/r	δ	w	c_1	c_2	c_3	Q
HUHPM-BA	Yes			0.00543	0.99482	0.94419	1.11489
HUHPM-BBV	Yes	1		0.12267	0.99959	0.46742	0.92043
	Yes	2		0.25161	0.99115	0.40904	1.07934
	Yes	4		0.39859	0.99497	0.29878	1.0556
	Yes	5		0.43949	0.99567	0.27692	1.07346
HUHPM-TDE	Yes		0.05	0.07608	0.96411	0.66542	0.9254
	Yes		0.3	0.11867	0.963	0.56915	0.94334
	Yes		0.7	0.2102	0.97282	0.54682	0.96281
	Yes		0.9	0.25726	0.97951	0.55727	0.95256

Appendix C

Table 3. A list of parameters for calculating the following unification formulas for relation of note strength and edged weight with weight parameters for the HUHPM-BBV and HUHPM-TDE networks: $\gamma_{\mathrm{BBV}}^{\mathrm{HUHPM}} = A_1 + \frac{A_2 - A_1}{\delta + \left(\frac{\frac{d}{r}}{A_3}\right)^{A_4}}$ and $\gamma_{\mathrm{TDE}}^{\mathrm{HUHPM}} = A_1 + \frac{A_2 - A_1}{w + \left(\frac{\frac{d}{r}}{A_3}\right)^{A_4}}$.

Note strength	δ	w	A_1	A_2	A_3	A_4
HUHPM-BBV	1		3.3907	2.46797	10.26797	1.26369
	2		3.40326	1.56594	5.71113	1.20425
	4		3.40545	−0.24169	3.26233	1.27381
	5		3.41523	−1.18226	2.38293	1.16523
HUHPM-TDE		0.05	5.47342	5.3528	311.78088	0.69321
		0.3	4.63495	4.14145	15.59356	0.69766
		0.5	4.28291	3.65149	5.16327	0.90909
		0.7	4.07814	3.28238	6.29068	0.69083
Edged weight	δ	w	A_1	A_2	A_3	A_4
HUHPM-BBV	1		3.54233	2.96284	1.22954	0.61649
	2		3.58692	2.55352	0.63613	0.56226
	4		3.67373	1.46053	0.25329	0.48664
	5		3.79415	0.67369	0.09348	0.32562
HUHPM-TDE		0.5	5.34441	4.4737	4.7959	0.51529
		0.7	5.29593	3.80841	14.4665	0.621
		0.9	5.08979	3.24461	14.08361	0.42047

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