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MONOTONICITY METHODS IN *PDE*

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ABSTRACT. In this paper, we renormalize the huts 5.1.3 and 6.1.1 in [1], so as to be more accessible, see more details in [4]. Roughly speaking, monotonicity is the natural substitution of convexity in building solutions of *PDE* .

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1. **Minty-Browder method in L^2.**

In this hut, we introduce the **monotonicity method** due to Minty and Browder. As as illustrative problem, we consider the following quasi-linear *PDE* :

$$\begin{cases} -\mathrm{div}\,(\boldsymbol{E}(Du)) = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{1}$$

where $\boldsymbol{E} : \mathbb{R}^n \to \mathbb{R}^n$ is given.

Observe that (1) can be solved by calculations of variations in case $\boldsymbol{E} = DF$ for some convex $F : \mathbb{R}^n \to \mathbb{R}$.

Key words and phrases. Monotonicity method, weak convergence method.

Our problem is then what natural conditions on $\boldsymbol{E}$ so that (1) may be directly tackled, when $\boldsymbol{E}$ is no longer the gradient of a convex function.

This is the work of Minty and Browder, who give

Definition 1. *A vector field $\boldsymbol{E}$ on $\mathbb{R}^n$ is called* **monotone** *if*

$$(\boldsymbol{E}(p) - \boldsymbol{E}(q)) \cdot (p - q) \geq 0, \ \forall\ p, q \in \mathbb{R}^n, \tag{2}$$

and show (1) can be tacitly worked out as

Theorem 2. *Assume $\boldsymbol{E}$ is monotone and satisfies the growth condition $|\boldsymbol{E}(p)| \leq C\,(1 + |p|)$, $p \in \mathbb{R}^n$.*

Let $\{u_k\} \in H_0^1(U)$ be weak solutions of the approximating problems

$$\begin{cases} -div\ \ (\boldsymbol{E}(Du_k)) = f_k, & in\ U, \\ u_k = 0, & on\ \partial U, \end{cases} \tag{3}$$

with $f_k \to f$ in $L^2(U)$.

Suppose $u_k \rightharpoonup u$ in $H_0^1(U)$. Then u is a weak solution of (1).

Proof. We first write down

$$\begin{aligned} 0 \ &\leq \ \int_U [\boldsymbol{E}(Du_k) - \boldsymbol{E}(Dv)]\,[Du_k - Dv]\,dx \ \ (Monotonicity) \\ &= \ \int_U [f_k u_k - f_k v - \boldsymbol{E}(Dv)(Du_k - Dv)]\,dx, \ \forall\ v \in H_0^1(U) \\ &\quad \big(\text{integration by parts and weak formulation}\big). \end{aligned}$$

Then taking $k \to \infty$ yields

$$0 \leq \int_U [f(u - v) - \boldsymbol{E}(Dv) \cdot (Du - Dv)]\,dx.$$

Choosing $v = u + \lambda w$, with $\lambda \in \mathbb{R}$, $w \in H_0^1(U)$ furthermore gives

$$0 \leq sgn(\lambda) \int_U [\boldsymbol{E}(Du + \lambda Dw) \cdot Dw - fw]\,dx.$$

Passing $\lambda \to 0$ finally, we have as desired

$$0 = \int_U [\boldsymbol{E}(Du) \cdot Dv - fw]\, dx, \ \forall\, w \in H_0^1(U).$$

□

2. **Minty-Browder method in L^∞.** We consider the strong solutions of *PDE* , instead of weak solutions in (1). Hence the Minty-Browder method moves from L^2 to L^∞.

To illustrate how it works, let us consider the following fully nonlinear *PDE* :

$$\begin{cases} F(D^2u) = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{4}$$

where $F : S^{n\times n} \to \mathbb{R}$ is given. Here $S^{n\times n}$ is the space of real, symmetric $n \times n$ matrices.

Definition 3. *The problem* (4) *is* **elliptic**, *if F is monotone decreasing with respect to matrix ordering on $S^{n\times n}$, and so*

$$F(S) \le F(R), \ \textit{if}\ S \ge R, \ S, R \in S^{n\times n}. \tag{5}$$

Remark 4. *This very definition of ellipticity, coincides with the classical ones. In fact, we say PDE*

$$Tr\,[A \cdot Du] = f$$

is **elliptic** *if A is a non-positive definite symmetric matrix. One then readily verifies*

$$\begin{aligned} S \ge R \quad &\Rightarrow \quad S - R \text{ non-negative definite} \\ &\Rightarrow \quad Tr\,[A \cdot (S - R)] \le 0 \\ &\Rightarrow \quad Tr[A \cdot S] \le Tr[A \cdot R], \ S, R \in S^{n\times n}. \end{aligned}$$

Now, suppose $f_k \to f$ uniformly, and consider the approximating problems

$$\begin{cases} F(Du_k) = f_k, & \text{in } U, \\ u_k = 0, & \text{on } \partial U. \end{cases} \tag{6}$$

Assume (6) has a smooth solution u_k, a priori bounded in $W^{2,\infty}(U)$. Then, up to a subsequence,

$$u_k \to u \text{ uniformly},\ D^2 u_k \overset{*}{\rightharpoonup} D^2 u \text{ in } L^\infty(U; S^{n\times n}),$$

for some u.

Our **problem** is then: does u satisfies (4)?

If F is uniformly elliptic and convex, then strong estimates are available and passing to limit is simple, see [3]. The main interest is consequently for the nonconvex F, as in hut 1.

Recall that in hut 1, the main assumption leading to the existence of a weak solution is the monotonicity inequality (2). We shall then furnish a similar monotonicity in this current circumstance, replacing the ellipticity of F.

For this purpose, we need

Proposition 5. *Let* $(X, \|\cdot\|)$ *be a Banach space. Then the limit*

$$[f, g] = \lim_{\lambda\to 0_+} \frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} \tag{7}$$

exists for all $f, g \in X$.

Proof. Writing

$$\frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} = \frac{\|g+\lambda f\| + \|g\|}{2} \cdot \frac{\|g+\lambda f\| - \|g\|}{\lambda},$$

we need only show that $\left\{\frac{\|g+\lambda f\| - \|g\|}{\lambda}\right\}_{\lambda>0}$ is bounded from below and increasing in λ. In fact, we have

1. $\dfrac{\|g+\lambda f\|-\|g\|}{\lambda} \geq \dfrac{-\lambda\|f\|}{\lambda} = -\|f\|;$
2. for $0<\lambda<\tilde{\lambda}$,

$$
\begin{aligned}
& \frac{\|g+\lambda f\|-\|g\|}{\lambda} - \frac{\left\|g+\tilde{\lambda} f\right\|-\|g\|}{\tilde{\lambda}} \\
= \ & \frac{\left\|\tilde{\lambda} g+\lambda\tilde{\lambda} f\right\| - \tilde{\lambda}\|g\| - \left\|\lambda g+\lambda\tilde{\lambda} f\right\| + \lambda\|g\|}{\lambda\tilde{\lambda}} \\
\leq \ & \frac{\left\|(\tilde{\lambda}-\lambda) g\right\| - (\tilde{\lambda}-\lambda)\|g\|}{\lambda\tilde{\lambda}} = 0.
\end{aligned}
$$

□

Remark 6. *In case X is a Hilbert space, $[f,g]$ is simply the inner product.*

We now give an useful property of $[\cdot,\cdot]$ as

Proposition 7. *The map $X\times X \ni \{f,g\} \mapsto [f,g]$ is upper semicontinous, that is,*

$$\limsup_{n\to\infty}\,[f_n,g_n] \leq [f,g]\,, \tag{8}$$

for all $f,g\in X$, $f_n\to f$, $g_n\to g$ in X.

Proof. Observe that in the proof of (7), we have $\left\{\dfrac{\|g+\lambda f\|-\|f\|}{\lambda}\right\}_{\lambda>0}$ is increasing in λ, for $f,g\in X$ fixed.

Thus

$$
\begin{aligned}
\limsup_{n\to\infty}\,[f_n,g_n] &= \limsup_{n\to\infty}\lim_{\lambda\to 0_+}\frac{\|g_n+\lambda f_n\|^2-\|g_n\|^2}{2\lambda} \\
&= \limsup_{n\to\infty}\left\{\lim_{\lambda\to 0_+}\left[\frac{\|g_n+\lambda f_n\|+\|g_n\|}{2}\cdot\frac{\|g_n+\lambda f_n\|-\|g_n\|}{\lambda}\right]\right\} \\
&= \limsup_{n\to\infty}\left[\|g_n\|\cdot\lim_{\lambda\to 0_+}\frac{\|g_n+\lambda f_n\|-\|g_n\|}{\lambda}\right] \\
&\leq \|g\|\cdot\limsup_{n\to\infty}\frac{\|g_n+\lambda f_n\|-\|g_n\|}{\lambda}
\end{aligned}
$$

$$\leq \quad \|g\| \cdot \frac{\|g+\lambda f\| - \|g\|}{\lambda}, \ \forall \lambda > 0.$$

Taking $\lambda \to 0_+$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} [f_n, g_n] &= \|g\| \cdot \lim_{\lambda\to 0_+} \frac{\|g+\lambda f\| - \|g\|}{\lambda} \\ &= \lim_{\lambda\to 0_+} \frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} \\ &= [f, g]. \end{aligned}$$

□

Then an explicit formula in case $X = C(\bar{U})$ as

Proposition 8. *Let $X = C(\bar{U})$, then*

$$[f, g] = \max\left\{ f(x_0)g(x_0);\ x_0 \in \bar{U}, |g(x_0)| = \|g\|_{C(\bar{U})} \right\},\ f, g \in C(\bar{U}). \tag{9}$$

Proof. Denote by

$$M_h = \left\{ x \in \bar{U};\ |h(x)| = \|h\| \right\},\ h \in C(\bar{U}).$$

Then

1. due to

$$\frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} \geq \frac{(g(x_0)+\lambda f(x_0))^2 - g(x_0)^2}{2\lambda} = g(x_0)f(x_0),\ \forall\ x_0 \in M_g,$$

we have

$$[f, g] \geq RHS \text{ of } (9).$$

2. for any sequence $\{\lambda_n\} \searrow 0$, $x_n \in M_{g+\lambda_n f}$,

$$\begin{aligned} \frac{\|g+\lambda_n f\|^2 - \|g\|^2}{2\lambda_n} &\leq \frac{(g(x_n)+\lambda_n f(x_n))^2 - g(x_n)^2}{2\lambda_n} \\ &= f(x_n)g(x_n) + \frac{\lambda_n}{2} f(x_n)^2 \\ &\to f(x_\infty)g(x_\infty),\ \text{as } n \to \infty, \end{aligned} \tag{10}$$

for some $\bar{U} \ni x_\infty \leftarrow x_n$.

Meanwhile, taking $n \to \infty$ in

$$|g(x_n) + \lambda_n f(x_n)| = \|g + \lambda_n f\|\,,$$

gives

$$|g(x_\infty)| = \|g\|\,.$$

This together with (10) shows that

$$[f, g] \leq RHS \text{ of } (9).$$

The proof is then completed. □

With this explicit formula for $[f, g]$, we show that monotonicity is a consequence of ellipticity as

Proposition 9. *If F is convex, then the operator $A[u] \equiv F(D^2 u)$ satisfies*

$$0 \leq [A[u] - A[v], u - v]\,,\ \forall\, u, v \in C_0^2(\bar{U}). \tag{11}$$

Here $C_0^2(\bar{U})$ is the subspace of $C^2(\bar{U})$, with vanishing boundary data.

Proof. Suppose $(u - v)\,(x_0) = \|u - v\|_{C(\bar{U})}$, $x_0 \in U$, then

$$\begin{aligned} & D^2(u - v)(x_0) \leq 0 \\ \Rightarrow\quad & F(D^2 u)(x_0) \geq F(D^2 v)(x_0)\ \big(\text{by ellipticity}\big) \\ \Rightarrow\quad & [A[u] - A[v], u - v] = \big(F(D^2 u) - F(D^2 v)\big)\,(x_0) \cdot (u - v)\,(x_0) \geq 0, \end{aligned}$$

by invoking (9).

The case $(v - u)\,(x_0) = \|u - v\|_{C(\bar{U})}$, $x_0 \in U$ is similarly treated. □

With all the above preparations above, we now state and prove our main result in this hut.

Theorem 10. *Consider problem (4) and its approximating problems (6). If $A[u] \equiv F(D^2u)$ satisfies the monotonicity inequality:*

$$0 \le [A[u] - A[v], u - v]\,,\ \forall\, u, v \in C_0^2(\bar{U}). \tag{12}$$

Then u solves (4) a.e..

Proof. 1. For the approximating solution $\{u_k\}$, we have

$$\begin{aligned} 0 &\le [A[u_k] - A[v], u_k - v] \\ &\le [f_k - A[v], u_k - v],\ \forall\, v \in C_0^2(\bar{U}). \end{aligned}$$

Taking $k \to \infty$ upon a subsequence, we obtain by invoking (8) that

$$0 \le [f - A[v], u - v]\,,\ \forall\, v \in C_0^2(\bar{U}). \tag{13}$$

2. Our strategy to prove u solves (4) is then to choose appropriate v in (13).

In fact, since $u \in W^{2,\infty}(U)$, Rademacher's theorem (see [2, 5]) implies then u is C^2 a.e.. Fix any $x_0 \in U$ where $D^2u(x_0)$ exists. We **handcraft** a C^2 function v having the form

$$v(x)\begin{cases} \begin{array}{c} = u(x_0) + Du(x_0)(x - x_0) \\ +\frac{1}{2}D^2u(x_0)(x - x_0, x - x_0) + \varepsilon\,|x - x_0|^2 - 1 \end{array}, & x \text{ near } x_0; \\ = 0, & x \in \partial U; \\ \in \left(u(x) - \frac{1}{2}, u(x) + \frac{1}{2}\right), & \text{otherwise}. \end{cases} \tag{14}$$

(The $\varepsilon > 0$ is chosen so that $u - v$ looks like a parabola for x near x_0.) Then $|u - v|$ attains its maximum over $\bar{U}$ only at x_0. But then (13) and (9) say $(f - A[u])\,(x_0) \ge 0$, that is,

$$f(x_0) \ge F\left(D^2u(x_0) + 2\varepsilon I\right).$$

Sending $\varepsilon \to 0_+$, we find

$$f(x_0) \geq F(D^2 u(x_0)).$$

The opposite inequality follows by replacing $\varepsilon |x - x_0|^2 - 1$ by $-\varepsilon |x - x_0|^2 + 1$ in (13). Consequently, we have

$$F(D^2 u(x_0)) = f(x_0), \ a.e. x_0 \in U.$$

□

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