华南理工大学2004数分

To my parents

1 求极限 $\lim\limits_{x\to 0}\dfrac{x^2e^x+2\cos x-2}{\tan x-\sin x}$.

解答. 由

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{1}{\cos x}\cdot\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{sinx}{2x}=\frac{1}{2},$$

知

$$\begin{aligned}\text{原极限}&=2\lim_{x\to 0}\frac{x^2e^x+2\cos x-2}{x^3}\\&=2\lim_{x\to 0}\frac{x^2\left[1+x+o(x)\right]+2\left[1-x^2/2+o(x^3)\right]-2}{x^3}\\&=2.\end{aligned}$$

2 设 $\dfrac{1}{2}\ln\left(x^2+y^2\right)=\arctan\dfrac{y}{x}$. 求 $\dfrac{d^2y}{dx^2}$.

解答. 对

$$\frac{1}{2}\ln\left(x^2+y^2\right)=\arctan\frac{y}{x}$$

两边求导, 有

$$\frac{x+yy'}{x^2+y^2}=\frac{1}{2}\cdot\frac{2x+2yy'}{x^2+y^2}=\frac{1}{1+(y/x)^2}\cdot\frac{y'x-y}{x^2}=\frac{y'x-y}{x^2+y^2},$$

即

$$x+yy'=y'x-y, \tag{1}$$

$$y'=\frac{x+y}{x-y}.$$

再对 (1) 两边求导, 得

$$1+\left[(y')^2+yy''\right]=(y''x+y')-y',$$

$$1+(y')^2 = y''(x-y),$$

$$y'' = \frac{1+(y')^2}{x-y} = \frac{1+[(x+y)/(x-y)]^2}{x-y} = \frac{(x-y)^2+(x+y)^2}{(x-y)^3} = 2\frac{x^2+y^2}{(x-y)^3}.$$

3 设 $x_1 > \sqrt{a} > 1$, $x_{n+1} = \dfrac{a+x_n}{1+x_n}$, $n = 1, 2, \cdots$. 试证: $\{x_n\}$ 收敛, 并求 $\lim\limits_{n\to\infty} x_n$.

证明. 注意到

$$x_{n+1} = \frac{a+x_n}{1+x_n} = 1 + \frac{a-1}{1+x_n} > 1,$$

我们有

- 当 $x_n > \sqrt{a}$ 时, $x_{n+1} < 1 + \dfrac{a-1}{1+\sqrt{a}} = 1 + (\sqrt{a}-1) = \sqrt{a}$;
- 当 $x_n < \sqrt{a}$ 时, $x_{n+1} > 1 + \dfrac{a-1}{1+\sqrt{a}} = 1 + (\sqrt{a}-1) = \sqrt{a}$.

于是由 $x_1 > \sqrt{a} > 1$ 知

$$x_{2n-1} > \sqrt{a},\ 1 < x_{2n} < \sqrt{a},\ n = 1, 2, \cdots.$$

往证 $\{x_{2n-1}\}$ 递减, $\{x_{2n}\}$ 递增. 实际上,

$$\begin{aligned} x_{k+2} - x_k &= \frac{a+x_{k+1}}{1+x_{k+1}} - \frac{x_{k+1}-a}{1-x_{k+1}} \\ &= \frac{2\,(x_{k+1}-\sqrt{a})\,(x_{k+1}+\sqrt{a})}{(x_{k+1}+1)\,(x_{k+1}-1)} \\ &\begin{cases} <0, & k \text{ 是奇数}, \\ >0, & k \text{ 是偶数}. \end{cases} \end{aligned}$$

从而由单调有界原理, 存在 $b, c \in [1, x_1]$, 使得

$$\lim_{n\to\infty} x_{2n-1} = b,\ \lim_{n\to\infty} x_{2n} = c.$$

现于

$$x_{2k+1} = \frac{a + x_{2n}}{1 + x_{2n}}, \ \ x_{2k} = \frac{a + x_{2n-1}}{1 + x_{2n-1}},$$

中令 $n \to \infty$, 有

$$b = \frac{a + c}{1 + c}, \ \ c = \frac{a + b}{1 + b}.$$

解得

$$b = c = \sqrt{a},$$

$$\lim_{n \to \infty} x_n = \sqrt{a}.$$

4 设 C 为单位圆周, 逆时针方向为正向, 求 $\displaystyle\oint_C \frac{(y + 9x)dx + (y - x)dy}{9x^2 + y^2}$.

解答. 由

$$\frac{\partial}{\partial y}\left(\frac{y + 9x}{9x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{y - x}{9x^2 + y^2}\right),$$

知

$$\begin{aligned}
& \oint_C \frac{(y + 9x)dx + (y - x)dy}{9x^2 + y^2} \\
= & \int_{L^+} \frac{(y + 9x)dx + (y - x)dy}{9x^2 + y^2} \ \ (L: \ 9x^2 + y^2 = \varepsilon^2, \ \varepsilon \text{ 充分小}) \\
= & \int_0^{2\pi} \frac{(\varepsilon \sin\theta + 3\varepsilon\cos\theta) \cdot (-\varepsilon \sin\theta/3) + (\varepsilon\sin\theta - \varepsilon\cos\theta/3) \cdot \varepsilon\cos\theta}{\varepsilon^2} d\theta \\
& (x = \varepsilon\cos\theta/3, \ \ y = \varepsilon\sin\theta) \\
= & \int_0^{2\pi} \left[-\frac{1}{3}\sin^2\theta - \cos\theta\sin\theta + \sin\theta\cos\theta - \frac{1}{3}\cos^2\theta\right] d\theta \\
= & -\frac{2\pi}{3}.
\end{aligned}$$

5 求 $\sum_{n=1}^{\infty}\frac{n+2}{n(n+1)}x^n$ 的收敛区间, 并求级数的和.

解答. 因为 $\lim_{n\to\infty}\sqrt[n]{\frac{n+2}{n(n+1)}}=1$, 而原级数的收敛半径为 1. 又

- 由 *Leibniz* 判别法, $\sum_{n=1}^{\infty}(-1)^n\frac{n+2}{n(n+1)}$ 收敛;
- 由 比较判别法, $\sum_{n=1}^{\infty}\frac{n+2}{n(n+1)}$ 发散;

故原级数的收敛区间为 $[-1,1)$. 往求原级数的和. 实际上, $\forall\ x\in[-1,0)\cup(0,1)$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+2}{n(n+1)}x^n &= \sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{1}{n+1}\right)x^n\\ &= 2\sum_{n=1}^{\infty}\frac{x^n}{n}-\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}\\ &= 2\sum_{n=1}^{\infty}\frac{x^n}{n}-\frac{1}{x}\left[\sum_{n=1}^{\infty}\frac{x^n}{n}-x\right]\\ &= \frac{1-2x}{x}\ln(1-x)+1\end{aligned}$$

$$\left(\sum_{n=1}^{\infty}\frac{x^n}{n}=\sum_{n=1}^{\infty}\int_0^x t^{n-1}dt=\int_0^x\sum_{n=0}^{\infty}t^{n-1}dt=\int_0^x\frac{1}{1-t}dt=-\ln(1-x)\right).$$

于是

$$\sum_{n=1}^{\infty}\frac{n+2}{n(n+1)}x^n=\begin{cases}\frac{1-2x}{x}\ln(1-x)+1, & x\in[-1,0)\cup(0,1),\\ 0, & x=0.\end{cases}$$

6 设 S 为单位球面的上半部分, 外侧为正向, 计算 $\iint_S x^2dydz+y^2dzdx+z^2dxdy$.

解答.

$$
\begin{aligned}
\text{原第二型曲面积分} &= \iiint_{x^2+y^2+z^2\le 1,\ z\ge 0} 2\,(x+y+z)\,dxdydz \ \ (Stokes\text{公式}) \\
&= 2\iiint_{x^2+y^2+z^2\le 1,\ z\ge 0} zdxdydz \ \ (\text{对称性}) \\
&= 2\int_0^1 z\cdot\pi\left(1-z^2\right)dz \ \ (\text{化重积分}) \\
&= \frac{\pi}{2}.
\end{aligned}
$$

7 令 $f(x,y)=\begin{cases} 0, & (x,y)=(0,0), \\ \frac{x^3}{x^2+y^2}, & (x,y)\ne(0,0), \end{cases}$ $\boldsymbol{\nu}$ 是 (x,y) 平面上的任一单位法向量.

7.1 求 f 在 $(0,0)$ 沿 $\boldsymbol{\nu}$ 的方向导数.

7.2 试讨论 f 在 $(0,0)$ 处的连续性与可微性.

解答. 7.1

$$
\begin{aligned}
\frac{\partial f}{\partial \boldsymbol{\nu}}(0,0) &= \lim_{l\to 0}\frac{f(l\boldsymbol{\nu})-f(0,0)}{l} \\
&= \lim_{l\to 0}\frac{(l\cos\theta)^3/l^2}{l} \ \ (\boldsymbol{\nu}=(\cos\theta,\sin\theta)) \\
&= \cos^3\theta=[\boldsymbol{\nu}\cdot\boldsymbol{e}_1]^3.
\end{aligned}
$$

7.2 由 $\left|\dfrac{x^3}{x^2+y^2}\right|\le|x|$ 知 f 在 $(0,0)$ 处连续. 而由

$$
f_x(0,0)=\frac{\partial f}{\partial \boldsymbol{e}_1}(0,0)=1,\ \ f_y(0,0)=\frac{\partial f}{\partial \boldsymbol{e}_2}(0,0)=0,
$$

及

$$
\frac{|f(x,kx)-f(0,0)-(f_x(0,0)x+f_y(0,0)kx)|}{(x^2+(kx)^2)^{1/2}}=\frac{k^2}{(1+k^2)^{3/2}},\ \forall\ k\in\mathbb{R},
$$

知 f 在 $(0,0)$ 处不可微.

8 设 $f(x)$ 连续, $y(x)=\int_0^x f(x-t)\sin t dt$. 试证: $y(x)$ 满足

$$\begin{cases} y''+y=f(x), \\ y(0)=y'(0)=0. \end{cases}$$

证明. 由

$$y(x)=\int_0^x f(x-t)\sin t dt=\int_0^x f(t)\sin(x-t)dt,$$

知

$$y'(x)=\int_0^x f(t)\cos(x-t)dt,\ y''(x)=-\int_0^x f(t)\sin(x-t)dt.$$

而 $y''+y=f$, 且 $y(0)=y'(0)=0$.

9 设函数 f 在 $[-1,1]$ 上三次可微, $f(-1)=f(0)=f'(0)=0$, $f(1)=1$. 试证: $\exists\,\xi\in(-1,1)$, $s.t.$ $f^{(3)}(\xi)\geq 3$.

证明. 当 $f''(0)\leq 1$ 时, 由 *Taylor* 展式, $\exists\,\xi_1\in(0,1)$, $s.t.$

$$1=f(1)=f(0)+f'(0)+\frac{f''(0)}{2}+\frac{f'''(\xi_1)}{6},$$

而

$$f'''(\xi_1)=6\left[1-\frac{f''(0)}{2}\right]\geq 3;$$

而当 $f''(0)>1$ 时, 亦由 *Taylor* 展式, $\exists\,\xi_2\in(-1,0)$, $s.t.$

$$0=f(-1)=f(0)-f'(0)+\frac{f''(0)}{2}-\frac{f'''(\xi_2)}{6},$$

而

$$f'''(\xi_2)=3f''(0)>3.$$

10 试讨论无穷级数 $f(x)=\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ 在 $(0,\infty)$ 上的一致收敛性, 以及 f 在 $(0,\infty)$ 上的有界性.

解答. 一方面, $f(x)=\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ 在 $(0,\infty)$ 不一致收敛. 这是因为

$$\exists\ \varepsilon_0=\frac{1}{2},\ \forall\ N,\ \exists\ x_N=\frac{1}{N^2},\ s.t.\ \sum_{n=N}^{\infty}\frac{1}{1+n^2x_N}\geq\frac{1}{2}=\varepsilon_0;$$

另一方面, f 在 $(0,\infty)$ 无界. 事实上,

$$f\left(\frac{1}{N^2}\right)\geq\sum_{n=N+1}^{2N}\frac{1}{1+(n/N)^2}>\sum_{n=N+1}^{2N}\frac{1}{5}=\frac{N}{5}\to\infty,\ \text{当}\ N\to\infty.$$

11 设 $f\geq 0$ 在 $(-\infty,\infty)$ 上连续, $\int_{-\infty}^{\infty}f(x)dx=1$, $f_\varepsilon(x)=\frac{1}{\varepsilon}f\left(\frac{x}{\varepsilon}\right)$. 试证明: 对每一个 有界连续函数 φ, 均有

$$\lim_{\varepsilon\to 0_+}\int_{-\infty}^{\infty}\varphi(x)f_\varepsilon(x)dx=\varphi(0).$$

证明. 直接写下:

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\varphi(x)f_\varepsilon(x)dx-\varphi(0)\right|\\
=&\left|\int_{-\infty}^{\infty}\varphi(\varepsilon y)f(y)dy-\varphi(0)\right|\quad(x=\varepsilon y)\\
=&\left|\int_{-\infty}^{\infty}\left[\varphi(\varepsilon y)-\varphi(0)\right]f(y)dy\right|\quad\left(\int_{-\infty}^{\infty}f(x)dx=1\right)\\
=&\left[\int_{|y|\leq\frac{1}{\varepsilon^{1/2}}}+\int_{|y|>\frac{1}{\varepsilon^{1/2}}}\right]\left[\varphi(\varepsilon y)-\varphi(0)\right]f(y)dy\\
\leq&\max_{|y|\leq\frac{1}{\varepsilon^{1/2}}}|\varphi(\varepsilon y)-\varphi(0)|+2\sup_{x\in\mathbb{R}}|\varphi(x)|\cdot\int_{|y|>\frac{1}{\varepsilon^{1/2}}}f(y)dy\\
=&\max_{|x|\leq\varepsilon^{1/2}}|\varphi(x)-\varphi(0)|+2\sup_{x\in\mathbb{R}}|\varphi(x)|\cdot\int_{|y|>\frac{1}{\varepsilon^{1/2}}}f(y)dy
\end{aligned}$$

$$\to \quad 0, \text{ 当 } \varepsilon \to 0_+$$
$$\left(\lim_{x\to 0} \varphi(x) = \varphi(0), \ \int_{-\infty}^{\infty} f(y)dy = 1 \right).$$

故有

$$\lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} \varphi(x) f_\varepsilon(x)dx = \varphi(0).$$

12 试证明:

$$\int_0^1 \ln \frac{1+x}{1-x}\frac{dx}{x} = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{4}.$$

证明. 直接写下

$$\begin{aligned}
& \int_0^1 \ln \frac{1+x}{1-x}\frac{dx}{x} = \int_0^1 [\ln(1+x) - \ln(1-x)] \frac{dx}{x} \\
= & \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right] \frac{dx}{x} \\
= & \int_0^1 \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}\frac{dx}{x} = 2\int_0^1 \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1} dx \\
= & 2\lim_{t\to 1_-} \int_0^t \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1} dx = 2\lim_{t\to 1_-} \sum_{n=0}^{\infty} \int_0^t \frac{x^{2n}}{2n+1} dx \\
= & 2\lim_{t\to 1_-} \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)^2} = 2\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
= & 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)}^2 = 2\left[\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \right] \\
= & 2\left[1 - \frac{1}{4}\right] \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{2}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{4}.
\end{aligned}$$

13 设 f, g, h 为 $[0,\infty)$ 上连续非负函数, 满足

$$g(t) \le f(t) + \int_0^t g(s)h(s)ds, \ t > 0;$$

$$f'(t) \geq 0, \int_0^{\infty} h(t)dt = A < \infty.$$

试证明: $g(t) \leq f(t)\left(1 + Ae^A\right)$.

证明. 由 $g(t) \leq f(t) + \int_0^t g(s)h(s)ds,\ t > 0;$ 知

$$g(t)h(t) \leq f(t)h(t) + \int_0^t g(s)h(s)ds \cdot h(t).$$

令 $F(t) = \int_0^t g(s)h(s)ds$, 则

$$F'(t) \leq f(t)h(t) + F(t)h(t),$$

$$\frac{d}{dt}\left[e^{-\int_0^t h(s)ds}F(t)\right] \leq e^{-\int_0^t h(s)ds}f(t)h(t),$$

而

$$e^{-\int_0^t h(s)ds}F(t) \leq \int_0^t e^{-\int_0^s h(\tau)d\tau}f(s)h(s)ds,$$

$$F(t) \leq f(t)\int_0^t e^{\int_s^t h(\tau)d\tau}h(s)ds \ \ (f' \geq 0)\,.$$

于是

$$g(t) \leq f(t) + F(t) \leq f(t)\left[1 + Ae^A\right].$$