RIEMANNIAN GEOMETRY

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To Professor Zhu
For better understanding on Lobatchevski Geometry...

Problem Set

0 Differentiable Manifolds

0.1 (Product Manifold). Let M and N be differentiable manifolds and let $\{(U_\alpha, x_\beta)\}, \{(V_\beta, y_\beta)\}$ differentiable structures on M and N, respectively. Consider the cartesian product $M \times N$ and the mapping

$$z_{\alpha\beta}(p,q) = (x_\alpha(p), y_\beta(q)),\ p \in U_\alpha,\ q \in V_\beta$$

a) Prove that $(U_\alpha \times V_\beta, z_{\alpha\beta})$ is a differentiable structure on $M \times N$ in which the projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are differentiable. With this differentiable structure $M \times N$ is called the product manifold of M with N.

b) Show that the product manifold $S^1 \times \cdots \times S^1$ of n circles S^1, where $S^1 \subset \mathbb{R}^2$ has the usual differentiable structure, is diffeomorphic to the $n-$torus T^n of example 4.9 a).

Proof. a) Clearly,

$$\begin{aligned} z_{\alpha\beta} : U_\alpha \times V_\beta &\to x_\alpha(U_\alpha) \times y_\beta(V_\beta) \subset M \times N \\ (p,q) &\mapsto (x_\alpha(p), y_\beta(q)) \end{aligned}$$

is injective. Moreover,

$$\bigcup_{\alpha,\beta} z_{\alpha\beta}(U_\alpha \times V_\beta) = \bigcup_\alpha x_\alpha(U_\alpha) \times \bigcup_\beta y_\beta(V_\beta) = M \times N$$

and if

$$z_{\alpha\beta}(U_\alpha \times V_\beta) \cap z_{\gamma\delta}(U_\gamma \times V_\delta) = W \neq \varnothing$$

then

$$z_{\gamma\delta}^{-1} \circ z_{\alpha\beta}(p,q) = z_{\gamma\delta}^{-1}(x_\alpha(p), y_\beta(q)) = (x_\gamma^{-1} \circ x_\alpha(p), y_\delta^{-1} \circ y_\beta(q))$$

is differentiable. Thus, by definition, with this differentiable structure, $M \times N$ is a differentiable manifold.

b) Recall $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let

$$\begin{aligned} F : S^1 \times \cdots \times S^1 &\to \mathbb{T}^n \\ (e^{i\alpha_j})_{j=1}^n &\mapsto \left(\frac{\alpha_j}{2\pi} + n_j\right)_{j=1}^n \end{aligned}$$

where $\alpha_j \in [0, 2\pi), n_j \in \mathbb{Z}$
We have

- F is injective, since

$$\frac{\alpha_j}{2\pi} + n_j = \frac{\beta_j}{2\pi} + m_j \Rightarrow \alpha_j - \beta_j = 2\pi(m_j - n_j) \Rightarrow e^{i\alpha_j} = e^{i\beta_j}$$

- F is surjective, just note that

$$\alpha_j \in [0, 2\pi) \Rightarrow \frac{\alpha_j}{2\pi} \in [0, 1)$$

- F and F^{-1} are differentiable, this is proved by a list of graphs. Indeed, one "$y^{-1} \circ F \circ x$" is of the form

$$f(t) = \frac{\arctan t}{\pi} - \frac{1}{4}$$

□

0.9 Let $G \times M \to M$ be a properly discontinuous action of a group G on a differentiable manifold M.

a) Prove that the manifold M/G (Example 4.8) is oriented if and only if there exists an orientation of M that is preserved by all the diffeomorphisms of G.
b) Use a) to show that the projective plane $P^2(\mathbb{R})$, the Klein bottle and the Mobius band are non-orientable.
c) Prove that $P^2(\mathbb{R})$ is orientable if and only if n is odd.

Proof. a) if part: Let (U_α, x_α) be an orientation of M that is preserved by all the diffeomorphisms of G, i.e.

$$W = U_\beta \cap g(U_\alpha) \neq \varnothing \Rightarrow \det(x_\beta^{-1} \circ g \circ x_\alpha) > 0$$

We claim that $(\pi(U_\alpha), \pi \circ x_\alpha)$ is an orientation of M/G. Indeed,

$$\pi(U_\alpha) \cap \pi(U_\beta) \neq \varnothing \Rightarrow \det((\pi \circ x_\beta)^{-1} \circ (\pi \circ x_\alpha)) = \det(x_\beta^{-1} \circ g \circ x_\alpha) > 0$$

for some $g \in G$.

Only if part: We know the atlas of M/G is induced from M, hence the conclusion follows from the reverse of the "if part".

b) Let $G = \{Id, A\}$ where A is the antipodal map. Recall that

$$\begin{aligned} \text{Projective } 2-\text{space } P^2(\mathbb{R}) &= S^2/G, \text{ where } S^2 = 2-\text{dim sphere} \\ \text{Klein bottle } K &= \mathbb{T}^2/G, \text{ where } \mathbb{T}^2 = 2-\text{dim torus} \\ \text{Mobius band } M &= C/G, \text{ where } C = 2-\text{dim cylinder} \end{aligned}$$

Clearly, $S^2, \mathbb{T}^2, C$ are orientable 2−dim manifols, but A reverse the orientation of $\mathbb{R}^3$, hence $S^2, \mathbb{T}^2, C$. The conclusion follows from a).

c) We've the following equivalence:

$$\begin{aligned} P^n(\mathbb{R})\text{is orientable} &\Leftrightarrow A \text{ preserves the orientation of } S^n (\text{by } a)) \\ &\Leftrightarrow A \text{ preserves the orientation of } \mathbb{R}^{n+1} \\ &\quad\ (\text{The orientation is induced from } \mathbb{R}^{n+1}) \\ &\Leftrightarrow (n+1) \text{ is even} \\ &\Leftrightarrow n \text{ is odd} \end{aligned}$$

□

1 **Riemannian Metrics**

1.1 Prove that the antipodal mapping $A : S^n \to S^n$ given by $A(p) = -p$ is an isometry of S^n. Use this fact to introduce a Riemannian metric on the real projective space $P^n(\mathbb{R})$ such that the natural projection $\pi : S^n \to P^n(\mathbb{R})$ is a local isometry.

Proof. a) A is an isometry of S^n.

We first claim that $T_pS^n = T_{A(p)}S^n$.

It is enough to prove $T_pS^n \subset T_{A(p)}S^n$, since

$$T_{A(p)}S^n \subset\subset T_{A\circ A(p)}S^n = T_pS^n$$

Indeed, for any $v \in T_pS^n, \exists c : (-\varepsilon, \varepsilon) \to S^n$ such that $c(0) = p, c'(0) = v$. Thus $A \circ c : (-\varepsilon, \varepsilon) \to S^n$ is a curve with $A \circ c(0) = A(p), (A \circ c)'(0) = dA_p(c'(0)) = -c'(0) = -v$. Hence $-v \in T_{A(p)}S^n$ and $v \in T_{A(p)}S^n$ since $T_{A(p)}S^n$ is a linear space.

Now the fact A is an isometry of S^n is clear.

$$< dA_p(v), dA_p(w) >_{A(p)} = < -v, -w >_{-p} = < v, w >_{-p} = < v, w >_p$$

b) Construction of a metric on $P^n(\mathbb{R})$ such that π is a local isometry.

For any $p \in S^n, \pi(p) \in P^n(\mathbb{R})$, define

$$< (d\pi)_p(v), (d\pi)_p(w) >_{\pi(p)} \triangleq < v, w >_p$$

Indeed,

- Because of surjectivity of π and the construction of atlas on $P^n(\mathbb{R})$, the vector "on" $P^n(\mathbb{R})$ is of the form $(d\pi)_p(v), p \in S^n, v \in T_p(S^n)$.
- It is well-defined. Indeed,$(d\pi)_p$ is surjective, thus injective, hence the one-to-one correspondence between $(d\pi)_p(v)$ and v. And if $\pi(p) = \pi(q)$, then $q = p$ or $q = A(p)$. In the latter case,

$$(d\pi)_p(v) = (d(\pi \circ A))_p(v) = (d\pi)_{A(p)} \circ (dA)_p(v) = (d\pi)_{A(p)}(-v)$$
$$(d\pi)_p(w) = (d\pi)_{A(p)}(-w)$$
$$< -v, -w >_{A(p)} = < v, w >_p$$

- Since the action of G on M is properly continuous, by definition, π is a local isometry.

$\square$

1.4 A function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = yt + x,\ t, x, y \in \mathbb{R},\ y > 0$, is called a proper affine function. The subset of all such function with respect to the usual composition law forms a Lie group G. As a differentiable manifold G is simply the upper half-plane $\{(x, y) \in \mathbb{R}^2;\ y > 0\}$ with the differentiable structure induced from $\mathbb{R}^2$. Prove that:

a) The left-invariant Riemannian metric on G which at the neutral element $e = (0, 1)$ coincides with Euclidean metric($g_{11} = 1 = g_{22}, g_{12} = 0 = g_{21}$) is given by $g_{11} = \dfrac{1}{y^2} = g_{22}, g_{12} = 0$, (this is the metric of the non-euclidean geometry of Lobatchevski).

b) Putting $(x, y) = z = x + i\ y, i = \sqrt{-1}$, the transformation

$$z \mapsto z' = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1$$

is an isometry of G.

Proof. a) • For any $g = (x, y) \in G$, $g^{-1} = \left(-\dfrac{x}{y}, \dfrac{1}{y}\right)$.

Indeed,

$$y\left(\frac{1}{y}t - \frac{x}{y}\right) + x = t = \frac{1}{y}(yt + x) - \frac{x}{y}, \quad \forall t \in \mathbb{R}$$

- Denote by

$$\partial_1 = \frac{\partial}{\partial x},\ \partial_2 = \frac{\partial}{\partial y}$$

then

$$dL_{g^{-1}}(\partial_1) = \left(\frac{1}{y}, 0\right),\ dL_{g^{-1}}(\partial_2) = \left(0, \frac{1}{y}\right)$$

Since

$$\gamma(s) = (x + s, y), s \in \mathbb{R}$$

is a curve in G with $\gamma(0) = g, \gamma'(0) = \partial_1$, we get

$$\begin{aligned} dL_{g^{-1}}(\partial_1) &= \frac{d}{ds}|_{s=0}\left[\frac{1}{y}(yt + x + s) - \frac{x}{y}\right] \\ &= \frac{d}{ds}|_{s=0}\left(\frac{s}{y}, 1\right) \\ &= \left(\frac{1}{y}, 0\right) \end{aligned}$$

And $dL_{g^{-1}}(\partial_2) = \left(0, \frac{1}{y}\right)$ follows from the same lines.

- The left-invariant Riemannian metric of G is given by

  $$< v, w >_g \triangleq \langle dL_{g^{-1}}(v), dL_{g^{-1}}(w)\rangle_e$$

  Hence

  $$g_{11} = \left\langle \left(\frac{1}{y}, 0\right), \left(\frac{1}{y}, 0\right)\right\rangle_e = \frac{1}{y^2}$$

  $$g_{22} = \left\langle \left(0, \frac{1}{y}\right), \left(0, \frac{1}{y}\right)\right\rangle_e = \frac{1}{y^2}$$

  $$g_{12} = g_{21} = \left\langle \left(0, \frac{1}{y}\right), \left(\frac{1}{y}, 0\right)\right\rangle_e = 0$$

  as desired.

b) Since

$$\begin{cases} z = x + i\ y \\ \overline{z} = x - i\ y \end{cases}$$

We get

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{-4dzdz'}{(z - \overline{z})^2}$$

Hence for the transform

$$z \mapsto z' = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1$$

we've

$$dz' = \frac{dz}{(cz + d)^2}$$

Thus

$$\frac{-4dz'd\overline{z'}}{(z' - \overline{z'})^2} = \frac{-4dzd\overline{z}}{(z - \overline{z})^2}$$

as desired.

□

1.5 Prove that the isometries of $S^n \subset \mathbb{R}^n$, with the induced metric, are the restrictions of S^n of the linear orthogonal maps of $\mathbb{R}^{n+1}$.

Proof. Denote by $Iso(S^n), Iso(\mathbb{R}^{n+1})$ the isometries of $S^n, \mathbb{R}^{n+1}$ respectively. The orthogonal maps of $\mathbb{R}^{n+1}$ is $O(n+1)$.
Clearly,$O(n+1) \subset Iso(S^n)$ because the metric on S^n is induced from $\mathbb{R}^{n+1}$. While for the converse, let $f \in Iso(S^n)$, define $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$F(x) = \begin{cases} 0, & \text{if } x = 0 \\ f\left(\frac{x}{||x||}\right) ||x||, & \text{if } x \neq 0 \end{cases}$$

then $F \in O(n+1)$ since

$$F(x) \cdot y = f\left(\frac{x}{||x||}\right) ||x|| \cdot y = f\left(\frac{x}{||x||}\right) \frac{y}{||y||} ||x|| ||y|| = \frac{x}{||x||} f\left(\frac{y}{||y||}\right) ||x|| ||y|| = x \cdot F(y)$$

if $0 \neq x, y \in \mathbb{R}^{n+1}$. □

2 Affine Connections; Riemannian Connections

2.2 Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c : I \to M$ be an integral curve of X through p, i.e. $c(t_0) = p$ and $\dfrac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M is

$$(\nabla_X Y)(p) = \frac{d}{dt}|_{t=t_0}(P^{-1}_{c,t_0,t}(Y(c(t)))$$

where $P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ is the parallel transport along c, from t_0 to t (this show how the connection can be reobtained from the concept of parallelism).

Proof. Let $(e_i)_{i=1}^n$ be an orthonormal basis for $T_pM, e_i(t) = P_{c,t_0,t}$, i.e. $\nabla_{c'(t)} e_i(t) = 0$, thus $(e_i(t))_{i=1}^n$ is an orthonormal basis for $T_{c(t)}M$. Indeed,

$$\nabla_{c'(t)} < e_i(t), e_j(t) >=< \nabla_{c'(t)} e_i(t), e_j(t) > + < e_i(t), \nabla_{c'(t)} e_j(t) >= 0$$

$$< e_i(t), e_j(t) >=< e_i, e_j >= \delta_i^j$$

Now, we can write

$$Y(c(t)) = Y^i(t) e_i(t)$$

and the calculation as follows

$$\begin{aligned} \frac{d}{dt}|_{t=t_0}(P^{-1}_{c,t_0,t}(Y(c(t))) &= \frac{d}{dt}|_{t=t_0}(P^{-1}_{c,t_0,t}(Y^i(t)e_i(t))) \\ &= \frac{d}{dt}|_{t=t_0}(Y^i(t)e_i) \\ &= \frac{d}{dt}|_{t=t_0}(Y^i(t))e_i \\ &= (\nabla_{c'(t)}(Y^i(t))e_i(t))|_{t=t_0} \\ &= (\nabla_{c'(t)}(Y^i(t)e_i(t))|_{t=t_0} \\ &= (\nabla_X Y)(p) \end{aligned}$$

□

2.3 Let $f : M^n \to \overline{M}^{n+k}$ be an immersion of a differentiable manifold M into a Riemannian manifold $\overline{M}$. Assume that M has the Riemannian metric induced by f (c.f. Example 2.5 of Chapter 1). Let $p \in M$ and let $U \subset M$ be a neighborhood of p such that $f(U) \subset \overline{M}$ is a submanifold of $\overline{M}$. Further, suppose that X, Y are differentiable vector fileds on $f(U)$ which extend to differentiable vector fields $\overline{X}, \overline{Y}$ on an open set of $\overline{M}$. Define $(\nabla_X Y)(p) =$ tangential component of $\overline{\nabla}_{\overline{X}}\overline{Y}(p)$, where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Prove that ∇ is the Riemannian connection of M.

Proof. Denote by

$$\nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^\top$$

then

- ∇ is compatible with the metric on M. For all $p \in M, f(p) \in f(M)$.

$$\begin{aligned} X < Y, Z > (p) &= \overline{X} < \overline{Y}, \overline{Z} > (p) \\ &= < \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z} > (p) + < \overline{Y}, \overline{\nabla}_{\overline{X}}\overline{Z} > (p) \\ &= < \overline{\nabla}_{\overline{X}}\overline{Y}, Z > (p) + < Y, \overline{\nabla}_{\overline{X}}\overline{Z} > (p) \\ &= < \nabla_X Y, Z > (p) + < Y, \nabla_X Z > (p) \end{aligned}$$

- ∇ is torsion-free. For all $p \in M, f(p) \in f(M)$.

$$(\nabla_X Y - \nabla_Y X)(p) = (\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X})^\top(p) = [\overline{X}, \overline{Y}]^\top(p) = [X, Y](p)$$

For the last equality, we see in local coordinate,

$$\begin{aligned} [\overline{X}, \overline{Y}]^\top(p) &= \left(\sum_{i,j=1}^{n+k} \left\{ \overline{X}^i \frac{\partial \overline{Y}^j}{\partial x^i} - \overline{Y}^i \frac{\partial \overline{X}^j}{\partial x^i} \right\} \frac{\partial}{\partial x^j} \right)^\top (p) \\ &= \left(\sum_{i=1}^{n} \sum_{j=1}^{n+k} \left\{ X^i \frac{\partial \overline{Y}^j}{\partial x^i} - Y^i \frac{\partial \overline{X}^j}{\partial x^i} \right\} \frac{\partial}{\partial x^j} \right)^\top (p) \\ &= \left(\sum_{i,j=1}^{n} \left\{ X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right\} \frac{\partial}{\partial x^j} \right) (p) \\ &= [X, Y](p) \end{aligned}$$

The third equality holds because $\nabla_X Y(p)$ depends only on $X(p)$ and $Y(c(t))$ where $c(t)$ is an integral curve for X through p.

Thus ∇ is the Riemannian connection of M. □

2.8 Consider the upper half-plane

$$\mathbb{R}^2_+ = \left\{ (x, y) \in \mathbb{R}^2;\ y > 0 \right\}$$

with the metric given by $g_{11} = \dfrac{1}{y^2} = g_{22}, g_{12} = 0 = g_{21}$ (metric of Lobatchevski's non-euclidean geometry).

a) Show that the Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0,\ \Gamma_{11}^2 = \frac{1}{y},\ \Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}$$

b) Let $v_0 = (0, 1)$ be a tangent vector at point $(0, 1)$ of $\mathbb{R}_+^2$ (v_0 is a unit vector on the $y-$axis with origin at $(0, 1)$). Let $v(t)$ be the parallel transport of v_0 along the curve $x = t, y = 1$. Show that $v(t)$ makes an angle t with the direction of $y-$axis, measured in the clockwise sense.

Proof. a) We've

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)\\ &= \frac{y^2}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)\\ &= \frac{y^2}{2}\cdot\frac{-2}{y^3}\left(\frac{\partial x^2}{\partial x^j}\delta_{ik} + \frac{\partial x^2}{\partial x^i}\delta_{kj} - \frac{\partial x^2}{\partial x^k}\delta_{ij}\right)\end{aligned}$$

Thus

$$\begin{cases}\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0\\ \Gamma_{11}^2 = \frac{1}{y}\\ \Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}\end{cases}$$

b) Let $v(t) = (a(t), b(t))$ be the parallel field along the curve $x = t, y = 1$ with

$$v(0) = (0, 1), \quad v'(0) = v_0 = (0, 1)$$

Then from the geodesic equations,we've

$$\begin{cases}\frac{da}{dt} + \Gamma_{12}^1 b = 0\\ \frac{db}{dt} + \Gamma_{11}^2 a = 0\end{cases}$$

Taking $a = \cos\theta(t),\ b = \sin\theta(t)$ (since parallel transport preserves inner product, we may just assume this.) then the above equations imply

$$\frac{d\theta}{dt} = -1$$

While we know $v_0 = (0, 1)$, thus

$$\theta_0 = \frac{\pi}{2}$$

Hence

$$\theta = \frac{\pi}{2} - t$$

as desired.

□

3 Geodesics; Convex Neighborhoods

3.7 (Geodesic frame). Let M be a Riemannian manifold of dimension n and let $p \in M$. Show that there exists a neighborhood $U \subset M$ of p and n vector fileds $E_1, \cdots, E_n \in \mathfrak{X}(U)$, orthonormal at each point of U, such that, at p, $\nabla_{E_i} E_j(p) = 0$.
Such a family $E_i, i = 1, \cdots, n$, of vector fields is called a (local) geodesic frame at p.

Proof. Let $U = \exp_p(B_\epsilon(0))$ be a normal neighborhood of p small enough, $(e_i)_{i=1}^n$ be an orthonormal basis of T_pM. For any $q \in U$, let γ be the radial geodesic joining p to q. Using parallel transport, we get

$$E_i \in \mathfrak{X}(U), i = 1, \cdots, n$$

defined by

$$E_i(q) = P_{\gamma,p,q}(e_i)$$

We have

- E_i orthonormal, since parallel transport preserves the inner product;
- $\nabla_{E_i} E_j(p) = 0$, since $\nabla_v E_i = 0, \forall v \in T_pM$.

□

3.9 Let M be a Riemannian manifold. Define an operator $\triangle : \mathfrak{D}(M) \to \mathfrak{D}(M)$(the Laplacian of M) by

$$\triangle f = div\ \nabla f,\ f \in \mathfrak{D}(M)$$

a) Let E_i be a geodesic frame at $p \in M, i = 1, \cdots, n =dimM$(see Exercise 7). Prove that

$$\triangle f(p) = \sum_i E_i(E_i(f))(p)$$

Conclude that if $M = \mathbb{R}^n$, $\triangle$ coincides with the usual Laplacian, namely, $\triangle f = \sum_i \frac{\partial^2 f}{\partial x^2}$.

b) Show that

$$\triangle(f \cdot g) = f\triangle g + g\triangle f + 2 < \nabla f, \nabla g >$$

Proof. a) Firstly, $\nabla f(p) = \sum_i E_i(f)E_i(p)$

$$< \nabla f, E_i > (p) = df_p(E_i) = E_i(p)f = (E_i f)(p)$$

Secondly, $\triangle f(p) = \sum_i E_i(E_i(f))(p)$

$$\triangle f(p) = (div\nabla f)(p) = (div(\sum_i (E_i f)E_i))(p) = \sum_i (\nabla_{E_i}(E_i f))(p) = \sum_i E_i(E_i(f))(p)$$

Lastly, if $M = \mathbb{R}^n$, since $\left(\dfrac{\partial}{\partial x_i}\right)_{i=1}^n$ is an orthonormal basis for $T_p\mathbb{R}^n, \forall p \in \mathbb{R}^n$, we get

$$\triangle f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

b) For $p \in M$, let $(E_i)_{i=1}^n$ be a geodesic frame at $p \in M$, then

$$\begin{aligned}\triangle(f \cdot g)(p) &= \sum_i E_i(E_i(f \cdot g))(p) \\ &= \sum_i E_i(g \cdot E_i f + f \cdot E_i g)(p) \\ &= \sum_i (E_i f \cdot E_i g + g \cdot E_i(E_i f) + E_i f \cdot E_i g + f \cdot E_i(E_i g))(p) \\ &= (f\triangle g + g\triangle f + 2 < \nabla f, \nabla g >)(p)\end{aligned}$$

The last equality follows from

$$< \nabla f, \nabla g > (p) =< \sum_i E_i(f)E_i, \sum_j E_j(g)E_j > (p) = \sum_{i,j}(E_i f \cdot E_j g)\delta_{ij} = \sum_i E_i f \cdot E_i g$$

□

4 **Curvature**

4.7 Prove the 2nd Bianchi Identity:

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0$$

for all $X, Y, Z, W, T \in \mathfrak{X}(M)$.

Proof. Since the objects involved are all tensors, it suffices to prove the equality at a point $p \in M$. If we choose a geodesic frame $(E_i)_{i=1}^n$ at p. We've

$$\nabla_{E_i} E_j(p) = 0, [E_i, E_j](p) = (\nabla_{E_i} E_j - \nabla_{E_j} E_i)(p) = 0, \ \forall i, j \in \{1, \cdots, n\}$$

And it suffices to prove in case

$$X = E_i, Y = E_j, Z = E_k, W = E_l, T = E_m$$

also.Hence

$$\begin{aligned}
\nabla R(E_i,E_j,E_k,E_l,E_m)(p) &= (\nabla_{E_m}R)(E_i,E_j,E_k,E_l)(p) \\
&\quad \text{(definition)} \\
&= \nabla_{E_m}(R(E_i,E_j,E_k,E_l))(p) \\
&\quad \text{(Leibniz formula and geodesic frame)} \\
&= \nabla_{E_m}(R(E_k,E_l,E_i,E_j))(p) \\
&\quad \text{(Riemann connection)} \\
&= \nabla_{E_m} < -\nabla_{E_k}\nabla_{E_l}E_i + \nabla_{E_l}\nabla_{E_k}E_i + \nabla_{[E_k,E_l]}E_i, E_j > (p) \\
&\quad \text{(definition)} \\
&= < -\nabla_{E_m}\nabla_{E_k}\nabla_{E_l} + \nabla_{E_m}\nabla_{E_l}\nabla_{E_k}E_i + \nabla_{E_m}\nabla_{[E_k,E_l]}E_i, E_j > (p) \\
&\quad \text{(metric and geodesic frame)}
\end{aligned}$$

and

$$\begin{aligned}
& R(E_i,E_j,E_k,E_l,E_m)(p) + R(E_i,E_j,E_l,E_m,E_k)(p) + R(E_i,E_j,E_m,E_k,E_l)(p) \\
= \; & < -\nabla_{E_m}\nabla_{E_k}\nabla_{E_l}E_i + \nabla_{E_m}\nabla_{E_l}\nabla_{E_k}E_i + \nabla_{E_m}\nabla_{[E_k,E_l]}E_i, E_j > (p) \\
& + < -\nabla_{E_k}\nabla_{E_l}\nabla_{E_m}E_i + \nabla_{E_k}\nabla_{E_m}\nabla_{E_l}E_i + \nabla_{E_k}\nabla_{[E_l,E_m]}E_i, E_j > (p) \\
& + < -\nabla_{E_l}\nabla_{E_m}\nabla_{E_k}E_i + \nabla_{E_l}\nabla_{E_k}\nabla_{E_m}E_i + \nabla_{E_l}\nabla_{[E_m,E_k]}E_i, E_j > (p) \\
= \; & < (-\nabla_{E_m}\nabla_{E_k} + \nabla_{E_k}\nabla_{E_m} + \nabla_{[E_m,E_k]})(\nabla_{E_l}E_i), E_j > (p) \\
& + < (-\nabla_{[E_m,E_k]}\nabla_{E_l} + \nabla_{E_l}\nabla_{[E_m,E_k]} + \nabla_{[[E_m,E_k],E_l]})E_i, E_j > (p) \\
& - < \nabla_{[[E_m,E_k],E_l]}E_i, E_j > (p) \\
& + < (-\nabla_{E_l}\nabla_{E_m} + \nabla_{E_m}\nabla_{E_l} + \nabla_{[E_l,E_m]})(\nabla_{E_k}E_i), E_j > (p) \\
& + < (-\nabla_{[E_l,E_m]}\nabla_{E_k} + \nabla_{E_k}\nabla_{[E_l,E_m]} + \nabla_{[[E_l,E_m],E_k]})E_i, E_j > (p) \\
& - < \nabla_{[[E_l,E_m],E_k]}E_i, E_j > (p) \\
& + < (-\nabla_{E_k}\nabla_{E_l} + \nabla_{E_l}\nabla_{E_k} + \nabla_{[E_k,E_l]})(\nabla_{E_m}E_i), E_j > (p) \\
& + < (-\nabla_{[E_k,E_l]}\nabla_{E_m} + \nabla_{E_m}\nabla_{[E_k,E_l]} + \nabla_{[[E_k,E_l],E_m]})E_i, E_j > (p) \\
& - < \nabla_{[[E_k,E_l],E_m]}E_i, E_j > (p) \\
= \; & R(E_m,E_k,\nabla_{E_l}E_i,E_j)(p) + R([E_m,E_k],E_l,E_i,E_j)(p) \\
& + R(E_l,E_m,\nabla_{E_k}E_i,E_j)(p) + R([E_l,E_m],E_k,E_i,E_j)(p) \\
& + R(E_k,E_l,\nabla_{E_m}E_i,E_j)(p) + R([E_k,E_l],E_m,E_i,E_j)(p) \\
& - < \nabla_{[[E_m,E_k],E_l]+[[E_l,E_m],E_k]+[[E_k,E_l],E_m]}E_i, E_j > (p)\text{(definition)} \\
& = 0\text{(geodesic and Jacobi identity)}
\end{aligned}$$

□

4.8 (Schur's Theorem). Let M^n be a connected Riemannian manifold with $n \geq 3$.Suppose that M is isotropic, that is, for each $p \in M$, the sectional curvature $K(p,\sigma)$ does not depend on $\sigma \subset T_pM$. Prove that M has constant sectional curvature, that is, $K(p,\sigma)$ also does not depend on p.

Proof. For any $p \in M$, choose a geodesic frame $(E_i)_{i=1}^n$ at p, i.e. $(E_i)_{i=1}^n$ orthonormal in a neighborhood of p and $\nabla_{E_i} E_j(p) = 0$. Denote by

$$R_{ijkl} = R(E_i, E_j, E_k, E_l)(p)$$

$$\nabla_m R_{ijkl} = (\nabla_{E_m} R)(E_i, E_j, E_k, E_l)(p) = \nabla_{E_m}(R(E_i, E_j, E_k, E_l))(p)$$

Since the sectional curvature uniquely determines the Riemann curvature, we've:
if $K(p, \sigma) = f(p)$, then

- $R_{ijkl} = f(p)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$
- $Ric_{ij} = \sum_k R_{ikjk} = f(p) \sum_k (\delta_{ij} - \delta_{ik}\delta_{kj}) = (n-1)f(p)\delta_{ij}$
- $R = \sum_i R_{ii} = n(n-1)f(p)$

From the 2nd Bianchi identity,

$$\nabla_i R_{ijkj} + \nabla_k R_{ijji} + \nabla_j R_{ijik} = 0$$

Summing over i, j over $\{1, \cdots, n\}$, one gets

$$\sum_i \nabla_i R_{ik} - \nabla_k R + \sum_j \nabla_j R_{jk} = 0$$

$$2\sum_i \nabla_i R_{ik} - \nabla_k R = 0$$

$$2(n-1)\nabla_k f(p) - n(n-1)\nabla_k f(p) = 0$$

$$(n-2)(n-1)\nabla_k f(p) = 0$$

Thus

$$\nabla_k f(p) = 0, \ \forall k$$

since $n \geq 3$. Finally,

$$K(p, \sigma) = f \equiv Const$$

since M is connected. □

5 Jacobi Fields

5.3 Let M be a Riemannian manifold with non-positive sectional curvature. Prove that, for all p, the conjugate locus $C(p)$ is empty.

Proof. For any $p \in M$, if $C(p) \neq$, i.e. $\exists q \in C(p)$, then

$$\exists \begin{cases} \text{geodesic } \gamma : [0, a] \to M \\ \text{Jacobi filed } J \neq 0 \end{cases} \quad s.t. \quad \begin{cases} \gamma(0) = p, \gamma(a) = q \\ J(0) = 0 = J(a) \end{cases}$$

From the Jacobi equation,

$$J'' + R(\gamma', J)\gamma' = 0$$

We know

$$\begin{aligned} < J', J >' &= < J'', J > + < J', J' > \\ &= - < R(\gamma', J)\gamma', J > + < J', J' > \\ &= -K_M(\gamma', J)||\gamma' \wedge J||^2 + < J', J' > \\ &\geq 0 \end{aligned}$$

Since M is of non-positive sectional curvature. Thus

$$0 =< J'(0), J(0) > \leq < J', J > \leq < J'(a), J(a) >= 0$$

$$< J', J >= 0$$

$$< J, J >' = 2 < J', J >= 0$$

$$||J||^2 = ||J(0)|| = 0$$

A contradiction. □

5.4 Let $b < 0$ and let M be a manifold with constant negative sectional curvature equal to b. Let $\gamma : [0, l] \to M$ be a normalized geodesic, and let $v \in T_{\gamma(l)}M$ such that $< v, \gamma'(l) >= 0$ and let $|v| = 1$. Since M has negative curvature, $\gamma(l)$ is not conjugate to $\gamma(0)$(See Exercise 3). Show that the Jacobi field J along γ determined by $J(0) = 0, J(l) = v$ is given by

$$J(t) = \frac{\sinh(t\sqrt{-b}}{\sinh(l\sqrt{-b})} w(t)$$

where $w(t)$ is the parallel transport along γ of the vector

$$w(0) = \frac{u_0}{|u_0|},\ u_0 = (dexp_p)^{-1}_{l\gamma'(0)}(v)$$

and where u_0 is considered as a vector $T_{\gamma(0)}M$ by the identification $T_{\gamma(0)}M \approx T_{l\gamma'(0)}(T_{\gamma(0)}M)$

Proof. • The Jacobi field $\tilde{J}$ along γ with $\tilde{J}(0) = 0, \tilde{J}'(0) = w(0) \in T_{\gamma(0)}M$ is of the form

$$\tilde{J}(t) = \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w(t)$$

where $w(t)$ is the parallel transport of $w(0)$ along γ.
Indeed, let $(E_i)_{i=1}^n$ be an orthonormal basis for $T_{\gamma(0)}M$,$(E_i(t))_{i=1}^n$ be parallel transport of E_i along γ. Then if we write

$$\tilde{J}(t) = \sum_i \tilde{J}_i(t)E_i(t) \in T_{\gamma(t)}M$$

$$w(0) = \sum_i w_i E_i \in T_{\gamma(0)}M$$

One gets from the Jacobi equation that

$$\begin{cases} \tilde{J}_i''(t) + b\tilde{J}_i(t) = 0 \\ \tilde{J}_i(0) = 0 \\ \tilde{J}_i'(0) = w_i \end{cases}$$

Hence

$$\tilde{J}(t) = \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w_i$$

$$\tilde{J}(t) = \sum_i J_i(t)E_i(t) = \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} \sum_i w_i E_i(t) = \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w(t)$$

- One can write $\tilde{J}(t) = (dexp_p)_{t\gamma'(0)}(tw(0))$
  This is just another saying that Jacobi filed is the variational field of geodesic.
- Since

$$J(l) = v = (dexp_p)_{l\gamma'(0)}(u_0) = (dexp_p)_{l\gamma'(0)}\left(l\frac{u_0}{|u_0|} \cdot \frac{|u_0|}{l}\right) = \frac{|u_0|}{l}\tilde{J}(l)$$

  We have

$$J(t) = \frac{u_0}{l}\tilde{J}(t) = \frac{u_0}{l}\frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w(t)$$

  Indeed,

  M is of negative sectional curvature
  $\Rightarrow C(\gamma(0)) = \varnothing$
  $\Rightarrow$ Jacobi field$J$ along γ is uniquely determined by $J(0), J(l)$

- Since

$$1 = |v| = |J(l)| = \frac{|u_0|}{l}\frac{\sinh(l\sqrt{-b})}{\sqrt{-b}}$$

  We have

$$\frac{|u_0|}{l} = \frac{\sqrt{-b}}{\sinh(l\sqrt{-b})}$$

  and finally

$$J(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(l\sqrt{-b})} w(t)$$

□

6 Isometric Immersions

6.3 Let M be a Riemannian manifold and let $N \subset K \subset M$ be a submanifolds of M. Suppose that N is totally geodesic in K and that K is totally geodesic in M. Prove that N is totally geodesic M.

Proof. From the hypothesis,we know, every geodesic in N is a geodesic in K, thus a geodesic in M, hence the assertion. □

6.11 Let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a differentiable function. Define the Hessian, $Hessf$ of f at $p \in \overline{M}$ as the linear operator

$$\begin{aligned} Hessf : T_p\overline{M} &\to T_p\overline{M} \\ (Hessf)Y &= \overline{\nabla}_Y \overline{\nabla} f, \ Y \in T_p\overline{M} \end{aligned}$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subset \overline{M}^{n+1}$ be the hypersurface in $\overline{M}$ defined by $M = \{p \in \overline{M}; f(p) = a\}$. Prove that

a) The Laplacian $\overline{\triangle} f$ is given by

$$\overline{\triangle} f = trac\ Hessf$$

b) If $X, Y \in \mathfrak{X}(\overline{M})$, then

$$< (Hessf)Y, X >=< Y, (Hessf)X >$$

Conclude that $Hessf$ is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}, p \in \overline{M}$, is given by

$$(Hessf)(X, Y) =< (Hessf)X, Y >, X, Y \in T_p\overline{M}$$

c) The mean curvature H of $M \subset \overline{M}$ is given by

$$nH = -div\left(\frac{\overline{\nabla} f}{|\overline{\nabla} f|}\right)$$

d) Observe that every embedded hypersurface $M^n \subset \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from c) that the mean curvature H of such a hypersurface is given by

$$H = -\frac{1}{n} divN$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

Proof. a) For any $p \in \overline{M}$, let $(E_i)_{i=1}^{n+1}$ be othonormal basis for $T_p\overline{M}$, then

$$\begin{aligned} \overline{\triangle} f &= div_{\overline{M}} \overline{\nabla} f \\ &= \sum_{i=1}^{n+1} < \overline{\nabla}_{E_i} \overline{\nabla} f, E_i > \\ &= \sum_{i=1}^{n+1} < (Hessf)E_i, E_i > \\ &= trace\ Hessf \end{aligned}$$

b)

$$\begin{aligned} <(Hessf)Y,X> &= <\overline{\nabla}_Y\overline{\nabla}f,X> \text{ (definition)} \\ &= Y<\overline{\nabla}f,X> - <\overline{\nabla}f,\overline{\nabla}_Y,X> \text{ (metric)} \\ &= YXf-(\overline{\nabla}_YX)f\text{(definition)} \\ &= XYf-(\overline{\nabla}_YX)f\text{(definition and torsion-free property)} \\ &= <Y,(Hessf)X> \end{aligned}$$

c) Take an orthonormal frame $E_1,\cdots,E_n,E_{n+1}=\dfrac{\overline{\nabla}f}{|\overline{\nabla}f|}=\eta$ in a neighborhood of $p\in M$ in $\overline{M}$,then

$$\begin{aligned} nH &= trace\ S_\eta \\ &= \sum_{i=1}^{n}<S_\eta(E_i),E_i> \\ &= -\sum_{i=1}^{n}<\overline{\nabla}_{E_i}\eta,E_i> - <\overline{\nabla}_\eta\eta,\eta> \\ &= -\sum_{i=1}^{n+1}<\overline{\nabla}_{E_i}\eta,E_i> \\ &= -div_{\overline{M}}\eta \\ &= -div\left(\frac{\overline{\nabla}f}{|\overline{\nabla}f|}\right) \end{aligned}$$

d) As a simple consequence of implicit function theorem, for any $p\in M$, there is a coordinate neighborhood (U,x) in $\overline{M}$ of p such that

$$\overline{M}\cap U = x\{x_{n+1}=0\}$$

[See S.S.Chern: Lectures on Differential Geometry, for example.]
If we take $f:\overline{M}\to\mathbb{R}$ defined locally by

$$f\circ x = x_{n+1}$$

then

$$\overline{\nabla}f\in(T_qM)^\perp,\ \forall q\in\overline{M}\cap U$$

Indeed,

$$<\overline{\nabla}f,\frac{\partial}{\partial x_i}>=\frac{\partial}{\partial x_i}f=dx\left(\frac{\partial}{\partial x_i}\right)f=\frac{\partial}{\partial x_i}(f\circ x)=\frac{\partial}{\partial x_i}x_{n+1}=0,\ \forall 1\le i\le n$$

Thus from c),

$$H=-\frac{1}{n}div\left(\frac{\overline{\nabla}f}{|\overline{\nabla}f|}\right)=-\frac{1}{n}div\ N$$

□

7 Complete Manifolds; Hopf-Rinow and Hadamard Theorems

7.6 A geodesic $\gamma : [0, +\infty) \to M$ in a Riemannian manifold M is called a ray starting from $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$, for any $s \in (0, \infty)$. Assume that M is complete, non-compact, and let $p \in M$. Show that M contains a ray starting from P.

Proof. Argue by contradiction.

M contains no ray starting from p
$\Leftrightarrow$ for any $\gamma : [0, \infty) \to M$ with $\gamma(0) = p, \exists s \in (0, \infty),\ s.t.\ \gamma|_{[0,s]}$
does not minimizes the distance between p and $\gamma(s)$
$\Leftrightarrow$ for any $v \in T_pM$ with $|v| = 1, \exists s \in (0, \infty),\ s.t.\ \exp_p(tv), t \in [0, s]$
does not minimizes the distance between p and $\exp_p(sv)$

Define

$$\begin{aligned} c : T_pM &\to \mathbb{R}^+ \\ v &\mapsto c(v) = \inf s < \infty \end{aligned}$$

where the inf is taken over all s such that $\exp_p(tv), t \in [0, s]$ does not minimizes the distance between p and $\exp_p(sv)$. Clearly,

- $c(v) = \inf s = \min s$;
- c is a continuous function of v.

This is done by careful analysis, see Chapter 13 for example.
Since $\{v \in T_pM; |v| = 1\}$ is a compact set, we know c is bounded, i.e. $\max c < \infty$, thus

$$M = B(p, \max c + 1)$$

Hence M is compact by Hopf-Rinow Theorem, a contradiction. □

7.7 Let M and $\overline{M}$ be Riemannian manifolds and let $f : M \to \overline{M}$ be a diffeomorphism. Assume that $\overline{M}$ is complete and that there exists a constant $c > 0$ such that

$$|v| \geq c|df_p(v)|$$

for all $p \in M$ and all $v \in T_pM$. Prove that M is complete.

Proof.
- $p, q \in M \Rightarrow d_M(p, q) \geq c \cdot d_{\overline{M}}(f(p), f(q))$
Indeed, for any piecewise differentiable curve γ joining p to q, $f \circ \gamma$ is

such one joining $f(p)$ to $f(q)$, thus

$$\begin{aligned} l(\gamma) &= \int_a^b |\gamma'(t)|dt \\ &\geq c\int_a^b |df(\gamma'(t))|dt \\ &= c\int_a^b |(f\circ\gamma)'(t)|dt \\ &\geq c\cdot d_{\overline{M}}(f(p),f(q)) \end{aligned}$$

Taking inf over all such curves, one gets

$$d_M(p,q) \geq c\cdot d_{\overline{M}}(f(p),f(q))$$

- M is complete as a metric space
  For any Cauchy sequence $(p_n)_{n=1}^{\infty} \subset M$, we've, from

  $$d_M(p_n,p_m) \geq c\cdot d_{\overline{M}}(f(p_n),f(p_m))$$

  that $(f(p_n))_{n=1}^{\infty} \subset \overline{M}$ a Cauchy sequence, hence converges to some point, $q\in\overline{M}$, say. Then

  $$p_n = f^{-1}(f(p_n)) \to f^{-1}(q) \in M \quad \text{as} \quad n\to\infty$$

□

7.10 Prove that the upper half-plane $\mathbb{R}^2_+$ with the Lobatchevski metric:

$$g_{11} = \frac{1}{y^2} = g_{22}, \qquad g_{12} = 0 = g_{21}$$

is complete.

Proof. We write $\mathbb{H}^2 = (\mathbb{R}^2, g)$.

- **Lemma** Let $f:(M,g)\to(\overline{M},\overline{g})$ be an isometry between two Riemannian manifolds, then

$$df\left(\nabla_X Y\right) = \overline{\nabla}_{df(X)}df(Y), \qquad \forall X,Y\in\mathfrak{X}(M)$$

where $\nabla,\overline{\nabla}$ are Riemann connections of $M,\overline{M}$ respectively.
In other words, isometries preserve Riemann connections.
Proof of the Lemma We simply use Koszul formula as follows.

$$\begin{aligned} & 2\overline{g}\left(df(\nabla_X Y), df(Z)\right)\right)\circ f \\ = \;& 2g(\nabla_X Y, Z)\ (isometry) \\ = \;& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])\ (Koszul formula) \\ = \;& X\left(\overline{g}(df(Y),df(Z))\circ f\right) - \cdots - \overline{g}\left(df(X), df([X,Y])\right)\circ f + \cdots \\ = \;& \left(df(X)\overline{g}\left(df(Y),df(Z)\right)\right)\circ f - \cdots - \overline{g}\left(df(X),[df(Y),df(Z)]\right)\circ f \\ = \;& 2\overline{g}\left(\nabla_{df(X)}df(Y), df(Z)\right)\circ f \end{aligned}$$

• **Claim**

$$\gamma(t)=(0,e^t)=ie^t,\qquad t\in[0,\infty)$$

is the geodesic with data $(e=(0,1)=i,\ dy=(0,1)=i)$.

Method 1 we've only to show each portion of γ minimize curve length. To this end,for $c:[a,b]\to\mathbb{H}^2$ with $c(a)=a\geq 1,\ c(b)=b\geq 1$,

$$\begin{aligned} l(c) &= \int_a^b\left|\frac{dc}{dt}\right|dt \\ &= \int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\frac{dt}{y} \\ &\geq \int_a^b\left|\frac{dy}{dt}\right|\frac{dt}{y} \\ &\geq \int_a^b\frac{dy}{y} \\ &= \int_{\ln a}^{\ln b}dt \\ &= l\left(\gamma|_{[\ln a,\ln b]}\right) \end{aligned}$$

Method 2 We just see γ satisfies the geodesic equation. Indeed, since the Christoffel symbols are

$$\begin{cases}\Gamma_{11}^1=\Gamma_{12}^2=\Gamma_{22}^1=0\\ \Gamma_{11}^2=\frac{1}{y}\\ \Gamma_{12}^1=\Gamma_{22}^2=-\frac{1}{y}\end{cases}$$

Thus

$$\frac{d^2}{dt^2}e^t+\Gamma_{22}^2\cdot e^t\cdot e^t=e^t-\frac{1}{e^t}\cdot e^{2t}=0$$

• **Claim**

$$\gamma_\theta(t)=\frac{\cos\frac{\theta}{2}\cdot ie^t-\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\cdot ie^t+\cos\frac{\theta}{2}},\qquad t\in[0,\infty)$$

is the geodesic in $\mathbb{H}^2$ with data $(e,\ v=(\sin\theta,\cos\theta))$, where $\theta\in[0,2\pi)$.Hence by Hopf-Rinow theorem, $\mathbb{H}^2$ is complete.

Proof of the Claim

✓ γ_θ, as the image of $\gamma_0=\gamma$ under the isometry of $\mathbb{H}^2$:

$$z\mapsto\frac{\cos\frac{\theta}{2}\cdot z-\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\cdot z+\cos\frac{\theta}{2}}$$

is geodesic;

✓ $\gamma_\theta(0)=i=(0,1)=e$;

✓

$$
\begin{aligned}
\gamma_\theta'(0) &= \frac{1}{\left(\sin\frac{\theta}{2}\cdot ie^t+\cos\frac{\theta}{2}\right)^2}\cdot ie^t|_{t=0}\\
&= i\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)^2\\
&= i(\cos\theta - i\sin\theta)\\
&= \sin\theta + i\sin\theta\\
&= v.
\end{aligned}
$$

□

Remark In the proof we construct all geodesics starting from $e=(0,1)$.

- If $v=(0,1)$, the geodesic being $(0,e^t)$;
- If $v=(0,-1)$, the geodesic being $0,e^{-t})$;
- If $v=(\sin\theta,\cos\theta),\ \theta\neq k\pi,\ k\in\mathbb{Z}$, we've the geodesic γ_θ satisfies

$$|\gamma_\theta(t)-\cot\theta| = |\csc\theta|$$

Indeed,

$$
\begin{aligned}
&|\gamma_\theta(t)|^2 - 2\Re(\gamma_\theta(t)\cdot\cot\theta)\\
=\ & \frac{\sin^2\frac{\theta}{2}+e^{2t}\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+e^{2t}\sin^2\frac{\theta}{2}} - 2\Re\left(\frac{\cos\frac{\theta}{2}\cdot ie^t-\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\cdot ie^t+\cos\frac{\theta}{2}}\right)\cdot\frac{1-\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}\\
=\ & \frac{\tan^2\frac{\theta}{2}+e^{2t}}{1+e^{2t}\tan^2\frac{\theta}{2}} - 2\frac{(e^{2t}-1)\tan\frac{\theta}{2}}{1+e^{2t}\tan^2\frac{\theta}{2}}\cdot\frac{1-\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}\\
=\ & \frac{1+e^{2t}\tan^2\frac{\theta}{2}}{1+e^{2t}\tan^2\frac{\theta}{2}}\\
=\ & 1
\end{aligned}
$$

Finally,since $\mathbb{H}^2$ is a Lie group, all geodesics in $\mathbb{H}^2$ is known.

8 Spaces of Constant Curvature

8.1 Consider, on a neighborhood in $\mathbb{R}^n, n>2$ the metric

$$g_{ij} = \frac{\delta_{ij}}{F^2}$$

where $F\neq 0$ is a function of $(x_1,\cdots,x_n)\in\mathbb{R}^n$. Denote by $F_i=\dfrac{\partial F}{\partial x_i}, F_{ij}=\dfrac{\partial^2 F}{\partial x_i\partial x_j}$,etc.

a) Show that a necessary and sufficient condition for the metric to have constant curvature K is

$$(*)\qquad \begin{cases} F_{ij}=0,\ i\neq j;\\ F(F_{jj}+F_{ii}) = K+\sum_{i=1}^n (F_i)^2. \end{cases}$$

b) Use $(*)$ to prove that the metric g_{ij} has constant curvature K if and only if

$$F = \sum_{i=1}^{n} G_i(x_i)$$

where

$$G_i(x_i) = ax_i^2 + b_i x_i + c_i$$

and

$$\sum_{i=1}^{n} (4c_i a - b_i^2) = K$$

c) Put $a = \frac{a}{4}, b_i = 0, c_i = \frac{1}{n}$ and obtain the formula of Riemann

$$(**) \qquad g_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{K}{4} \sum x_i^2\right)^2}$$

for a metric g_{ij} of constant curvature K. If $K < 0$ the metric g_{ij} is defined in a ball of radius $\sqrt{\frac{4}{-K}}$.

d) If $K > 0$, the metric $(**)$ is defined on all of $\mathbb{R}^n$. Show that such a metric on $\mathbb{R}^n$ is not complete.

Proof. a) The metric and its inverse are

$$g_{ij} = \frac{\delta_{ij}}{F^2}, \qquad g^{ij} = F^2 \delta_{ij}$$

Thus the Christoffel symbols

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2} g^{kl} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ij}\right) \\
&= \frac{1}{2} F^2 \left(\partial_j g_{ik} + \partial_i g_{kj} - \partial_k g_{ij}\right) \\
&= \frac{1}{2} F^2 \cdot \frac{-2}{F^3} \left(\delta_{ik} F_j + \delta_{kj} F_i - \delta_{ij} F_k\right) \\
&= -\delta_{ik} f_j - \delta_{kj} f_i + \delta_{ij} f_k
\end{aligned}$$

where

$$f = \log F$$

Write down precisely,

$$\begin{cases}
\Gamma_{ij}^k = 0, & \text{if } i \neq j, j \neq k, k \neq i \\
\Gamma_{ii}^j = f_j, \ \Gamma_{ij}^i = -f_j, & \text{if } i \neq j \\
\Gamma_{ii}^i = -f_i
\end{cases}$$

Hence the Riemannian curvature ($i \neq j$)

$$\begin{aligned}
R_{ijij} &= \langle -\nabla_i \nabla_j i + \nabla_j \nabla_i i, j\rangle \\
&= \left\langle -\nabla_i \left(\Gamma_{ij}^k k\right) + \nabla_j \left(\Gamma_{ii}^k k\right), j\right\rangle \\
&= \left\langle -\partial_i \Gamma_{ij}^k k - \Gamma_{ij}^k \Gamma_{ik}^l l + \partial_j \Gamma_{ii}^k k + \Gamma_{ii}^k \Gamma_{jk}^l l, j\right\rangle \\
&= -\partial_i \Gamma_{ij}^k g_{kj} + \partial_j \Gamma_{ii}^k g_{kj} - \Gamma_{ij}^k \Gamma_{ik}^l g_{lj} + \Gamma_{ii}^k \Gamma_{jk}^l g_{lj} \\
&= \frac{1}{F^2}\left(-\partial_i \Gamma_{ij}^j + \partial_j \Gamma_{ii}^j - \Gamma_{ij}^k \Gamma_{ik}^j + \Gamma_{ii}^k \Gamma_{jk}^j\right) \\
&= \frac{1}{F^2}\left[f_{ii} + f_{jj} + (f_j^2 - f_i^2) + \left(f_i^2 - f_j^2 - \sum_{k\neq i,j} f_k^2\right)\right] \\
&= \frac{1}{F^2}\left(f_{ii} + f_{jj} - \sum_k f_k^2 + f_i^2 + f_j^2\right)
\end{aligned}$$

Finally, the sectional curvature

$$\begin{aligned}
K(i,j) &= \frac{R_{ijij}}{< i,i >< j,j > - < i,j >^2} \\
&= F^2(f_{ii} + f_{jj} - \sum_{k=1}^{n} f_k^2 + f_i^2 + f_j^2) \\
&= FF_{ii} - F_i^2 + FF_{jj} - F_j^2 - \sum_k F_k^2 + F_i^2 + F_j^2 \\
&= F(F_{ii} + F_{jj}) - \sum_k F_k^2
\end{aligned}$$

Now we prove a). The sufficiency is obvious. For the necessity, we need only to show

$$F_{ij} = 0, \quad \forall i \neq j$$

Indeed, since $K(i,j) = K = Const$,

$$F_{ii} = c, \quad \forall i$$

Thus

$$K = 2Fc - \sum_k F_k^2$$

Differentiating w.r.t. l twice, we obtain

$$0 = 2F_l - \sum_k 2F_k F_{kl}$$

$$\sum_{k\neq l} F_k F_{kl} = 0$$

$$\sum_{k\neq l} (F_{kl})^2 = \sum_{k\neq l} (F_{kl})^2 + F_k F_{kll} = 0$$

$$F_{kl} = 0 \quad \forall k \neq l$$

Remark For simplicity and type convenience, we use i for $\partial_i = \dfrac{\partial}{\partial x_i}$. And there is no confusion between ∇_i and F_i.

b) **Claim** From $(*)$,

$$\begin{cases} F_{ij} = 0, & \forall i \neq j \\ F_{ii} = 2a = Const, & \forall i \end{cases}$$

We have

$$F = \sum_{i=1}^{n} G_i(x_i)$$

where

$$G_i(x_i) = ax_i^2 + b_i x_i + c$$

Indeed, $F_{ii} = 2a$ implies

$$F_i = 2ax_i + g(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

while $F_{ij} = 0, \forall j \neq i$ implies

$$0 = F_{ij} = \partial_j g, \quad \forall j \neq i$$

$$g = b_i = Const$$

$$F_i = 2ax_i + b_i$$

Hence

$$F = ax_i^2 + b_i x_i + h_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

Thus

$$ax_i^2 + b_i x_i + h_i = ax_j^2 + b_j x_j + h_j, \quad \forall j \neq i$$

$$h_i - (ax_j^2 + b_j x_j) = h_j - (ax_i^2 + b_i x_i)$$

Since the r.h.s. of the equality above doesn't have the x_j-term, we have

$$h_i = \sum_{j \neq i} (ax_j^2 + b_j x_j) + c$$

Hence the claim.

Now,

$$\begin{aligned} K &= F(F_{ii} + F_{jj}) - \sum_k (F_k)^2 \\ &= 4a \sum_k (ax_i^2 + b_i x_i + c_i) - \sum_k (2ax_i + b_i)^2 \\ &= \sum_k (4c_i a - b_i^2) \end{aligned}$$

c) Put $a = \dfrac{K}{4}, b_i = 0, c_i = \dfrac{1}{n}$, we obtain the formula of Riemann

$$g_{ij} = \frac{\delta_{ij}}{\left[\sum_i \left(\frac{K}{4}x_i^2 + \frac{1}{n}\right)\right]^2} = \frac{\delta_{ij}}{\left(1 + \frac{K}{4}\sum_i x_i^2\right)^2}$$

If $K < 0$, we should have

$$\sum_i x_i^2 \leq \left(\sqrt{\frac{4}{-K}}\right)^2$$

i.e. g_{ij} are defined in a ball of radius $\sqrt{\dfrac{4}{-K}}$.

d) If $K > 0$, the metric $(**)$ is defined on all of $\mathbb{R}^n$. We shall show $(\mathbb{R}^n, g_{ij})$ is not complete.

Indeed, for any $p = (x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned}
d_g(0,p) &\leq |0p|_g \\
&= \int_0^1 \sqrt{\sum_i \frac{x_i^2}{\left[1 + \frac{K}{4}\sum_k (tx_k)^2\right]^2}}dt \\
&= \int_0^1 \frac{D}{1 + \frac{K}{4}Dt^2}dt \quad \left(D = \sqrt{\sum_i x_i^2}\right) \\
&= \frac{2}{K}\arctan\frac{\sqrt{K}}{2} \\
&\leq \frac{2}{K}\cdot\frac{\pi}{2} \\
&= \frac{\pi}{\sqrt{K}} \\
&< \infty
\end{aligned}$$

Hence $(\mathbb{R}^n, g_{ij})$ is bounded, also, it is closed as a whole space, but we know $\mathbb{R}^n$ is non-compact (Note that compactness is a topological property.). Thus, $(\mathbb{R}^n, g_{ij})$ is not complete by Hopt-Rinow Theorem. □

8.4 Identity $\mathbb{R}^4$ with $\mathbb{C}^2$ by letting (x_1, x_2, x_3, x_4) correspond to (x_1+ix_2, x_3+ix_4). Let

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2 + |z_2|^2 = 1\right\}$$

and let $h : S^3 \to S^3$ be given by

$$h(z_1, z_2) = \left(e^{\frac{2\pi i}{q}}z_1, e^{\frac{2\pi i r}{q}}z_2\right), \qquad (z_1, z_2) \in S^3$$

where q and r are relatively prime integers, $q > 2$.

a) Show that $G = \{id, h, \cdots, h^{q-1}\}$is a group of isometries of the sphere S^3, with the usual metric, which operates in a totally discontinuous manner. The manifold S^3/G is called a lens space.

b) Consider S^3/G with metric induced by the projection $p : S^3 \to S^3/G$. Show that all the geodesics of S^3/G is closed but can have different lengths.

Proof. a) **Claim 1** Each h^k is an isometry of S^3.

Indeed, denote by

$$\alpha = \frac{2\pi}{q}, \quad \beta = \frac{2\pi r}{q}$$

then

$$h^k(z_1, z_2) = \left(e^{ik\alpha} z_1, e^{ik\beta} z_2\right)$$
$$dh^k_{(z_1,z_2)} = \left(e^{k\alpha i} dz_1, e^{k\beta i} dz_2\right)$$

For any $p \in S^3, u = (u_1, u_2), v = (v_1, v_2) \in T_pS^3$, where

$$u_j = u_{j1} + iu_{j2}, \quad v_j = v_{j1} + iv_{j2}, \quad j = 1, 2$$

We have

$$\begin{aligned}
& \left\langle dh^k(u) dh^k(v)\right\rangle_{h^k(p)} \\
= & \left\langle \begin{pmatrix} e^{ik\alpha} u_1 \\ e^{ik\beta} u_2 \end{pmatrix}, \begin{pmatrix} e^{ik\alpha} v_1, e^{ik\beta} v_2 \end{pmatrix} \right\rangle \\
= & \left\langle \begin{pmatrix} u_{11} \cos k\alpha - u_{12} \sin k\alpha \\ +i\left(u_{11} \sin k\alpha + u_{12} \cos k\alpha\right) \end{pmatrix}, \begin{pmatrix} v_{11} \cos k\alpha - v_{12} \sin k\alpha \\ +i\left(v_{11} \sin k\alpha + v_{12} \cos k\alpha\right) \end{pmatrix} \right\rangle \\
= & \left(u_{11} \cos k\alpha - u_{12} \sin k\alpha\right)\left(v_{11} \cos k\alpha - v_{12} \sin k\alpha\right) \\
& + \left(u_{11} \sin k\alpha + u_{12} \cos k\alpha\right)\left(v_{11} \sin k\alpha + v_{12} \cos k\alpha\right) \\
& + \left(u_{11} \cos k\alpha - u_{12} \sin k\alpha\right)\left(v_{11} \cos k\beta - v_{12} \sin k\beta\right) \\
& + \left(u_{11} \sin k\beta + u_{12} \cos k\beta\right)\left(v_{11} \sin k\beta + v_{12} \cos k\beta\right) \\
= & \left\langle (u_1, u_2), (v_1, v_2)\right\rangle \\
= & \langle u, v\rangle_p
\end{aligned}$$

Claim 2 G operates on S^3 in a properly discontinuous manner.

Just note that for any $(z_1, z_2) \in S^3$,

$$h_k(z_1, z_2) = \left(e^{ik\alpha} z_1, e^{ik\beta} z_2\right), \quad k \in \{1, \cdots, q-1\}$$

are continuous, and $\neq (z_1, z_2)$, Hence

$$\exists U \ni x, \ s.t. \ h^k U \cap U \neq \varnothing, \forall k \in \{1, \cdots, q-1\}$$

Indeed,

♠ $h^k(z_1, z_2) \neq (z_1, z_2)$
Since q and r are relatively prime,

$$\exists\ s, t,\ s.t.\ sq + tr = 1$$

if some $k \in \{1, \cdots, q-1\}$ satisfies

$$e^{ik\alpha} = 1 \quad \text{or} \quad e^{ik\beta} = 1$$

then we have $k = mq$, a contradiction; or $kr = mq$ for some $m \in \mathbb{Z}$, a contradiction again since

$$k = skq + tkr = skq + tmq = (sk + tm)q$$

♠ The existence of such U.
Set $p = (z_1, z_2), q_k = h^k(p)$, then by Hausdorff property,

$$\exists U \ni p,\ V_k \ni q,\ s.t.\ U \cap V_k = \varnothing$$

Since h is continuous, we may retract U such that

$$h^k(U) \subset V_k, \quad \forall k \in \{1, \cdots, q-1\}$$

This U verifies.

b) Since G is a group of isometry, we can introduce the metric on S^3/G such that $p : S^3 \to S^3/G$ is a local isometry. Thus the geodesics are preserved. Now the geodesics on S^3 are all closed, the geodesics of S^3/G are close also, but they may have different length. Consider, for example,

$$\begin{cases} \gamma_1 = (e^{i\theta}, 0) \\ \gamma_2 = (0, e^{i\theta}) \end{cases} \qquad \theta \in [0, 2\pi]$$

the geodesics on S^3, but we have

$$\begin{cases} l(p(\gamma_1)) = \alpha \\ l(p(\gamma_2)) = \beta \end{cases}$$

when $\alpha \neq \beta$,i.e. $r \neq 1$, these two are different.

□

8.5 (Connections of conformal metrics) Let M be a differentiable manifold. Two Riemannian metrics g and $\overline{g}$ on M are conformal if there exists a positive function $\mu : M \to \mathbb{R}$ such that $\overline{g}(X, Y) = \mu g(X, Y)$, for all $X, Y \in \mathfrak{X}(M)$. Let ∇ and $\overline{\nabla}$ be the Riemannian connections of g and $\overline{g}$, respectively. Prove that

$$\overline{\nabla}_X Y = \nabla_X Y + S(X, Y)$$

where

$$S(X, Y) = \frac{1}{2\mu} \{(X\mu)Y + (Y\mu)X - g(X, Y)\nabla\mu\}$$

and $\nabla\mu$ is calculated in the metric g, that is,

$$X(\mu) = g(X, \nabla\mu)$$

Proof. By Koszul Formula,

$$\begin{aligned}
\mu g(\overline{\nabla}_X Y, Z) &= \overline{g}(\nabla_X Y, Z) \\
&= \frac{1}{2}\left\{ \begin{array}{c} X\overline{g}(Y,Z) + Y\overline{g}(Z,X) - Z\overline{g}(X,Y) \\ -\overline{g}(X,[Y,Z]) + \overline{g}(Y,[Z,X]) + \overline{g}(Z,[X,Y]) \end{array} \right\} \\
&= \frac{1}{2}\{(X\mu g(Y,Z) + Yg(Z,X) - Zg(X,Y)\} \\
&\quad + \frac{\mu}{2}\left\{ \begin{array}{c} Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \end{array} \right\} \\
&= \frac{1}{2}\{g((X\mu)Y + (Y\mu)X - g(X,Y)\nabla\mu, Z)\} + \mu g(\nabla_X Y, Z) \\
&= \mu g(S(X,Y) + \nabla_X Y, Z)
\end{aligned}$$

□

9 **Variations of Energy**

9.1 Let M be a complete Riemannian manifold, and let $N \subset M$ be a closed submanifold of M. Let $p_0 \in M, p_0 \notin N$, and let $d(p_0, N)$ be the distance from p_0 to N. Show that there exists a point $q_0 \in N$ such that $d(p_0, q_0) = d(p_0, N)$ and that a minimizing geodesic which joins p_0 to q_0 is orthogonal to N at q_0.

Proof.

- Existence of such $q_0 \in N$.
  Let $\{q_i\} \subset N$, $s.t.\ d(p_0, q_i) \to d(p_0, N)$, then $\{q_i\}$ is bounded, and by Hopf-Rinow therorem,

  $$\exists\ \{j\} \subset \{i\},\ s.t.\ q_j \to q_0$$

  for some $q_0 \in M$. But N is closed, we have $q_0 \in N$ and $d(p_0, q_0) = d(p_0, N)$.
- Orthogonality.
  Let $\gamma : [0, l] \to M$ be a minimizing geodesic joining p_0 to q_0. We shall show $\gamma'(l) \perp N$, i.e. $\gamma'(0) \perp N$, $\forall v \in T_{q_0}N$.
  Indeed, for $v \in T_{q_0}N$, let $\zeta : (-\varepsilon, \varepsilon) \to M$ be a geodesic with data q_0, v(i.e. $\zeta(0) = q_0, \zeta'(0) = v$) and consider the variation $f : (-\varepsilon, \varepsilon) \times [0, l] \to M$ such that $f(s, 0) = p_0,\ f(s, l) = \zeta(s)$. If we denote by $V(s) = \frac{\partial f}{\partial s}|_{s=0}$, then from the formula for the first variation of energy,

$$\begin{aligned}
0 &= \frac{1}{2}E'(0) \\
&= -\int_0^l \langle V(t), \gamma''(t)\rangle\, dt - \langle V(0), \gamma'(0)\rangle + \langle V(l), \gamma'(l)\rangle \\
&= \langle v, \gamma'(l)\rangle
\end{aligned}$$

□

9.2 Introduce a complete Riemannian metric on $\mathbb{R}^2$. Prove that

$$\lim_{r\to\infty}\left(\inf_{x^2+y^2\geq r^2} K(x,y)\right)\leq 0$$

where $(x,y)\in\mathbb{R}^2$ and $K(x,y)$ is the Gauss curvature of the given metric at (x,y).

Proof. Argue by contradiction. Denote the complete metric on $\mathbb{R}^2$ by g and suppose

$$\lim_{r\to\infty}\left(\inf_{x^2+y^2\geq r^2} K(x,y)\right)> 0$$

Then

$$\exists\begin{cases} c>0 \\ r>0\end{cases} \quad s.t. \inf_{x^2+y^2\geq r^2} K(x,y)\geq c>0$$

Hence by Bonnet-Myers Theorem,

$$\left(\left\{(x,y);\ x^2+y^2\geq r^2\right\},g\right)$$

$(\subset\mathbb{R}^2$, complete) is compact. Thus

$$\mathbb{R}^2=\left\{(x,y);\ x^2+y^2\leq r^2\right\}\cup\left\{(x,y);\ x^2+y^2\geq r^2\right\}$$

as the union of two compact sets, is compact. A contradiction! □

9.3 Prove the following generalization of the Theorem of Bonnet-Myers: Let M^n be a complete Riemannian manifold. Suppose that there exists constants $a>0$ and $c\geq 0$ such that for all pairs of points in M^n and for all minimizing geodesics $\gamma(s)$, parametrized by arc length s, joining these points, we have

$$Ric(\gamma'(s))\geq a+\frac{df}{ds}, \quad \text{along } \gamma$$

where f is a function of s, satisfying $|f(s)|\leq c$ along γ. Then M^n is compact.

Proof. We claim that

$$diam(M)\leq\frac{\pi^2}{\sqrt{c^2+\pi^2 a}-c}\triangleq L$$

Thus by Hopf-Rinow Theorem, M is compact.
Indeed, if not, then

$$\exists\begin{cases} p,q\in M \\ \text{minimizing geodesic } \gamma:[0,l]\to M\end{cases} \quad s.t.\ \gamma \text{ joing } p \text{ to } q \text{ with } l(\gamma)=l>L$$

Now choose a parallel orthonormal field

$$e_1(s),\cdots,e_{n-1}(s),e_n(s)=\gamma'(s)$$

along γ, and consider the proper variations V_j defined by

$$V_j(s)=\sin\frac{\pi s}{l}, \quad j=1,\cdots,n-1$$

Then from the formula for the second variation of energy,

$$\begin{aligned}\frac{1}{2}E''(V_j)(0) &= \int_0^l \left\langle V_j, V_j'' + R(\gamma', V_j)\gamma'\right\rangle ds \\ &= \int_0^l \sin^2\frac{\pi s}{l}\left[\frac{\pi^2}{l^2} - K_{\gamma(s)}(\gamma', e_j)\right] ds\end{aligned}$$

Summing j over $\{1, \cdots, n-1\}$, we get

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{n-1} E''(V_j)(0) &= \int_0^l \sin^2\frac{\pi s}{l}\left[\frac{(n-1)\pi^2}{l} - (n-1)Ric(\gamma')\right] ds \\ &\leq (n-1)\int_0^l \sin^2\frac{\pi s}{l}\left[\frac{\pi^2}{l} - a - \frac{df}{ds}\right] ds \\ &= (n-1)\left[\left(\frac{\pi^2}{l} - a\right)\frac{l}{2} + \int_0^l \sin\frac{2\pi s}{l}\cdot\frac{\pi}{l}\cdot f\ ds\right] \\ &\leq (n-1)\left[\frac{\pi^2}{2l} - \frac{al}{2} + \frac{c\pi}{l}\cdot\frac{2l}{\pi}\right] \\ &= -\frac{n-1}{2l}\left[al^2 - 2cl - \pi^2\right] \\ &< 0\end{aligned}$$

As a result,

$$\exists\ j,\ s.t.\ E''(V_j)(0) < 0$$

which contradicts the fact that γ is minimizing. □

Remark The theorem above has application to Relativity, see G.J.Galloway, "A generalization of Myer's Theorem and an application to relativistic cosmology", J.Diff. Geometry, 14(1979), 105-116

9.4 Let M be an orientable Riemannian manifold with positive (sectional) curvature and even dimension. Let γ be a closed geodesic in M, that is, γ is an immersion of the circle S^1 in M that is geodesic at all of its points. Prove that γ is homotopic to a closed curve whose length is strictly less than that of γ.

Proof. We have only to show $\exists$ a variation field V along γ such that $E''_V(0) < 0$(the second variation of energy concerning V).
Indeed, since M is orientable, if we denote by P_γ the parallel transport along γ, then

- P_γ is an isometry $\Rightarrow$ $\det P_\gamma = \pm 1$;
- P_γ preserves orientation $\Rightarrow$ $\det P_\gamma = 1$;
- $P_\gamma(\gamma'(0)) = \gamma'(2\pi) = \gamma'(0)$ $\Rightarrow$ P_γ leaves some $v(\perp \gamma'(0))$ invariant!

Thus, we may choose variation field $V(t) = P_\gamma(v)$, and by the formula for the second variation of energy,

$$\begin{aligned} \frac{1}{2}E''_V(0) &= -\int_0^{2\pi} \langle V, V'' + R(\gamma', V)\gamma' \rangle \, dt \\ &= -|v|^2| \cdot \gamma'(0)|^2 \cdot \int_0^{2\pi} K_{\gamma(t)}(v(t), \gamma'(t))dt \\ &< 0 \end{aligned}$$

as asserted. □

Remark Note that in this settting, in the formula for the second variation of energy, the last four terms offset! Just because we consider closed geodesic...

9.5 Let N_1 and N_2 be two close disjoint submanifolds of a compact Riemannnian manifold M.

a) Show that the distance between N_1 and N_2 is assumed by a geodesic γ perpendicular to both N_1 and N_2.

b) Show that, for any orthogonal variation $h(t, s)$ of γ, with $h(0, s) \in N_1$ and $h(l, s) \in N_2$, we have the following expression for the formula for the second variation

$$\frac{1}{2}E''(0) = I_l(V, V) + \left\langle V(l), S^{(2)}_{\gamma'(l)}(V(l)) \right\rangle - \left\langle V(0), S^{(1)}_{\gamma'(0)}(V(0)) \right\rangle$$

where V is the variational vector and $S^{(i)}_{\gamma'}$ is the linear map associated to the second fundamental form of N_i in the direction γ', $i = 1, 2$.

Proof. a) Let $\{p_i\} \subset N_1, \{q_i\} \subset N_2$ be such that $d(p_i, q_i) \to d(N_1, N_2)$. Since M is compact, we can find (common) $\{j\} \subset \{i\}$, *s.t.*

$$p_j \to p \in N_1, \quad q_j \to q \in N_2$$

then

$$d(p, q) = d(N_1, N_2)$$

Since d is continuous.

b) Now let $\gamma : [0, l] \to M$ be a minimizing geodesic joining p to q, then

$$\gamma'(0) \perp T_pN_1, \quad \gamma'(l) \perp T_qN_2$$

from the result of Exercise 1.

b)

$$\begin{aligned}
\frac{1}{2}E''(0) &= I_l(V,V) - \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt} \right\rangle(0,0) + \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt} \right\rangle(0,a) \\
&\quad - \left\langle V(0), \frac{DV}{dt}(0) \right\rangle + \left\langle V(a), \frac{DV}{dt}(a) \right\rangle \\
&= I_l(V,V) - \left\langle B\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right), \frac{d\gamma}{dt} \right\rangle(0,0) + \left\langle B\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right), \frac{d\gamma}{dt} \right\rangle(0,a) \\
&\quad (\text{ by } a) \text{ and orthogonality of } h) \\
&= I_l(V,V) - \left\langle S_{\gamma'(0)}(V(0)), V(0) \right\rangle + \left\langle S_{\gamma'(l)}(V(l)), V(l) \right\rangle
\end{aligned}$$

□

10 The Rauch Comparison Theorem

10.3 Let M be a complete Riemannian manifold with non-positive sectional curvature. Prove that

$$|(d\exp_p)_v(w)| \geq |w|$$

for all $p \in M$, all $v \in T_pM$ and all $w \in T_v(T_pM)$.

Proof. Let $\tilde{M} = (T_pM = \mathbb{R}^n, \delta_{ij})$ and

- $\tilde{\gamma}(t) = tv,\ \gamma(t) = \exp_p(tv)$;
- $\tilde{J}(t) = tw,\ J(t) = (d\exp_p)_{tv}(tw)$.

Then by Rauch Comparison Theorem, using $K_M \leq 0$, that

$$|d(\exp_p)_v(w)| \geq |w|$$

□

10.5 (The Sturm Comparison Theorem). In this exercise we present a direct proof of Rauch's Theorem in dimension two, without using material from the present chapter. We will indicate a proof of the Theorem of Sturm mentioned in the Introduction to the chapter. Let

$$\begin{cases} f''(t) + K(t)f(t) = 0,\ f(0) = 0,\ t \in [0,l]; \\ \tilde{f}''(t) + \tilde{K}(t)\tilde{f}(t) = 0,\ \tilde{f}(0) = 0,\ t \in [0,l]. \end{cases}$$

be two ordinary differential equations. Suppose that $\tilde{K}(t) \geq K(t)$ for $t \in [0,l]$, and that $f'(0) = \tilde{f}'(0) = 1$.

a) Show that for all $t \in [0,l]$,

$$(1)\ 0 = \int_0^t \{\tilde{f}(f'' + Kf) - f(\tilde{f}'' + \tilde{K}\tilde{f})\}dt = [\tilde{f}f' - f\tilde{f}']_0^t + \int_0^t (K - \tilde{K})f\tilde{f}dt$$

Conclude from this that the first zero of f does not occur before the first zero of $\tilde{f}$.

b) Suppose that $\tilde{f}(t) > 0$ on $(0, l]$. Use (1) and the fact that $f(t) > 0$ on $(0, l]$ to show that $f(t) \geq \tilde{f}(t),\ t \in [0, l]$, and that the equality is verified for $t = t_1 \in (0, l]$ if and only if $K(t) = \tilde{K}(t),\ t \in [0, t_1]$.
Verify that this is the Theorem of Rauch in dimension two.

Proof. a) First, integration by parts gives (1). Second, we prove that the first zero of f does not occur before the first zero of $\tilde{f}$. Indeed, if $t \in (0, l]$ is such that

$$\tilde{f}(t) > 0 \text{ on } (0, t_0), \qquad \tilde{f}(t_0) = 0$$

and if $f(t_1) = 0$ for some $t_1 \in (0, t_0)$, then

$$\tilde{f}(t_1) > 0, \qquad f'(t_1) < 0$$

contradicting (1) with t replaced by t_1.

b) We know from (1) that

$$\tilde{f}f' - f\tilde{f}' \geq 0$$

i.e.

$$\frac{f'}{f} \geq \frac{\tilde{f}'}{\tilde{f}}$$

$$(\ln f)' \geq (\ln \tilde{f})'$$

Integrating from t_0 to t ($0 < t_0 < t \leq l$), we obtain

$$\ln f(t) - \ln f(t_0) \geq \ln \tilde{f}(t) - \ln \tilde{f}(t_0)$$

$$\ln \frac{f(t)}{\tilde{f}(t)} \geq \ln \frac{f(t_0)}{\tilde{f}(t_0)}$$

$$\frac{f(t)}{\tilde{f}(t)} \geq \frac{f(t_0)}{\tilde{f}(t_0)}$$

But

$$\lim_{t_0 \to 0} \frac{f(t_0)}{\tilde{f}(t_0)} = \lim_{t_0 \to 0} \frac{f'(t_0)}{\tilde{f}'(t_0)} = 1$$

we've

$$f(t) \geq \tilde{f}(t)$$

as required.
And if the equality is valid for some $t = t_1 \in (0, l]$,then

$$f(t) = \tilde{f}(t), \qquad \forall t \in [0, t_1]$$

(Otherwise, $\exists\ t^* \in (0, t_1)$ satisfies $f(t^*) > \tilde{f}(t^*)$, then

$$1 = \frac{f(t_1)}{\tilde{f}(t_1)} \geq \frac{f(t^*)}{\tilde{f}(t^*)} > 1$$

A contradiction!)
Thus

$$f'(t_1) = 0 = \tilde{f}'(t_1)$$

Hence by (1),

$$0 = \int_0^{t_1} (K - \tilde{K}) f^2 dt$$

$$K = \tilde{K}, \qquad \forall t \in [0, t_1]$$

($f > 0, \forall t \in (0, l]$ and continuity of the K's).

□

11 The Morse Index Theorem

11.2 Prove the following inequality on real functions (Wirtinger's inequality). Let $f : [0, \pi] \to \mathbb{R}$ be a real function of class C^2 such that $f(0) = 0 = f(\pi)$. Then

$$\int_0^{\pi} f^2 dt \le \int_0^{\pi} (f')^2 dt$$

and equality occurs if and only if $f(t) = c \sin t$, where c is a constant.

Proof. Let $\gamma : [0, \pi] \to S^2$ be a normalized geodesic joining $\gamma(0) = p$ to $\gamma(\pi) = -p$, and let v be a parallel field along γ with $< v, \gamma' >= 0,\ |v| = 1$. Set $V = fv$, then

$$\begin{aligned} 0 &\le I_\pi(V, V) \text{ (Morse Index Theorem)} \\ &= \int_0^{\pi} \left\{ |f'|^2 - |f|^2 \right\} dt \ (K_{S^2} = 1) \end{aligned}$$

as required. And

$$\begin{aligned} \text{equality occurs} &\Leftrightarrow V \text{ is a Jacobi field. } (f(0) = 0 = f(\pi), n = 2) \\ &\Leftrightarrow f'' + f = 0 \ (K_{S^2} = 1) \\ &\Leftrightarrow f = c \sin t \ (f(0) = 0 = f(\pi)) \end{aligned}$$

□

11.4 Let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $a(t) \ge 0, t \in \mathbb{R}$, and $a(0) > 0$. Prove that the solution to the differential equation

$$\frac{d^2\varphi}{dt^2} + a\varphi = 0$$

with initial conditions $\varphi(0) = 1, \varphi'(0) = 0$, has at least one positive zero and one negative zero.

Proof. We need only to prove φ has at least one positive zero, the other assertion being similar. Argue by contradiction, if

$$t \in (0, \infty) \Rightarrow \varphi(t) > 0$$

then

$$\varphi'' = -a\varphi \le 0$$

i.e.

$$\varphi' \text{ is non-increasing}$$

But now, $a(0) > 0, \varphi(0) = 1$,

$$\varphi''(0) = -a(0)\varphi(0) < 0$$

$$\exists \varepsilon > 0, \ s.t. \ t \in (0, \varepsilon] \Rightarrow \varphi''(t) < 0 \Rightarrow \varphi'(t) < \varphi'(0) = 0$$

Thus

$$\begin{aligned} \varphi(T) &= \varphi(0) + \int_0^T \varphi'(t)dt \\ &= 1 + \int_0^\varepsilon \varphi'(t)dt + \int_\varepsilon^T \varphi'(t)dt \\ &< 1 + \int_\varepsilon^T \varphi'(t)dt \\ &\leq 1 + \varphi'(\varepsilon)(T - \varepsilon) \\ &< 0 \end{aligned}$$

if T is large enough. A contradiction! □

11.5 Suppose M^n is complete Riemannian manifold with sectional curvature strictly positive and let $\gamma : (-\infty, \infty) \to M$ be a normalized geodesic in M. Show that there exists $t_0 \in \mathbb{R}$ such that the segment $\gamma([-t_0, t_0])$ has index greater or equal to $n - 1$.

Proof. Let Y be a parallel field along γ with $< Y, \gamma' >= 0, |Y| = 1$. Set

$$\varphi_Y = \langle R(\gamma', Y)\gamma', Y \rangle$$

$$K(t) = \inf_Y \varphi_Y(t) > 0$$

and let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$0 \leq a(t) \leq K(t), \quad 0 < a(0) < K(0), \quad t \in \mathbb{R}$$

Let φ be the solution of the system

$$\begin{cases} \varphi'' + a\varphi = 0 \\ \varphi(0) = 1, \varphi'(0) = 0 \end{cases}$$

and let $-t_1, t_2$ be the two zeros of this system. If we denote by $X = \varphi Y$, then

$$\begin{aligned}
& I_{[-t_1,t_2]}(X,X) \\
= & \int_{-t_1}^{t_2} \{\langle X', X'\rangle - \langle R(\gamma', X)\gamma', X\rangle\}\, dt \\
= & -\int_{-t_1}^{t_2} \langle X'' + R(\gamma', X)\gamma', X\rangle\, dt \ (\varphi(-t_1) = 0 = \varphi(t_2)) \\
= & -\int_{-t_1}^{t_2} [\varphi''\varphi + \varphi^2 \varphi_Y] dt \\
\le & -\left(\int_{-t_1}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{-\varepsilon}^{t_2}\right) [\varphi''\varphi + \varphi^2 K] dt \\
< & -\left(\int_{-t_1}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{-\varepsilon}^{t_2}\right) [\varphi''\varphi + \varphi^2 a] \ (K(0) > a(0), K(t) \ge a(t)) \\
= & -\int_{-t_1}^{t_2} [\varphi'' + a\varphi]\varphi dt \\
= & \ 0
\end{aligned}$$

Thus if $t_0 = max\{t_1, t_2\}$, then

$$Index\left(\gamma|_{[-t_0,t_0]}\right) \ge Index\left(\gamma|_{[-t_1,t_2]}\right) \ge n-1 \ (t_0 = t_1 \text{ or } t_2)$$

□

11.6 A line in a complete Riemannian manifold is a geodesic

$$\gamma : (-\infty, \infty) \to M$$

which minimizes the arc length between any two of its points. Show that if the sectional curvature K of M is strictly positive, M does not have any lines. By an example show that the theorem is false if $K \ge 0$.

Proof. Of course, we take $n \ge 2$. By Exercise 5,

$$\exists\ t_0 \in \mathbb{R},\ \exists\ X \in \mathfrak{V}(-t_0, t_0),\ s.t.\ I_{[-t_0,t_0]}(X,X) < 0$$

Then by the formula for the second variation of energy,

$$\gamma|_{[-t_0,t_0]} \text{ is not minimizing}$$

Thus M does not have any rays.
If $K \ge 0$, the theorem is false, because any "line" is Euclidean flat space $(\mathbb{R}^n, \delta_{ij})$ is indeed a line! □

Concluding Remarks—Lobatchevski Geometry

- 1.4
  As a Lie group, endowed with left-invariant metric, the isometry of which...
- 2.8
  The Christoffel symbols, a beautiful parallel field...
- 7.10
  As a complete manifold, all the geodesics are calculated...
- 8.1
  Some extensions...