§17 Monotonicity Formula And Basic Consequences

In this section we assume that U is open in $\mathbb{R}^{n+k}$, $V = \underline{\underline{v}}(M,\theta)$ has the generalized mean curvature $\underline{\underline{H}}$ in U (see 16.5), and we write μ for μ_V ($= H^n \angle \theta$ as in 15.1).

Our aim is to obtain information about V by making appropriate choice of X in the formula (see 16.5)

17.1 $$\int div_M X \, d\mu = -\int X \bullet \underline{\underline{H}} \, d\mu, \quad X \in C_c^1\left(U; \mathbb{R}^{n+k}\right)$$

First we choose $X_x = \gamma(r)(x-\xi)$, where $\xi \in U$ is fixed, $r = |x-\xi|$, and γ is a $C^1(\mathbb{R})$ function with

$$\gamma'(t) \le 0 \quad \forall t; \gamma(t) \equiv 1 \quad for\ t \le \rho/2; \gamma(t) \equiv 0 \quad for\ t > \rho$$

where $\rho > 0$ is such that $\bar{B}_\rho(\xi) \subset U$. (Here and subsequently $B_\rho(\xi)$ denotes the open ball in $\mathbb{R}^{n+k}$ with center ξ and radius ρ.)

For any $f \in C^1(U)$ and any $x \in M$ such that $T_x M$ exists (see 11.4 – 11.6) we have (by 12.1) $\nabla^M f(x) = \sum_{j,l=1}^{n+k} e^{jl} D_l f(x) e_j$, where $D_l f(x)$ denotes the partial derivative $\dfrac{\partial f}{\partial x^l}$ of f taken in U and where $\left(e^{jl}\right)$ is the matrix of the orthogonal projection of $\mathbb{R}^{n+k}$ onto $T_x M$. Thus writing $\nabla_j^M = e_j \bullet \nabla^M$ (as in

Section 16), with the above choice of X we deduce

$$div_M X=\sum_{j=1}^{n+k}\nabla_j^M X_j=\gamma(r)\sum_{j=1}^{n+k}e^{jj}+r\gamma'(r)\sum_{j,l=1}^{n+k}\frac{x^j-\xi^j}{r}\bullet\frac{x^l-\xi^l}{r}e^{jl}$$

More precise,

$$\nabla^M f(x)=P_{T_xM}\left(grad_{\mathbb{R}^{n+k}}f(x)\right)=P_{T_xM}\left(\sum_{l=1}^{n+k}D_l f(x)e_l\right)$$
$$=\sum_{l=1}^{n+k}D_l f(x)P_{T_xM}e_l=\sum_{j,l=1}^{n+k}D_l f(x)e^{lj}e_j=\sum_{j,l=1}^{n+k}e^{jl}D_l f(x)e_j$$

and

$$div_M X=\sum_{j=1}^{n+k}\nabla_j^M X^j$$
$$=\sum_{j=1}^{n+k}e_j\bullet\nabla^M X^j$$
$$=\sum_{j=1}^{n+k}e_j\bullet\sum_{k,l=1}^{n+k}e^{kl}D_lX^je_k$$
$$=\sum_{j,l=1}^{n+k}e^{jl}D_lX^j$$
$$=\sum_{j,l=1}^{n+k}e^{jl}\left[\gamma'(r)\frac{x^l-\xi^l}{r}\bullet\left(x^j-\xi^j\right)+\gamma(r)\delta_l^j\right]$$
$$=r\gamma'(r)\sum_{j,l=1}^{n+k}e^{jl}\frac{x^l-\xi^l}{r}\bullet\frac{x^j-\xi^j}{r}+\gamma(r)\sum_{j=1}^{n+k}e^{jj}$$
$$=r\gamma'(r)\left\langle\sum_{j=1}^{n+k}\left(\sum_{l=1}^{n+k}e^{jl}\frac{x^l-\xi^l}{r}\right)e_j,\sum_{j=1}^{n+k}\frac{x^j-\xi^j}{r}e_j\right\rangle+n\gamma(r)$$
$$=r\gamma'(r)\left\langle P_{T_xM}Dr,Dr\right\rangle+n\gamma(r)$$
$$=r\gamma'(r)\left[1-\left|D^{\perp}r\right|^2\right]+n\gamma(r)$$

Since $\left(e^{jl}\right)$ represents the orthogonal projection onto T_xM

we have $\sum_{j=1}^{n+k} e^{jj} = n$ and $\sum_{j,l=1}^{n+k} \frac{x^j - \xi^j}{r} \bullet \frac{x^l - \xi^l}{r} e^{jl} = 1 - \left|D^{\perp} r\right|^2$, where $D^{\perp} r$ denotes the orthogonal projection of Dr (which is a vector of length $= 1$) onto $(T_x M)^{\perp}$.

The formula 17.1 thus yields

(*) $$\begin{aligned} & n \int \gamma(r)\, d\mu + \int r\gamma'(r)\, d\mu \\ & = -\int \underline{\underline{H}} \bullet (x - \xi)\, \gamma(r)\, d\mu + \int r\gamma'(r) \left|D^{\perp} r\right|^2 d\mu \end{aligned}$$

provided $\bar{B}_{\rho}(\xi) \subset U$. Now take ϕ such that $\phi(t) \equiv 1$ for $t \le \frac{1}{2}$, $\phi(t) = 0$ for $t \ge 1$ and $\phi'(t) \le 0$ for all t. Then we can use (*) with $\gamma(r) = \phi\left(\frac{r}{\rho}\right)$. Since $r\gamma'(r) = \frac{r}{\rho}\phi'\left(\frac{r}{\rho}\right) = -\rho \frac{\partial}{\partial \rho} \phi\left(\frac{r}{\rho}\right)$ this gives

$$nI(\rho) - \rho I'(\rho) = -\rho J'(\rho) - L(\rho)$$

where

$$I(\rho) = \int \phi\left(\frac{r}{\rho}\right) d\mu,\ L(\rho) = \int \phi\left(\frac{r}{\rho}\right) (x - \xi) \bullet \underline{\underline{H}}\, d\mu$$

$$J(\rho) = \int \phi\left(\frac{r}{\rho}\right) \left|D^{\perp} r\right|^2 d\mu$$

Thus, multiply by ρ^{-n-1} and rearranging we have

17.2 $$\frac{d}{d\rho}\left[\frac{I(\rho)}{\rho^n}\right] = \frac{J'(\rho)}{\rho^n} - \frac{L(\rho)}{\rho^{n+1}}$$

Thus letting ϕ increase to the characteristic function of the interval $(-\infty, -1)$, we obtain, in the distribution sense, 17.3 $$\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]=\frac{d}{d\rho}\int_{B_\rho(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu+\frac{1}{\rho^{n+1}}\int_{B_\rho(\xi)}(x-\xi)\bullet\underline{\underline{H}}\,d\mu$$

This is the fundamental monotonicity identity. Since $\mu\left(B_\rho(\xi)\right)$ and $\int_{B_\rho(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu$ are increasing in ρ it also holds in the classical sense for $a.e.\ \rho>0$ such that $\overline{B}_\rho(\xi)\subset U$.

More precise, $\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]=\frac{\mu_\rho^{'}\left(B_\rho(\xi)\right)\rho^n-\mu\left(B_\rho(\xi)\right)n\rho^{n-1}}{\rho^{2n}}$

and

$$\begin{aligned}&\frac{1}{\rho^n}\frac{d}{d\rho}\int_{B_\rho(\xi)}\left|D^\perp r\right|^2d\mu\\&=\frac{1}{\rho^n}\lim_{m\to\infty}\frac{\int_{B_\rho(\xi)-B_{\rho-\frac{\rho}{m}}(\xi)}\left|D^\perp r\right|^2d\mu}{\frac{1}{m}}\\&=\lim_{m\to\infty}\frac{\int_{B_\rho(\xi)-B_{\rho-\frac{1}{m}}(\xi)}\left(\frac{r}{\rho}\right)^n\frac{\left|D^\perp r\right|^2}{r^n}d\mu}{\frac{1}{m}}\end{aligned}$$

where we can estimate the last quantity as follows,

$$\begin{aligned}&\left(1-\frac{1}{m}\right)^n\frac{\int_{B_\rho(\xi)-B_{\rho-\frac{\rho}{m}}(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu}{\frac{1}{m}}\\&\le\frac{\int_{B_\rho(\xi)-B_{\rho-\frac{\rho}{m}}(\xi)}\left(\frac{r}{\rho}\right)^n\frac{\left|D^\perp r\right|^2}{r^n}d\mu}{\frac{1}{m}}\le\frac{\int_{B_\rho(\xi)-B_{\rho-\frac{1}{m}}(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu}{\frac{1}{m}}\end{aligned}$$

Notice that if $\underset{=}{H} \equiv 0$ then 17.3 tells us that the ratio $\dfrac{\mu(B_\rho(\xi))}{\rho^n}$ is non-decreasing in ρ. Generally, by integrating with respect to ρ in 17.3 we get the identity

17.4
$$\frac{\mu(B_\sigma(\xi))}{\sigma^n} = \frac{\mu(B_\rho(\xi))}{\rho^n} - \int_{B_\rho(\xi)-B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n} d\mu - \frac{1}{n}\int (x-\xi)\bullet \underset{=}{H}\left(\frac{1}{r_\sigma^n} - \frac{1}{\rho^n}\right) d\mu$$

for all $0 < \sigma \le \rho$ with $\bar{B}_\rho(\xi) \subset U$, where $r_\sigma = \max\{r, \sigma\}$, so that if $\underset{=}{H} = 0$ we have the particularly interesting identity

17.5
$$\frac{\mu(B_\sigma(\xi))}{\sigma^n} = \frac{\mu(B_\rho(\xi))}{\rho^n} - \int_{B_\rho(\xi)-B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n} d\mu$$

More precise, integrating 17.3 with respect to ρ, we get

$$\begin{aligned}
&\frac{\mu(B_\rho(\xi))}{\rho^n} - \frac{\mu(B_\sigma(\xi))}{\sigma^n} \\
&= \int_{B_\rho(\xi)-B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n} d\mu + \int_\sigma^\rho \frac{1}{\tau^{n+1}} \int_{B_\tau(\xi)} (x-\xi)\bullet \underset{=}{H}\, d\mu d\tau \\
&= \int_{B_\rho(\xi)-B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n} d\mu + \int_\sigma^\rho \frac{1}{\tau^{n+1}} \int_{B_\rho(\xi)} \chi_{B_\tau(\xi)}(x)(x-\xi)\bullet \underset{=}{H}\, d\mu d\tau \\
&= \int_{B_\rho(\xi)-B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n} d\mu + \int_{B_\rho(\xi)} (x-\xi)\bullet \underset{=}{H} \int_\sigma^\rho \frac{\chi_{B_\tau(\xi)}(x)}{\tau^{n+1}} d\tau d\mu \\
&= \int_{B_\rho(\xi)-B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n} d\mu + \int_{B_\rho(\xi)} (x-\xi)\bullet \underset{=}{H} \int_{\max\{\sigma,r\}}^\rho \frac{1}{\tau^{n+1}} d\tau d\mu \\
&= \int_{B_\rho(\xi)-B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n} d\mu + \frac{1}{n}\int_{B_\rho(\xi)} (x-\xi)\bullet \underset{=}{H}\left(\frac{1}{r_\sigma^n} - \frac{1}{\rho^n}\right) d\mu
\end{aligned}$$

We now want to examine the important question of what 17.3

tells us in case we assume boundedness and L^p-conditions on $\underset{=}{H}$.

17.6 Theorem If $\xi \in U$, $0<\alpha\le 1$, $\Lambda \ge 0$, and if

(*) $$\frac{1}{\alpha}\int_{B_\rho(\xi)}\left|\underset{=}{H}\right| d\mu \le \Lambda\left(\frac{\rho}{R}\right)^{\alpha-1}\mu\left(B_\rho(\xi)\right) \quad \text{for all } \rho\in(0,R)$$

where $\bar{B}_R(\xi)\subset U$, then $e^{\Lambda R^{1-\alpha}\rho^\alpha}\dfrac{\mu\left(B_\rho(\xi)\right)}{\rho^n}$ is a non-decreasing function of $\rho\in(0,R)$, and in fact

(1) $$e^{\Lambda R^{1-\alpha}\sigma^\alpha}\frac{\mu\left(B_\sigma(\xi)\right)}{\sigma^n}\le e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}-\int_{B_\rho(\xi)-B_\sigma(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu$$

Whenever $0<\sigma<\rho\le R$. Also,

(2) $$e^{-\Lambda R^{1-\alpha}\sigma^\alpha}\frac{\mu\left(B_\sigma(\xi)\right)}{\sigma^n}\ge e^{-\Lambda R^{1-\alpha}\rho^\alpha}\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}-\int_{B_\rho(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu$$

Proof From 17.3, multiply by the integrating factor $e^{\Lambda R^{1-\alpha}\rho^\alpha}$ we get

$$\begin{aligned}
&\frac{d}{d\rho}\int_{B_\rho(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu \le e^{\Lambda R^{1-\alpha}\rho^\alpha}\int_{B_\rho(\xi)}\frac{\left|D^\perp r\right|^2}{r^n}d\mu\\
&= e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]-e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{1}{\rho^{n+1}}\int_{B_\rho(\xi)}(x-\xi)\bullet\underset{=}{H}\,d\mu\\
&\le e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]+e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{1}{\rho^{n}}\int_{B_\rho(\xi)}\left|\underset{=}{H}\right|d\mu\\
&\le e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]+e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{1}{\rho^{n}}\alpha\Lambda\left(\frac{\rho}{R}\right)^{\alpha-1}\mu\left(B_\rho(\xi)\right)\\
&\le e^{\Lambda R^{1-\alpha}\rho^\alpha}\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]+\left[e^{\Lambda R^{1-\alpha}\rho^\alpha}\Lambda R^{1-\alpha}\alpha\rho^{\alpha-1}\right]\bullet\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]
\end{aligned}$$

$$= \frac{d}{d\rho}\left[e^{\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right]$$

Thus integrate with respect to ρ on the interval (σ, ρ), (1) follows,

$$\int_{B_\rho(\xi)} \frac{\left|D^\perp r\right|^2}{r^n} d\mu \le e^{\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} - e^{\Lambda R^{1-\alpha}\sigma^{\alpha}} \frac{\mu\left(B_\sigma(\xi)\right)}{\sigma^n}$$

For (2), we just use the same method,

$$\frac{d}{d\rho}\int_{B_\rho(\xi)} \frac{\left|D^\perp r\right|^2}{r^n} d\mu \ge e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{d}{d\rho}\int_{B_\rho(\xi)} \frac{\left|D^\perp r\right|^2}{r^n} d\mu$$

$$= e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right] - e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{1}{\rho^{n+1}} \int_{B_\rho(\xi)} (x-\xi)\bullet \underset{=}{H}\, d\mu$$

$$\ge e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right] - e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{1}{\rho^{n+1}} \int_{B_\rho(\xi)} r\left|\underset{=}{H}\right| d\mu$$

$$\ge e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right] - e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{1}{\rho^{n+1}} \alpha\Lambda\left(\frac{\rho}{R}\right)^{\alpha-1} \mu\left(B_\rho(\xi)\right)$$

$$= e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right] + \left[e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \left(-\Lambda R^{1-\alpha}\alpha\rho^{\alpha-1}\right)\right]\bullet\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right]$$

$$= \frac{d}{d\rho}\left[e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right]$$

17.7 Theorem If $\xi \in U$ and $\left(\int_{B_R(\xi)} \left|\underset{=}{H}\right|^p d\mu\right)^{\frac{1}{p}} \le \Gamma$, where $\overline{B}_R(\xi) \subset U$ and $p > n$, then

$$\left[\frac{\mu\left(B_\sigma(\xi)\right)}{\sigma^n} \right]^{\frac{1}{p}} \le \left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n} \right]^{\frac{1}{p}} + \frac{\Gamma}{p-n}\left(\rho^{1-\frac{n}{p}} - \sigma^{1-\frac{n}{p}} \right)$$

whenever $0 < \sigma < \rho \le R$.

Proof From 17.3, we get

$$\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right] \ge \frac{1}{\rho^{n+1}}\int_{B_\rho(\xi)}(x-\xi)\bullet \underset{=}{H}\,d\mu$$

$$\ge -\frac{1}{\rho^{n+1}}\int_{B_\rho(\xi)}|x-\xi|\bullet\left|\underset{=}{H}\right|d\mu$$

$$\ge -\frac{1}{\rho^n}\left(\int_{B_\rho(\xi)}\left|\underset{=}{H}\right|^p d\mu\right)^{\frac{1}{p}}\mu\left(B_\rho(\xi)\right)^{1-\frac{1}{p}}$$

$$\ge -\Gamma\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]^{1-\frac{1}{p}}\rho^{-\frac{n}{p}}$$

That is,

$$\frac{d}{d\rho}\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]^{\frac{1}{p}} \ge -\frac{\Gamma}{p}\rho^{-\frac{n}{p}}$$

Integrating with respect to ρ, we have

$$\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}-\frac{\mu\left(B_\sigma(\xi)\right)}{\sigma^n} \ge -\frac{\Gamma}{p-n}\left(\rho^{1-\frac{n}{p}}-\sigma^{1-\frac{n}{p}}\right)$$

17.8 **Corollary** If $\underset{=}{H} \in L^p_{loc}(\mu)$ in U for some $p > n$, then the density $\Theta^n(\mu,x) = \lim_{\rho\downarrow 0}\frac{\mu\left(\overline{B}_\rho(x)\right)}{\omega_n\rho^n}$ exists at every point $x \in U$, and $\Theta^n(\mu,\bullet)$ is an upper-semi-continuous function in U:

$$\Theta^n(\mu,x) \ge \limsup_{y\to x}\Theta^n(\mu,y), \quad \forall x \in U$$

Proof The inequality 17.7 tells us that $\left[\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\right]^{\frac{1}{p}} + \frac{\Gamma}{p-n}\rho^{1-\frac{n}{p}}$

is a non-decreasing function of ρ, hence $\lim_{\rho\downarrow 0}\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}$ exists (and is the same as $\lim_{\rho\downarrow 0}\frac{\mu\left(\overline{B}_\rho(\xi)\right)}{\rho^n}$). [$\lim_{\rho\downarrow 0}\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\leq\lim_{\rho\downarrow 0}\frac{\mu\left(\overline{B}_\rho(\xi)\right)}{\rho^n}$

$$\leq\lim_{\rho\downarrow 0}\frac{\mu\left(B_{\rho\left(1+\frac{1}{m}\right)}(\xi)\right)}{\left[\rho\left(1+\frac{1}{m}\right)\right]^n}\left(1+\frac{1}{m}\right)^n=\lim_{\rho\downarrow 0}\frac{\mu\left(B_\rho(\xi)\right)}{\rho^n}\left(1+\frac{1}{m}\right)^n,\ \forall m\in\mathbb{N}\ ]$$

Now for the proof of corollary,

$$\Theta^n(\mu,x)\geq\limsup_{y\to x}\Theta^n(\mu,y)$$

$$\Leftrightarrow\Theta^n(\mu,x)\geq\lim_{\varepsilon\downarrow 0}\sup_{|y-x|<\varepsilon}\Theta^n(\mu,y)$$

$$\Leftrightarrow\forall\delta\in(0,1),\frac{\Theta^n(\mu,x)}{\delta}>\lim_{\varepsilon\downarrow 0}\sup_{|y-x|<\varepsilon}\Theta^n(\mu,y)$$

$$\Leftrightarrow\forall\delta\in(0,1),\exists\varepsilon>0,s.t.\ \ if\ |y-x|<\varepsilon,\ then\ \Theta^n(\mu,y)<\frac{\Theta^n(\mu,x)}{\delta}$$

$$\Leftrightarrow\forall\delta\in(0,1),\exists\varepsilon>0,s.t.\ \ if\ |y-x|<\varepsilon,\ then\ \left[\Theta^n(\mu,y)\right]^{\frac{1}{p}}<\left[\frac{\Theta^n(\mu,x)}{\delta}\right]^{\frac{1}{p}}$$

Thus we prove the last assertion, to this end, let $\sigma\downarrow 0$, we get

$$\begin{aligned}\left[\Theta^n(\mu,y)\right]^{\frac{1}{p}}&\leq\left[\frac{\mu\left(B_\rho(y)\right)}{\omega_n\rho^n}\right]^{\frac{1}{p}}+\frac{\Gamma}{(p-n)\omega_n^{\frac{1}{p}}}\rho^{1-\frac{n}{p}}\\&\leq\left[\frac{\mu\left(B_{\rho+\varepsilon}(x)\right)}{\omega_n(\rho+\varepsilon)^n}\right]^{\frac{1}{p}}\left[1+\frac{\varepsilon}{\rho}\right]^{\frac{n}{p}}+c(x,p,n)\rho^{1-\frac{n}{p}}\\&<\left[\frac{\Theta^n(\mu,x)}{\delta}\right]^{\frac{1}{p}}\end{aligned}$$

if we take $\varepsilon\ll\rho$ small enough.

17.9 **Remarks**

(1) If $\theta \geq 1 \ \mu - a.e. U$, $\underset{=}{H} \in L^p_{loc}(U)$, $p > n$ as in 17.8, then $\Theta^n(\mu, x) \geq 1$ at each point of $spt\mu \cap U$, and hence we can write $V \angle U = \underset{=}{v}(M_*, \theta_*)$ where $M_* = spt\mu \cap U$, $\theta_* = \Theta^n(\mu, x)$, $x \in U$.

Thus $V \angle U$ is represented in terms of a relatively closed countably $n-$rectifiable set with upper-semi-continuous multiplicity function.

Observe that

$$\mathcal{H}^n(A \cap M) = \int_{A\cap M} d\mathcal{H}^n = \int_{A\cap M} \frac{1}{\theta(x)} \theta(x) d\mathcal{H}^n(x) = \int_{A\cap M} \frac{1}{\theta(x)} d\mu(x)$$

for any $\mathcal{H}^n -$measurable subset of $\mathbb{R}^{n+k}$. $\mu - a.e.$ is equivalent to $\mathcal{H}^n - a.e.$ on M, the Remark is "understandable".

- ✓ For density

$$\Theta^n(\mu, x) = \lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n} = \lim_{\rho \downarrow 0} \frac{\int_{B_\rho(x)\cap M} \theta \ d\mathcal{H}^n}{\omega_n \rho^n}$$
$$= \lim_{\rho \downarrow 0} \frac{\int_{B_\rho(x)\cap N_j} \theta \ d\mathcal{H}^n}{\omega_n \rho^n} \geq 1$$

where N_j is C^1-submanifold of $\mathbb{R}^{n+k}$ with $x \in N_j$, such N_j exists by 11.7.

- ✓ For varifold

$$\mathcal{H}^n\left(M \cap \left(U \sim spt\ \mu\right)\right) = \int_{M\cap U \sim spt\ \mu} d\mathcal{H}^n = \int_{U \sim spt\ \mu} \frac{1}{\theta} d\mu = 0$$

(2) If $\xi \in U$, $\Theta^n(\mu,\xi) \geq 1$, and $\left(\frac{1}{\omega_n}\int_{B_R(\xi)} \left|\underline{\underline{H}}\right|^p d\mu\right)^{\frac{1}{p}} \leq \Gamma\left(1-\frac{n}{p}\right)$,

where $\overline{B}_R(\xi) \subset U$ and $p > n$, then both inequalities 17.6(1), (2)

hold with $\Lambda = 2\Gamma R^{-\frac{n}{p}}$ and $\alpha = 1-\frac{n}{p}$, provided $\Gamma R^{1-\frac{n}{p}} \leq \frac{1}{2}$.

Proof Notice first that

$$\int_{B_\rho(\xi)} \left|\underline{\underline{H}}\right| d\mu \leq \left(\int_{B_\rho(\xi)} \left|\underline{\underline{H}}\right|^p d\mu\right)^{\frac{1}{p}} \mu\left(B_\rho(\xi)\right)^{1-\frac{1}{p}} \leq \Gamma\left(1-\frac{n}{p}\right)\omega_n^{\frac{1}{p}}\mu\left(B_\rho(\xi)\right)^{1-\frac{1}{p}}$$

We estimate $\mu\left(B_\rho(\xi)\right)$ using 17.7 as follows (letting $\sigma \downarrow 0$)

$$1 \leq \left[\frac{\mu\left(B_\rho(\xi)\right)}{\omega_n\rho^n}\right]^{\frac{1}{p}} + \frac{\Gamma\left(1-\frac{n}{p}\right)\omega_n^{\frac{1}{p}}}{p-n}\frac{\rho^{1-\frac{n}{p}}}{\omega_n^{\frac{1}{p}}}$$

$$\leq \left[\frac{\mu\left(B_\rho(\xi)\right)}{\omega_n\rho^n}\right]^{\frac{1}{p}} + \Gamma\rho^{1-\frac{n}{p}} \leq \left[\frac{\mu\left(B_\rho(\xi)\right)}{\omega_n\rho^n}\right]^{\frac{1}{p}} + \frac{1}{2}$$

Then $\left[\mu\left(B_\rho(\xi)\right)\right]^{\frac{1}{p}} \geq \frac{1}{2}\left[\omega_n\rho^n\right]^{\frac{1}{p}}$, hence,

$$\int_{B_\rho(\xi)} \left|\underline{\underline{H}}\right| d\mu \leq \Gamma\left(1-\frac{n}{p}\right)\omega_n^{\frac{1}{p}} \bullet 2 \bullet \left[\omega_n\rho^n\right]^{-\frac{1}{p}} \bullet \mu\left(B_\rho(\xi)\right)$$

$$\leq \left(1-\frac{n}{p}\right)2\Gamma\rho^{-\frac{n}{p}}\mu\left(B_\rho(\xi)\right) = \left(1-\frac{n}{p}\right)\bullet\left(2\Gamma R^{-\frac{n}{p}}\right)\left(\frac{\rho}{R}\right)^{-\frac{n}{p}}\mu\left(B_\rho(\xi)\right)$$

Thus the hypothesis of 17.6 hold with $\Lambda = 2\Gamma R^{-\frac{n}{p}}$ and $\alpha = 1-\frac{n}{p}$.

(3) Notice that either 17.6(1) or 17.7 give bounds of the form $\mu\left(B_\sigma(\xi)\right) \leq \beta\sigma^n$, $0 < \sigma < R$ for suitable constant β. Such an

inequality implies

$$\int_{B_\rho(\xi)} |x-\xi|^{\alpha-n} \, d\mu \le \frac{n\beta\rho^\alpha}{\alpha}$$

for any $\rho \in (0,R)$ and $0<\alpha<n$.

Proof We just apply the follow lemma.

$$\int_{B_\rho(\xi)} |x-\xi|^{\alpha-n} \, d\mu = \int_{B_\rho(\xi)} \left(\frac{1}{|x-\xi|^{n-\alpha}} - \frac{1}{\rho^{n-\alpha}} \right) d\mu + \frac{1}{\rho^{n-\alpha}} \mu\left(B_\rho(\xi)\right)$$

$$= (n-\alpha) \int_{\frac{1}{\rho}}^{\infty} t^{n-\alpha-1} \mu\left\{ x; \frac{1}{|x-\xi|} > t \right\} dt + \frac{1}{\rho^{n-\alpha}} \mu\left(B_\rho(\xi)\right)$$

$$= (n-\alpha) \int_{\frac{1}{\rho}}^{\infty} t^{n-\alpha-1} \mu\left(B_{\frac{1}{t}}(\xi) \right) dt + \frac{1}{\rho^{n-\alpha}} \mu\left(B_\rho(\xi)\right)$$

$$\le (n-\alpha) \int_{\frac{1}{\rho}}^{\infty} t^{n-\alpha-1} \beta \frac{1}{t^n} \, dt + \frac{1}{\rho^{n-\alpha}} \beta\rho^n$$

$$= (n-\alpha)\beta \frac{1}{\alpha} \left(\frac{1}{\rho} \right)^{-\alpha} + \beta\rho^\alpha$$

$$= \frac{n\beta\rho^\alpha}{\alpha}$$

17.10 **Lemma** If X is an abstract space, μ is a measure on X, $\alpha>0$. $f \in L^1(\mu), f \ge 0$ and if $A_t = \left\{ x \in X; f(x) > t \right\}$, then

$$\alpha \int_0^\infty t^{\alpha-1} \mu(A_t) \, dt = \int_{A_{t_0}} f^\alpha \, d\mu$$

More generally

$$\alpha \int_{t_0}^\infty t^{\alpha-1} \mu(A_t) \, dt = \int_{A_{t_0}} \left(f^\alpha - t_0^\alpha \right) d\mu$$

Proof It is a by-product of Fubini's theorem. More precise, we have

$$\alpha\int_{t_0}^{\infty} t^{\alpha-1}\mu(A_t)\,dt = \alpha\int_{t_0}^{\infty} t^{\alpha-1}\left(\int_{A_t} d\mu\right)dt$$
$$= \alpha\int_{t_0}^{\infty} t^{\alpha-1}\left(\int_{A_{t_0}} \chi_{A_t}\,d\mu\right)dt = \alpha\int_{A_{t_0}}\left(\int_{t_0}^{\infty} t^{\alpha-1}\chi_{A_t}\,dt\right)d\mu$$
$$= \int_{A_{t_0}}\left(\int_{t_0}^{f(x)} \alpha t^{\alpha-1}dt\right)d\mu(x) = \int_{A_{t_0}}\left(f^{\alpha} - t_0^{\alpha}\right)d\mu$$

17.11 **Lemma** Suppose $\theta \geq 1$ $\mu - a.e.$ in U, $\underset{=}{H} \in L^p_{loc}(\mu)$ in U for some $p > n$. If the approximate tangent space $T_x V$ (see Section 15) exists at a given point $x \in U$, then $T_x V$ is a "classical" tangent plane for $spt\ \mu$ in the sense that

$$\lim_{\rho\downarrow 0}\left[\sup_{y\in B_\rho(x)\cap spt\ \mu}\frac{dist(y, T_x V)}{\rho}\right] = 0$$

Proof For sufficiently small R (with $B_{2R}(x) \subset U$), 17.7,17.8 (with $\sigma \downarrow 0$) evidently imply

(1) $\quad \mu\left(B_\rho(\xi)\right) \geq \frac{1}{2}\omega_n\rho^n, \quad 0 < \rho < R, \quad \xi \in B_\rho(x) \cap spt\ \mu$

More precise, since $\underset{=}{H} \in L^p_{loc}(\mu)$, we choose $R, \Gamma > 0$ such that $B_{2R}(x) \subset U$ and $\left(\int_{B_{2R}(x)} \left|\underset{=}{H}\right|^p d\mu\right)^{\frac{1}{p}} \leq \Gamma$. Then for any $\xi \in B_\rho(x) \cap spt\ \mu, 0 < \rho < R$, $\left(\int_{B_\rho(\xi)} \left|\underset{=}{H}\right|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_{B_{2R}(x)} \left|\underset{=}{H}\right|^p d\mu\right)^{\frac{1}{p}} \leq \Gamma$, and by virtue of 17.7, 17.8, 17.9(1), we have

$1 \leq \left[\frac{\mu\left(B_\rho(x)\right)}{\omega_n\rho^n}\right]^{\frac{1}{p}} + \frac{\Gamma}{p-n}\rho^{1-\frac{n}{p}}$, thus $\mu\left(B_\rho(x)\right) \geq \frac{1}{2}\omega_n\rho^n$ if

$\left(1-\frac{\Gamma}{p-n}R^{1-\frac{n}{p}}\right)^p \le \frac{1}{2}$, this is possible since we can first fix some Γ for some R_0, then let $R_0 > R \downarrow 0$ with Γ unchanged.

Using this we are going to prove that if $\alpha \in \left(0, \frac{1}{2}\right)$ and $\rho \in (0, R)$, then

(2)
$$\mu\left(B_\rho(x) \sim \left\{y; dist\left(y, T_x V\right) < \varepsilon\rho\right\}\right) < \frac{1}{2}\omega_n \alpha^n \rho^n$$
$$\Rightarrow B_{\frac{\rho}{2}(x)} \cap spt\ \mu \subset \left\{y; dist\left(y, T_x V\right) < (\varepsilon+\alpha)\rho\right\}$$

Indeed if $\xi \in \left[B_{\frac{\rho}{2}}(x) \cap spt\ \mu\right] \cap \left\{y; dist\left(y, T_x V\right) \ge (\varepsilon+\alpha)\rho\right\}$, then $B_{\alpha\rho}(\xi) \subset B_\rho(x) \sim \left\{y; dist\left(y, T_x V\right) < \varepsilon\rho\right\}$ and hence the hypothesis of (2) implies $\mu\left(B_{\alpha\rho}(\xi)\right) \le \frac{1}{2}\omega_n \alpha^n \rho^n$. On the other hand, (1) implies $\mu\left(B_{\alpha\rho}(\xi)\right) \ge \frac{1}{2}\omega_n \alpha^n \rho^n$, so we have a contradiction. Thus (2) is proved, and (2) evidently leads immediately to the required result.

More precise, by (5) in the Proof of Theorem 11.8, we've for any $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$\lim_{\rho \downarrow 0} \frac{\mu\left(B_\rho(x) \cap X_{\sqrt{1-\varepsilon^2}}\left(\pi_x, x\right)\right)}{\omega_n \rho^n} = 0$$

[This is cone condition]

where $\pi_x = \left(P_x\right)^\perp$. Thus we can take ρ small enough such that

$$\frac{\mu\left(B_\rho(x)\sim\left\{y;\, dist\left(y,T_xV\right)<\varepsilon\rho\right\}\right)}{\omega_n\rho^n}<\frac{\varepsilon^n}{2}$$

[This is cylinder condition, you may take a picture to see it clearly.]

Hence

$$y\in B_{\frac{\rho}{2}}(x)\cap spt\ \mu\Rightarrow\frac{dist\left(y,T_xV\right)}{\frac{\rho}{2}}<2(\varepsilon+\varepsilon)=4\varepsilon$$