§16 First Variation

Suppose $\{\phi_t\}_{-\varepsilon<t<\varepsilon}(\varepsilon>0)$ is a 1-parameter family of diffeomorphisms of an open set U of $\mathbb{R}^{n+k}$ satisfying

16.1 $$\begin{aligned}&(i)\,\phi_0=\underset{=U}{1},\exists\ compact\ K\subset U\ such\ that\ \phi_t|_{U\sim K}=\underset{=U\sim K}{1},\forall t\in(-\varepsilon,\varepsilon)\\&(ii)\,(x,t)\to\phi_t(x)\ is\ a\ smooth\ map\ U\times(-\varepsilon,\varepsilon)\to U\end{aligned}$$

Then if $V=\underset{=}{v}(M,\theta)$ is a rectifiable n-varifold and if $K\subset U$ is comact as in (i) above, according to 15.7 above,

$$\underset{=}{M}\left(\phi_{t\#}(V\angle K)\right)=\int_{M\cap K}\theta(J_M f)\,d\mathcal{H}^n$$

and we can compute the first variation $\frac{d}{dt}|_{t=0}\ \underset{=}{M}\left(\phi_{t\#}(V\angle K)\right)$ exactly as in Section 9. We thus deduce

16.2 $$\frac{d}{dt}|_{t=0}\ \underset{=}{M}\left(\phi_{t\#}(V\angle K)\right)=\int_M div_M X\ d\mu_V$$

where $X_x=\frac{\partial}{\partial t}|_{t=0}\ \phi_t(x)$ is the initial velocity vector for the family $\{\phi_t\}$ and where $div_M X$ is as in Section 7:

$$div_M X=\nabla_j^M X^j\left(\equiv e_j\bullet\left(\nabla^M X^j\right)\right)$$

($\nabla^M X^j$ as in Section 12) we can therefore make the following definition.

16.3 **Definition** $V=\underset{=}{v}(M,\theta)$ is stationary in U if $\int_U div_M X\ d\mu_V=0$ for any C^1 vector field X on U having compact support in U.

More generally if N is a $(n+k_1)-$dimensional submanifold of $\mathbb{R}^{n+k}$ $(k_1 \le k)$,if U is an open subset of N, if $M \subset N$, and if the family $\{\phi_t\}$ is as in 16.1, then by Lemma 9.6, it is reasonable to make the following definition (in which $\overline{B}$ is the second fundamental form of N)

16.4 **Definition** If $U \subset N$ is open and $M \subset N$ is as above, then we say $V = \underset{=}{v}(M,\theta)$ is stationary in U if

$$\int_U div_M X \; d\mu_V = -\int_U X \bullet \overline{\underset{=}{H}}_M \; d\mu_V$$

whenever X is a C^1 vector field on U with compact support in U; here $\overline{\underset{=}{H}}_M = \sum_{i=1}^{n} \overline{B}_x(\tau_i, \tau_i)$, $\tau_1, \cdots, \tau_n$ any orthonormal basis for the approximate tangent space $T_x M$ of M at x. (Notice that by 16.2, which remains valid when $U \subset N$, this is equivalent to $\frac{d}{dt}|_{t=0} \; \underset{=}{M}\left(\phi_{t\#}(V \angle K)\right) = 0$ whenever $\{\phi_t\}$ are as in 16.1 with $U \subset N$).

It will be convient to generalize these notions of stationarity in the following way:

16.5 **Definition** Suppose $\underset{=}{H}$ is a locally μ_V-integrable function on $M \cap U$ with values in $\mathbb{R}^{n+k}$. We say that $V\left(= \underset{=}{v}(M,\theta)\right)$ has generalized mean curvature $\underset{=}{H}$ in U (U open in $\mathbb{R}^{n+k}$) if

$$\int_U div_M X \, d\mu_V = -\int_U X \bullet \underset{=}{H} \, d\mu_V$$

whenever X is a C^1 vector field on U with compact support in U.

16.6 **Remarks**

(1)Notice that in case M is smooth with $\left(\bar{M} \sim M\right) \cap U = \varnothing$,and when $\theta \equiv 1$, the generalized mean curvature of V is exactly the ordinary mean curvature of M as described in Section 7 (see in particular 7.6).

(2)V is stationary in U (U open in $\mathbb{R}^{n+k}$) in the sense of 16.3 precisely when it has zero generalized mean curvature in U, and V is stationary in U (U open in N) in the sense of 16.4 precisely when it has generalized mean curvature in $\underset{= M}{\bar{H}}$.