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An exactly solvable phase transition model: generalized statistics and generalized Bose–Einstein condensation

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Abstract. In this paper, we present an exactly solvable phase transition model in which the phase transition is purely statistically derived. The phase transition in this model is a generalized Bose–Einstein condensation. The exact expression for the thermodynamic quantity, which can be used to simultaneously describe both the gas phase and the condensed phase, is solved with the help of the homogeneous Riemann–Hilbert problem, so one can judge whether there exists a phase transition and determine the phase transition point mathematically rigorously. A generalized statistics in which the maximum occupation numbers of different quantum states can take on different values is introduced, as a generalization of Bose–Einstein and Fermi–Dirac statistics.

Keywords: rigorous results in statistical mechanics, fractional states (theory), Bose Einstein condensation (theory)

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Contents

1. Introduction

A few exactly solvable models play important roles in phase transition theory, since most, if not all, of our understanding of phase transitions comes from studying models [1]. In this paper, we present a purely statistically derived solvable phase transition model. In the model, the exactly solved thermodynamic quantity can be used to simultaneously describe different phases. Therefore, whether there is a phase transition can be judged mathematically rigorously, and the phase transition temperature can be calculated directly by analyzing the discontinuity in the thermodynamic quantities or their derivatives.

Bose–Einstein condensation (BEC) is the first purely statistically derived example of a phase transition. The phase transition in the present model is a generalized Bose–Einstein condensation; in other words, the phase transition is a BEC type phase transition.

The BEC type phase transition is a sudden change in the microscopic particle distribution: in the gas phase, no quantum state is macroscopically occupied, while in the condensed phase, there is a quantum state being occupied by a macroscopic number of particles. The microscopic particle distribution determines the macroscopic behavior of a thermodynamic system, or the macroscopic behavior reflects the average contribution of all quantum states in the system. In the condensed phase, the macroscopic behavior of the system is to a certain extent determined by the single quantum state that is macroscopically occupied, since the number of particles in such a state is of the same order of magnitude as the total number of particles. As a macroscopic manifestation of such a sudden change in the particle distribution, there is a singularity in the thermodynamic quantity.

We will show that, beyond the Bose case, there are still other systems that can display BEC type phase transitions.

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First, we will introduce a generalized statistics in which different quantum states have different maximum occupation numbers, and Bose–Einstein and Fermi–Dirac statistics are its special cases. In particular, we will pay attention to a special case of the generalized statistics in which at least one quantum state's maximum occupation number is infinite and show that the BEC type phase transition may occur in such systems. For example, we will show that a BEC type phase transition can occur in an ideal gas of any dimension, obeying the generalized statistics in which the maximum occupation number of the ground state is infinity, like that in the Bose–Einstein case, and the maximum occupation number of all other quantum states is 1, like that in the Fermi–Dirac case. For comparison, recall that the BEC can occur only in three-dimensional ideal Bose gases, and cannot occur in one- and two-dimensional cases.

The mathematical method for solving the model is based on the homogeneous Riemann–Hilbert problem—the boundary problem of analytic functions, which comes from the theory of singular integral equations [2, 3].

Moreover, our result also shows that a phase transition occurs only in the thermodynamic limit, i.e., the total number of particles and the volume must be infinite. The common proof for this result depends on an assumption that a finite volume can accommodate at most a finite number of particles, which is, of course, only valid for non-ideal gases [4]. Our result provides an example where this result holds also for ideal gases.

In section 2, we introduce the generalized statistics. In section 3, we construct and solve the phase transition model. Discussions and an outlook are given in section 4.

2. The generalized statistics

In this section, we introduce a generalized statistics in which the maximum occupation number of a quantum state can take on unrestricted integer values or infinity and the maximum occupation numbers of different states may be different.

Let n_i be the maximum occupation number of the ith quantum state, where n_i can take on an integer value or ∞. The grand partition function is

$$\Xi(T,V,\mu)=\prod_{i=0}^{\infty}\frac{1-\mathrm{e}^{-\beta(n_i+1)(\varepsilon_i-\mu)}}{1-\mathrm{e}^{-\beta(\varepsilon_i-\mu)}},\tag{1}$$

where T is the temperature, V the volume, μ the chemical potential, ε_i the energy of the ith state, and $\beta=1/(k_{\mathrm{B}}T)$. Then the equation of state reads

$$\frac{PV}{k_{\mathrm{B}}T}=\sum_{i=0}^{\infty}\ln\frac{1-z^{n_i+1}\mathrm{e}^{-(n_i+1)\beta\varepsilon_i}}{1-z\mathrm{e}^{-\beta\varepsilon_i}},\tag{2}$$

$$N=\sum_{i=0}^{\infty}\left[\frac{1}{z^{-1}\mathrm{e}^{\beta\varepsilon_i}-1}-\frac{n_i+1}{(z^{-1}\mathrm{e}^{\beta\varepsilon_i})^{n_i+1}-1}\right].\tag{3}$$

The equations of state for Bose–Einstein, Fermi–Dirac, and Gentile [5, 6] cases can be recovered by setting $n_i=\infty$, $n_i=1$, and $n_i=n$, respectively.

In this paper, we consider an ideal gas obeying the generalized statistics in which the maximum occupation number of only one quantum state is ∞, but that for all other

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quantum states is n, a given integer, i.e., $n_k = \infty$ and $n_i = n (i \neq k)$. The equation of state for such an ideal gas with the dispersion relation $\varepsilon = p^s/(2m)$ in ν dimensions can be obtained from equations (2) and (3):

$$\frac{PV}{k_B T} = N_\lambda h_{\nu/s+1}(z) - \ln\left[1 - (z\mathrm{e}^{-\beta\varepsilon_k})^{n+1}\right], \tag{4}$$

$$N = N_\lambda h_{\nu/s}(z) + \frac{n+1}{(z^{-1}\mathrm{e}^{\beta\varepsilon_k})^{n+1} - 1}, \tag{5}$$

where $z = \mathrm{e}^{\beta\mu}$ is the fugacity, $N_\lambda = (2\Gamma(\nu/s)/(s\Gamma(\nu/2)))(V/\lambda^\nu)$, and $\lambda = h/(2\pi^{s/2} m k_B T)^{1/s}$ is the mean thermal wavelength. $h_\sigma(z)$ can be expressed using the Bose–Einstein integral, $g_\sigma(z)$, as

$$h_\sigma(z) = g_\sigma(z) - (n+1)^{-(\sigma-1)} g_\sigma(z^{n+1}),$$

and in the limit $n \to \infty$ or $n = 1$, $h_\sigma(z)$ returns to the Bose–Einstein integral $g_\sigma(z)$ or the Fermi–Dirac integral $f_\sigma(z)$, respectively [5].

We will show that such an ideal gas system may display the BEC type phase transition, and whether the phase transition occurs or not rests on the value of k, the position of the state with an infinite maximum occupation number in the spectrum.

3. The phase transition

In this section, we consider two cases which can display BEC type phase transitions: the ideal gases obeying the generalized statistics with $n_0 = \infty$, $n_i = n$ $(i \neq 0)$ and with $n_k = \infty$ $(k \neq 0)$, $n_i = n$ $(i \neq k)$. An interesting case is $n = 1$. In this case, the maximum occupation number of only one state is the same as that in the Bose case, but for all other states it is the same as that in the Fermi case. We will show that even systems in which only one state's maximum occupation number is infinite can still display BEC type phase transitions.

3.1. The explicit expression for the fugacity

For judging whether there is a phase transition or not and determining the phase transition temperature, we first solve the exact explicit expression for the fugacity from the equation of state, and, then, analyze the discontinuity in the derivative of the fugacity.

On the basis of the homogeneous Riemann–Hilbert problem [3], we can solve the explicit expression for the fugacity from equation (5) exactly. (A brief introduction to the method of addressing the Riemann–Hilbert problem see [7].)

For the case of $n_k = \infty$ and $n_i = n$ $(i \neq k)$, introduce a complex function

$$\Psi(\zeta) = \frac{N_\lambda}{N} h_{\nu/s}(\zeta) + \frac{1}{N}\frac{n+1}{(\zeta^{-1}\mathrm{e}^{\beta\varepsilon_k})^{n+1} - 1} - 1, \tag{6}$$

where $h_\sigma(\zeta)$ is an analytic continuation of $h_\sigma(z)$. From equation (5), we can see that the fugacity is a zero of $\Psi(\zeta)$ on the real axis. Therefore, the problem of solving the fugacity z is converted into the problem of seeking the real zero of the complex function $\Psi(\zeta)$.

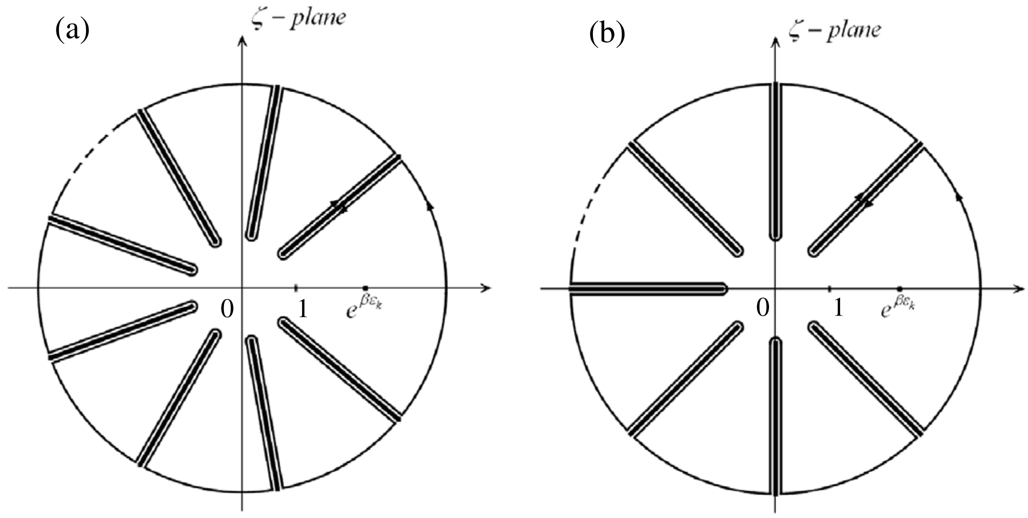

Figure 1. The analytic region of $\Psi(\zeta)$. The cases of n = even and n = odd are illustrated in (a) and (b), respectively.

We can express $\Psi(\zeta)$ as

$$\Psi(\zeta) = \eta \frac{(\zeta - z)\prod_{i=1}^{n_{\mathrm{zero}}-1}(\zeta - \omega_i)}{\prod_{j=1}^{n_\rho}(\zeta - \rho_j)\prod_{m=1}^{n_{\mathrm{b}}}(\zeta - c_m)^{\kappa_m}}\varphi(\zeta), \tag{7}$$

where $\varphi(\zeta)$ is the fundamental solution of the homogeneous Riemann–Hilbert problem, which has no zeros and singularities, z (the fugacity) and ω_i are zeros of $\Psi(\zeta)$, n_{zero} is the number of zeros, ρ_j is a pole of $\Psi(\zeta)$, n_ρ is the number of poles, c_m is an endpoint that is different from infinity of the boundary of the analytic region of $\Psi(\zeta)$ (in the present case, the boundary of the analytic region of $\Psi(\zeta)$ is a set of rays (see figure 1) with the origins c_m), n_{b} is the number of the endpoints different from infinity of the boundary (in the present case n_{b} is the number of rays), the constant κ_m is introduced to equal the degrees of divergence of the two sides of this equation at the mth endpoint c_m, and η is a constant.

From equation (7), it is easy to see that we can in principle obtain an explicit expression for the fugacity z. For this purpose, we need to first determine the fundamental solution $\varphi(\zeta)$, the endpoints c_m, and the poles ρ_j, etc.

The analytic region. For determining c_m, the endpoints of the boundary of the analytic region, we first analyze the analytic region of $\Psi(\zeta)$.

The boundary of the analytic region of $\Psi(\zeta)$ is determined by the analytic region of

$$h_\sigma(\zeta) = g_\sigma(\zeta) - \frac{1}{(n+1)^{\sigma-1}} g_\sigma(\zeta^{n+1}), \tag{8}$$

where $g_\sigma(\zeta)$ is the analytically continued Bose–Einstein integral which is just the Jonquiére function [8]:

$$g_\sigma(\zeta) = Li_\sigma(\zeta).$$

The boundary of the analytic region of $Li_\sigma(\zeta)$ is the positive real axis from 1 to ∞ [8]. Consequently, the boundary of the analytic region of $h_\sigma(\zeta)$ and $\Psi(\zeta)$ consists of n rays

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with origins on the unit circle (figure 1), denoted as L_m, $m = 1, 2, \ldots, n$, i.e., the mth ray L_m is $[\mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}, \infty \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))})$. It should be emphasized that $h_\sigma(\zeta)$ has no singularity on the positive real axis, or $h_\sigma(\zeta)$ is analytic on the positive real axis. Then, the endpoints that are different from infinity of the boundary (the origins of the rays) are

$$c_m = \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}, \qquad m = 1, 2, \ldots, n. \tag{9}$$

Therefore we have $n_\mathrm{b} = n$.

The fundamental solution of the homogeneous Riemann–Hilbert problem. Now, we calculate the fundamental solution of the homogeneous Riemann–Hilbert problem, $\varphi(\zeta)$.

Using the result of the homogeneous Riemann–Hilbert problem [3], we have

$$\varphi(\zeta) = \mathrm{e}^{\gamma(\zeta)} \prod_{m=1}^{n} (\zeta - c_m)^{\lambda_m}. \tag{10}$$

Here

$$\gamma(\zeta) = \frac{1}{2\pi \mathrm{i}} \sum_{m=1}^{n} \int_{L_m} \mathrm{d}x \, \frac{\ln G(x)}{x - \zeta} = \frac{1}{2\pi \mathrm{i}} \sum_{m=1}^{n} \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))} \int_1^\infty \mathrm{d}x \, \frac{\ln G\left(x \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}\right)}{x \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))} - \zeta}, \tag{11}$$

where

$$G(\zeta) = \frac{\varphi^+(\zeta)}{\varphi^-(\zeta)} \tag{12}$$

is the jump of $\varphi(\zeta)$ on the boundary, and λ_m is an integer determined by the condition

$$\begin{aligned} &\mp \mathrm{Re} \, \frac{\ln G(c_m)}{2\pi \mathrm{i}} + \lambda_m = 0, && \text{if } \mp \mathrm{Re} \, \frac{\ln G(c_m)}{2\pi i} \text{ is an integer}, \\ &-1 < \mp \mathrm{Re} \, \frac{\ln G(c_m)}{2\pi \mathrm{i}} + \lambda_m < 0, && \text{otherwise}. \end{aligned} \tag{13}$$

We first need to analyze the analytic region of the fundamental solution $\varphi(\zeta)$. From equation (7), we can see that the boundary of the analytic region of $\varphi(\zeta)$ consists of the non-isolated singularities of $\Psi(\zeta)$, and the jump of $\varphi(\zeta)$ on the boundary is the same as that of $\Psi(\zeta)$, i.e.,

$$G(\zeta) = \frac{\Psi^+(\zeta)}{\Psi^-(\zeta)}. \tag{14}$$

The value of $\Psi(\zeta)$ on the two sides of L_m is determined by the value of the function $h_\sigma(\zeta)$ on the two sides of L_m:

$$\begin{aligned} h_\sigma^\pm(x \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}) &= g_\sigma(x \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}) - \frac{1}{(n+1)^{\sigma-1}} \mathfrak{g}_\sigma(x^{n+1}) \mp \mathrm{i} \frac{\pi}{\Gamma(\sigma)} (\ln x)^{\sigma-1}, \\ m &= 1, 2, \ldots, n, \end{aligned} \tag{15}$$

where $\mathfrak{g}_\sigma(\zeta)$ is the Cauchy principal value of $g_\sigma(\zeta)$ on the boundary [9]. Then

$$\begin{aligned} \Psi^\pm(x \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}) = {} & \frac{N_\lambda}{N} \left[g_{\nu/s}(x \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}) - \frac{1}{(n+1)^{\nu/s-1}} \mathfrak{g}_{\nu/s}(x^{n+1}) \right] \\ & + \frac{1}{N} \frac{n+1}{(x^{-1} \mathrm{e}^{\beta \varepsilon_k})^{n+1} - 1} - 1 \mp \mathrm{i} \frac{\pi}{\Gamma(\nu/s)} \frac{N_\lambda}{N} (\ln x)^{\nu/s-1}. \end{aligned} \tag{16}$$

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Note that, in the case of $n_k = \infty$ and $n_i = 1$ $(i \neq k)$, equation (16) reduces to

$$\Psi^{\pm}(-x) = \frac{N_\lambda}{N}\mathfrak{f}_{\nu/s}(-x) + \frac{1}{N}\frac{2}{(x^{-1}\mathrm{e}^{\beta\varepsilon_k})^2 - 1} - 1 \mp \mathrm{i}\frac{\pi}{\Gamma(\nu/s)}\frac{N_\lambda}{N}(\ln x)^{\nu/s-1}, \tag{17}$$

where $\mathfrak{f}_{\nu/s}(\xi)$ is the Cauchy principal value of the analytically continued Fermi–Dirac integral.

Next, we calculate λ_m from equations (13).

At the endpoints of the boundary (including both the endpoints that are different from infinity and the infinity), we have

$$G(\mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}) = G(\infty) = 1. \tag{18}$$

Choosing $\ln G(\infty) = 0$ gives

$$\ln G(\mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}) = \mathrm{i}\arg G(\mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}) = -\mathrm{i}2\pi. \tag{19}$$

Then we have

$$\lambda_m = -1. \tag{20}$$

Therefore, the fundamental solution is

$$\varphi(\zeta) = \mathrm{e}^{\gamma(\zeta)}\prod_{m=1}^{n}\frac{1}{\zeta - \mathrm{e}^{\mathrm{i}(2\pi m/(n+1))}} = \mathrm{e}^{\gamma(\zeta)}\frac{\zeta - 1}{\zeta^{n+1} - 1}. \tag{21}$$

The value of κ_m. The parameter κ_m is chosen to guarantee that the degrees of divergence of the two sides of equation (7) at the endpoint c_m are the same.

At the origin of L_m, $c_m = \mathrm{e}^{\mathrm{i}2\pi m/(n+1)}$, when $\nu/s > 1$, $\mathfrak{g}_{\nu/s}(x^{n+1})$ and $(\ln x)^{\nu/s-1}$ are convergent, and when $\nu/s \leq 1$, the degrees of divergence of $\mathfrak{g}_{\nu/s}(x^{n+1})$ and $(\ln x)^{\nu/s-1}$ are less than 1. Thus we have

$$\kappa_m = 0. \tag{22}$$

The isolated singularity. $\Psi(\zeta)$ has only one isolated singularity,

$$\rho = \mathrm{e}^{\beta\varepsilon_k}. \tag{23}$$

The number of zeros of $\Psi(\zeta)$. Substituting the above result into equation (7) gives

$$\eta(\zeta - z)\prod_{i=1}^{n_{\mathrm{zero}}-1}(\zeta - \omega_i) = \mathrm{e}^{-\gamma(\zeta)}\frac{\zeta^{n+1} - 1}{\zeta - 1}(\zeta - \mathrm{e}^{\beta\varepsilon_k})\Psi(\zeta). \tag{24}$$

In principle, if ω_i and η are known, one can obtain the explicit expression for z directly. Nevertheless, the difficulty of finding the zeros ω_i is often the same as the difficulty of finding the zero z. Alternatively, we can construct a set of equations for z, ω_i, and η, and obtain z by solving the equations.

For solving z, we need $n_{\mathrm{zero}}+1$ equations. Since the number of the isolated singularities of $\Psi(\zeta)$ is already known, the number of the zeros, n_{zero}, can be determined by the argument principle, the contour being chosen as in figure 1. The result shows that $\Psi(\zeta)$ has $n+1$ zeros on the ζ-plane, so we need $n+2$ equations for determining z.

The case of $n_k = \infty$ *and* $n_i = 1$ $(i \neq k)$. For simplicity, we consider the case of $n_k = \infty$ and $n_i = 1$ $(i \neq k)$; the solutions for more general cases can also be obtained exactly but with more complex forms. When $n = 1$, equation (24) becomes

$$\eta(\zeta - z)(\zeta - \omega) = \mathrm{e}^{-\gamma(\zeta)}(\zeta + 1)(\zeta - \mathrm{e}^{\beta\varepsilon_k})\Psi(\zeta). \tag{25}$$

In this case, for solving z, we need three equations.

Substituting $\zeta = 0$ into equation (25) gives

$$\eta z \omega = \mathrm{e}^{\beta\varepsilon_k - \gamma(0)}; \tag{26}$$

substituting equation (26) into the derivative of equation (25) and setting $\zeta = 0$ gives

$$\gamma'(0) - 1 + \mathrm{e}^{-\beta\varepsilon_k} - \frac{1}{z} - \frac{1}{\omega} = -\frac{N_\lambda}{N}; \tag{27}$$

and substituting $\zeta = \mathrm{e}^{\beta\varepsilon_k}$ into equation (25) gives

$$\eta(\mathrm{e}^{\beta\varepsilon_k} - z)(\mathrm{e}^{\beta\varepsilon_k} - \omega) = -\mathrm{e}^{-\gamma(\mathrm{e}^{\beta\varepsilon_k})}(\mathrm{e}^{\beta\varepsilon_k} + 1)\frac{\mathrm{e}^{\beta\varepsilon_k}}{N}. \tag{28}$$

Solving the equations (26)–(28), we have

$$z = 2\left[\eta_\lambda + \mathrm{e}^{-\beta\varepsilon_k} + \sqrt{(\eta_\lambda - \mathrm{e}^{-\beta\varepsilon_k})^2 + \frac{4\mathrm{e}^{-\beta\varepsilon_k}(1 + \mathrm{e}^{-\beta\varepsilon_k})}{N\mathrm{e}^{\gamma(\mathrm{e}^{\beta\varepsilon_k}) - \gamma(0)}}}\right]^{-1}, \tag{29}$$

where $\eta_\lambda = N_\lambda/N + \gamma'(0) - 1$ and

$$\gamma(\zeta) = \frac{1}{2\pi\mathrm{i}}\int_1^\infty \mathrm{d}x\,\frac{\ln G(-x)}{x + \zeta}. \tag{30}$$

3.2. The phase transition and the necessary condition for phase transitions—the thermodynamic limit

Equation (29) is an exact expression for the fugacity, which can be used to simultaneously describe both the gas phase and the condensed phase, and, of course, it can be used to describe the transition between these two phases. From equation (29), we can directly see that the thermodynamic limit, $N \to \infty$, is the necessary condition for the phase transition.

The fugacity given by equation (29) is a smooth function and there is no singularity. Therefore, there is no phase transition regardless of how low the temperature is. However, in the thermodynamic limit, i.e., $N \to \infty$, equation (29) becomes

$$z = 2\frac{1}{\eta_\lambda + \mathrm{e}^{-\beta\varepsilon_k} + |\eta_\lambda - \mathrm{e}^{-\beta\varepsilon_k}|} = \begin{cases} \mathrm{e}^{\beta\varepsilon_k}, & \text{when } \eta_\lambda < \mathrm{e}^{-\beta\varepsilon_k}, \\ \dfrac{1}{\eta_\lambda}, & \text{when } \eta_\lambda > \mathrm{e}^{-\beta\varepsilon_k}. \end{cases} \tag{31}$$

The discontinuity may appear in the first-order derivative of the fugacity and the phase transition may occur. The discontinuous point appears at

$$\eta_\lambda = \mathrm{e}^{-\beta\varepsilon_k}, \tag{32}$$

which is just the phase transition point.

In the phase transition theory, there is a fundamental law: the necessary condition for a phase transition is that the system must be infinite, i.e., the thermodynamic limit. The proof of this statement depends on the assumption that a finite volume can accommodate

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at most a finite number of particles. If the number of particles is finite, the partition function will be an analytic function and, consequently, there is no singularity in the thermodynamic function and there is no phase transition [4, 10, 11]. Such an assumption is equivalent to requiring that the particle must have a nonzero volume. Clearly, this assumption does not hold for ideal gases. That is to say, though this conclusion is valid for all realistic systems (realistic gases are non-ideal gases), this proof is not valid for the idealized model—ideal gases. Our above result shows that for ideal gas systems, the thermodynamic limit is still a condition for a phase transition.

3.3. The case of $n_0 = \infty$ and $n_i = 1$ ($i \neq 0$): the phase transition temperature

We first consider the case of $n_0 = \infty$ and $n_i = 1$ ($i \neq 0$), i.e., the state whose maximum occupation number is infinite is the ground state, $\varepsilon_k = \varepsilon_0 = 0$.

In any dimension, there must exist a phase transition. This can be verified directly by observing the discontinuity in the derivative of the fugacity z from equation (32) and the transition temperature is determined by

$$\eta_\lambda = 1. \tag{33}$$

Then the phase transition temperature reads

$$T_c = \frac{h^s}{2\pi^{s/2} m k_{\mathrm{B}}} \left[\frac{N}{V} \frac{s\Gamma(\nu/2)}{2\Gamma(\nu/s)} \frac{1}{(1 - 2^{1-\nu/s})\zeta(\nu/s)} \right]^{s/\nu}. \tag{34}$$

Now let us see what happens when a phase transition occurs. The total number of the excited particles, from equation (5), is

$$N_{\mathrm{e}} = N_\lambda f_{\nu/s}(z). \tag{35}$$

Comparing the expressions for T_c and N_{e} gives that when the phase transition occurs,

$$N_{\mathrm{e}} = N. \tag{36}$$

This is just the condition that one determines the phase transition temperature for a BEC in an ideal Bose gas. In our case, however, this result comes from a mathematically rigorous calculation rather than being put in by hand.

This result indicates that when the phase transition occurs, the macroscopic properties of the system will begin to be controlled, to a certain extent, by a unique quantum state (here the state is the ground state). Such a phase transition is a sudden change in the particle distribution: in the gas phase, the macroscopic behavior of the system is a mean contribution of all quantum states, but in the condensed phase, the quantum state with an infinite maximum occupation number dominates. This is a BEC type phase transition. More concretely, when the phase transition begins, the number of excited particles decreases as the temperature decreases, while the number of particles in the ground state increases as the temperature decreases:

$$\frac{N_{\mathrm{e}}}{N} = \left(\frac{T}{T_c}\right)^{\nu/s}, \qquad \frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{\nu/s}. \tag{37}$$

In contrast to the case for the BEC in an ideal Bose gas, the BEC type phase transition can occur in any dimension in the ideal gases obeying the statistics in which the maximum

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occupation number of the ground state is ∞ and of all other states is finite, since the phase transition can occur for any positive value of ν/s. In the Bose case, however, the BEC only occurs when $\nu/s > 1$, and, as a result, the BEC only occurs in three-dimensional Bose gases. This is because the Bose–Einstein integral in the Bose case is replaced by $h_{\nu/s}(z)$ in the present case, while $h_{\nu/s}(z)$ is always bounded for $0 \leq z \leq 1$. That is to say, in such an ideal generalized statistics gas, the occurrence of the phase transition is easier to obtain than that in a Bose system.

3.4. The case of $n_k = \infty$ and $n_i = 1$ ($i \neq k$): the phase transition temperature and the Fermi energy

Next, we consider the case where the only state with an infinite maximum occupation number is not the ground state, i.e., $n_k = \infty$ and $n_i = 1$ ($k \neq 0$ and $i \neq k$). More general cases can be treated by the same procedure.

The phase transition temperature. From equation (31), we can see that the discontinuous point of the derivative of the fugacity appears at $\eta_\lambda = \mathrm{e}^{-\beta\varepsilon_k}$. Equation (31) indicates that the phase transition appears at $z = \omega = \mathrm{e}^{\beta\varepsilon_k}$. Substituting $\zeta = \mathrm{e}^{\beta\varepsilon_k}$ into the derivative of equation (25), when $N \to \infty$, gives

$$\Lambda(k_\mathrm{B}T_c)^{\nu/s} f_{\nu/s}(\mathrm{e}^{\varepsilon_k/(k_\mathrm{B}T_c)}) = 1, \tag{38}$$

where $\Lambda = (2\Gamma(\nu/s)/(s\Gamma(\nu/2)))((2\pi^{s/2}m)^{\nu/s}/h^\nu)(V/N)$. This result indicates that when the phase transition occurs, the number of particles in all the states except the infinite-maximum-occupation-number kth state equals the total number of particles of the system, i.e., $N_{n\neq\infty} = N$.

On the basis of the homogeneous Riemann–Hilbert problem, we can solve the phase transition temperature from equation (38). For simplicity, we only give the result for the case of $s = \nu$.

Introduce a complex function

$$\phi(\tau) = \frac{2}{s\Gamma(s/2)}\frac{V}{N}\frac{2\pi^{s/2}m}{h^s}k_\mathrm{B}\tau f_1\left(\mathrm{e}^{\varepsilon_k/(k_\mathrm{B}\tau)}\right) - 1. \tag{39}$$

The phase transition temperature T_c is a zero of $\phi(\tau)$ on the real axis.

The analytic region. We first analyze the analytic region of $\phi(\tau)$ on the τ-plane. The analytic region of $\phi(\tau)$ is determined by the behavior of the analytically continued Fermi–Dirac integral, $f_1(\mathrm{e}^{\varepsilon_k/(k_\mathrm{B}\tau)})$, which is illustrated in figure 2(a). The boundary of this region is complex. Introducing a transformation

$$\xi = \frac{1}{k_\mathrm{B}\tau}, \tag{40}$$

we have

$$\psi(\xi) = \frac{2}{s\Gamma(s/2)}\frac{V}{N}\frac{2\pi^{s/2}m}{h^s}\frac{1}{\xi}f_1(\mathrm{e}^{\varepsilon_k\xi}) - 1. \tag{41}$$

The boundary of the analytic region of $\psi(\xi)$ on the ξ-plane is

$$\begin{aligned} &\mathrm{Re}\,\xi \geq 0, \\ &\mathrm{Im}\,\xi = \frac{(2q+1)\pi}{\varepsilon_k}, \qquad q = 0, \pm1, \pm2, \ldots, \end{aligned} \tag{42}$$

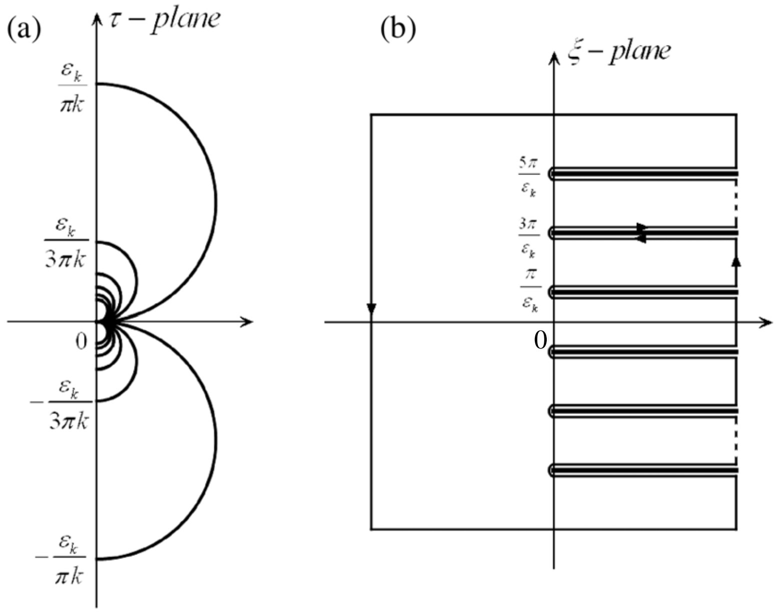

Figure 2. (a) The analytic region of $\phi(\tau)$; (b) the analytic region of $\psi(\xi)$.

as illustrated in figure 2(b), which is a set of rays running parallel to the real axis with origins

$$c_q = \left(0, \frac{(2q+1)\pi}{\varepsilon_k}\right). \tag{43}$$

The fundamental solution of the homogeneous Riemann–Hilbert problem. We can also express $\psi(\xi)$ in the form of equation (7), and, then, solve the explicit expression for the phase transition temperature. First, we seek for the fundamental solution of the homogeneous Riemann–Hilbert problem. According to equation (10), the fundamental solution can be written in the following form:

$$\varphi(\xi) = \mathrm{e}^{\gamma(\xi)} \prod_{q=-\infty}^{\infty} (\xi - c_q)^{\lambda_q}, \tag{44}$$

where

$$\gamma(\xi) = \frac{1}{2\pi\mathrm{i}} \int_{\Sigma_q L_q} \mathrm{d}\chi \, \frac{\ln G(\chi)}{\chi - \xi}, \tag{45}$$

and the integral is along the boundary of the analytic region,

$$L_q\colon \xi = x + \mathrm{i}\frac{(2q+1)\pi}{\varepsilon_k}, \qquad x \in [0, \infty) \quad \text{and} \quad q = 0, \pm 1, \pm 2, \ldots. \tag{46}$$

The constant λ_q is an integer satisfying the condition (13).

The jump on the boundary of the fundamental solution $\varphi(\xi)$ is the same as that of $\psi(\xi)$:

$$G(\xi) = \frac{\varphi^+(\xi)}{\varphi^-(\xi)} = \frac{\psi^+(\xi)}{\psi^-(\xi)}. \tag{47}$$

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$\psi^{\pm}(\xi)$, the value of $\psi(\xi)$ on the two sides of the boundary, is determined by the behavior of the analytically continued Fermi–Dirac integral,

$$f_\sigma^{\pm}\left(\mathrm{e}^{[x+\mathrm{i}((2q+1)\pi/\varepsilon_k)]\varepsilon_k}\right) = \mathfrak{f}_\sigma(-\mathrm{e}^{x\varepsilon_k}) \mp \mathrm{i}\frac{\pi}{\Gamma(\sigma)}(x\varepsilon_k)^{\sigma-1}. \tag{48}$$

Then,

$$\begin{aligned}\psi^{\pm}\left(x+\mathrm{i}\frac{(2q+1)\pi}{\varepsilon_k}\right) &= \frac{\Lambda'}{x^2+[((2q+1)\pi/\varepsilon_k)]^2}\left\{x\mathfrak{f}_1(-\mathrm{e}^{x\varepsilon_k}) \mp \pi\frac{(2q+1)\pi}{\varepsilon_k}\right\} - 1 \\ &\quad + \mathrm{i}\frac{\Lambda'}{x^2+[((2q+1)\pi/\varepsilon_k)]^2}\left\{\mp\pi x - \mathfrak{f}_1(-\mathrm{e}^{x\varepsilon_k})\frac{(2q+1)\pi}{\varepsilon_k}\right\},\end{aligned} \tag{49}$$

where $\Lambda' = (2/(s\Gamma(s/2)))(2\pi^{s/2}m/h^s)(V/N)$. From equations (47) and (49), we can see that $G(\infty+\mathrm{i}(2q+1)\pi/\varepsilon_k) = 1$. The constant λ_q is determined by the condition (13). Choosing $\ln G(\infty) = 0$ gives

$$\lambda_q = 0. \tag{50}$$

Consequently, the fundamental solution is

$$\varphi(\xi) = \mathrm{e}^{\gamma(\xi)}. \tag{51}$$

The value of κ_q. At the endpoints $c_q = (0, (2q+1)\pi/\varepsilon_k)$, we have $\psi(\mathrm{i}(2q+1)\pi/\varepsilon_k) \sim \ln[\xi - \mathrm{i}(2q+1)\pi/\varepsilon_k]$. Then,

$$\kappa_q = 0. \tag{52}$$

The isolated singularity of $\psi(\xi)$. $\psi(\xi)$ has only one isolated singularity,

$$\rho = 0. \tag{53}$$

The number of the zeros of $\psi(\xi)$. By the argument principle, the contour being illustrated in figure 2(b), we can determine that $\psi(\xi)$ has only one zero, $\xi = \beta_c = 1/(k_{\mathrm{B}}T_c)$, which is on the real axis.

Introducing $\Phi(\xi) = \xi\psi(\xi)$, we have

$$\Phi(\xi) = \upsilon(\xi - \beta_c)\mathrm{e}^{\gamma(\xi)}, \tag{54}$$

where υ is a constant. Substituting $\xi = 0$ into equation (54) and its first-order derivative gives two equations. Solving these equations gives

$$\begin{aligned}T_c &= \frac{h^s}{2\pi^{s/2}mk_{\mathrm{B}}}\frac{N}{V}\frac{s}{2}\Gamma\left(\frac{s}{2}\right)\frac{1}{\ln 2} - \frac{1}{2\ln 2}\frac{\varepsilon_k}{k_{\mathrm{B}}} \\ &\quad + \frac{1}{k_{\mathrm{B}}}\frac{1}{2\pi\mathrm{i}}\sum_{q=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}x\,\frac{\ln G(x+\mathrm{i}(2q+1)\pi/\varepsilon_k)}{(x+\mathrm{i}(2q+1)\pi/\varepsilon_k)^2}.\end{aligned} \tag{55}$$

The last term of equation (55) is small when ε_k is small, and is roughly proportional to ε_k^2.

The explicit expression for the phase transition temperature shows that T_c depends on the value of ε_k. In particular, $T_c = 0$ appears at

$$\varepsilon_k = \frac{h^s}{2\pi^{s/2}m}\left[\Gamma\left(\frac{\nu}{2}+1\right)\frac{N}{V}\right]^{s/\nu} \equiv \epsilon_{\mathrm{F}}, \tag{56}$$

i.e., when $\varepsilon_k \geq \epsilon_{\mathrm{F}}$, there will be no phase transition. Note that this result holds also for the case of $s \neq \nu$. It is not difficult to recognize the physical meaning of ϵ_{F}: it is just the Fermi energy of a ν-dimensional ideal Fermi gas with the dispersion relation $\varepsilon = p^s/(2m)$ [5]. The reason why there is no phase transition when $\varepsilon_k > \epsilon_{\mathrm{F}}$ is that if $\varepsilon_k > \epsilon_{\mathrm{F}}$, the states below ϵ_{F} can accommodate all particles in the system and, then, there are not enough particles accumulating in the kth state, i.e., the BEC type phase transition cannot occur.

4. Discussion and outlook

In this paper, we construct an exactly solvable phase transition model. We first consider a generalized statistics in which the maximum occupation numbers of different quantum states can take on different values. When the maximum occupation numbers of all the states are the same, e.g., equaling ∞, 1, or an arbitrary integer, the generalized statistics returns to Bose–Einstein, Fermi–Dirac, or Gentile statistics [5, 6, 12], respectively. The model constructed in this paper is an ideal gas obeying the generalized statistics in which the maximum occupation number of only one state is infinite, but that of all other states is finite. The phase transition which occurs in such systems is the BEC type phase transition. For judging whether the phase transition can occur and determining the phase transition point, we calculate the exact explicit solution for the fugacity with the help of the mathematical result of the homogeneous Riemann–Hilbert problem. By observing the discontinuity in the derivative of the fugacity, we analyze the phase transition rigorously. From this phase transition model, we can see that the thermodynamic limit is a necessary condition for a phase transition of an ideal system.

For constructing the solvable phase transition model, we introduce a kind of intermediate statistics. Various generalized exclusion statistics models play important roles in many fields [6], [13]–[18], since many physical systems may behave as neither Bose–Einstein nor Fermi–Dirac systems. Though Nature realizes only bosons and fermions, there are many composite particle systems, e.g., the Cooper pair in the theory of superconductivity, the Fermi gas superfluid [19], the exciton [20], the magnon [21], etc. For example, a boson consists of two fermions obeying Bose–Einstein statistics. However, when two such bosons come closer together, the fermions in the composite bosons may ‘feel’ each other, and the statistics may somewhat deviate from Bose–Einstein statistics. In this case, such a composite system can be effectively viewed as obeying a kind of intermediate statistics. It is shown in a recent study [22] that the fermion pairs in the one-dimensional Fermi gases obey generalized exclusion statistics.

The experimental and theoretical research of BEC is a branch of the statistical physics of a rapidly growing importance [23]. The BEC of ideal Bose gases is a special case of the generalized BEC phase transition. By studying this exactly solvable model, we can also obtain a deeper insight into the BEC of ideal Bose gases. We can conclude that the

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conditions for the BEC type phase transition are as follows:

(1) There must exist a *low enough* quantum state with an infinite maximum occupation number, where 'low enough' means that when the temperature tends to the absolute zero, there are still a macroscopic number of particles in this state. In other words, this condition requires that the total capacity of all states below such a state must be small enough that this state can be macroscopically occupied when the temperature is low. As a result, the energy of this state, denoted as $\varepsilon_\infty^{\min}$, must be the lowest one among the states whose maximum occupation numbers are infinite. Such a condition is of course satisfied by a Bose system since the maximum occupation number of the ground state is infinite. However, for the systems obeying the generalized statistics, as discussed above, in the case of $n_0 = \infty$ and $n_i = n$ $(i \neq 0)$, this condition can always be satisfied, but in more general cases, e.g., the case of $n_k = \infty$ and $n_i = n$ $(i \neq k)$, this condition can be satisfied only when $\varepsilon_k < \epsilon_{\mathrm{F}}$.

(2) The state that will be macroscopically occupied when a BEC type phase transition occurs must be isolated from other infinite-maximum-occupation-number states, where 'isolated' means that the state density of this state is a δ-function, i.e., the state density of the states with infinite maximum occupation numbers (not the state density of the system) must take the form of $\rho_\infty(\varepsilon) = \eta_\infty(\varepsilon) + \delta(\varepsilon - \varepsilon_\infty^{\min})$ and $\eta_\infty(\varepsilon_\infty^{\min}) = 0$, where $\rho_\infty(\varepsilon)$ is the density of the infinite-maximum-occupation-number states. In the examples of the generalized statistics that we considered above, this condition is satisfied naturally, since there is only one infinite-maximum-occupation-number state, for $n_0 = \infty$ and $n_i = n$ $(i \neq 0)$, $\rho_\infty(\varepsilon) = \delta(\varepsilon)$, and for $n_k = \infty$ and $n_i = n$ $(i \neq k)$, $\rho_\infty(\varepsilon) = \delta(\varepsilon - \varepsilon_k)$. However, for the case of ideal Bose gases, this condition is not always satisfied. In ideal Bose gas systems, the maximum occupation number of all states is infinite, i.e., the state density of the system $\rho(\varepsilon) = \rho_\infty(\varepsilon)$, and then the lowest infinite-maximum-occupation-number state is the ground state, i.e., $\varepsilon_\infty^{\min} = 0$. In three dimensions, the state density is $\rho_\infty(\varepsilon) = \eta_\infty(\varepsilon) + \delta(\varepsilon)$, where $\eta_\infty(\varepsilon) \propto \sqrt{\varepsilon}$, so $\eta_\infty(0) = 0$. The condition is satisfied, and the BEC phase transition can occur in a three-dimensional ideal Bose gas. In one and two dimensions, the state densities are $\rho_\infty(\varepsilon) = \eta_\infty(\varepsilon) \propto 1/\sqrt{\varepsilon}$ and $\rho_\infty(\varepsilon) = \eta_\infty(\varepsilon) = \mathrm{const}$, respectively; the above condition is not satisfied, and there are no BEC phase transitions in one- and two-dimensional Bose gases.

Furthermore, many physical systems possess other kinds of statistics beyond Bose–Einstein and Fermi–Dirac statistics. For example, the Calogero–Sutherland model is shown to possess fractional statistics [15], a spinless fermion system in two dimensions may obey exclusion statistics [17], and bound pairs of fermions form hard-core bosons obeying generalized exclusion statistics [22]. Moreover, in the model constructed in the present paper, there are both bosonic and fermionic states in a system. In a Bose system, if each boson consists of two fermions, then in the system there must simultaneously exist both bosons and fermions due to the fact that there exists an 'ionization' energy. As long as the temperature of the system is not the absolute zero, there are always a certain proportion of particles having energies larger than the 'ionization' energy and behaving as fermions. In such a case, the particle in the low-lying state behaves as a boson and the particle in the high-lying state behaves as a fermion. That is to say, a composite system will not accurately possess Bose–Einstein or Fermi–Dirac statistics. In such a composite system, our model may work. We will address this issue in future work. Moreover, a system consisting of both bosons and fermions has also been studied in the literature [24].

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