The basic theory of K41 of the velocity field: structure function

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So, nat's ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller yet to bite 'em,
And so proceed ad infinitum.
Thus every poet, in his kind,
Is bit by him that comes behind.

—— Jonathan Swift: On Poetry: Rhapsody (1733)

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity
(in the molecular sense).

—— Lewis Richardson (1922)

1 Physical Picture

We must emphasize here that **the Kolmogorov theory is a phenomenology theory**. The physical picture of this theory is a very simple sketch that it is not responsible for describing the real turbulence exactly and there may be some places even unreal. However, as we shall see, this theory grasps the essential features of the turbulence and can give some valid predictions.

In K41 theory, the turbulence can be considered to be composed of eddies of different scales. The characteristic velocity of the biggest eddies $v_{(1)}$ and the scale $r_{(1)}$ are always large. Thus the Reynolds number of the biggest eddies

$$Re_{(1)} = \frac{v_{(1)} r_{(1)}}{\nu}$$

is very large. The biggest eddies are unsteady and then will smash into smaller eddies with smaller characteristic velocity. The characteristic velocity and the scale of these eddies are denoted as $v_{(2)}$ and $r_{(2)}$. The smashing processes will continue until the nth Reynolds number of the eddies

$$Re_{(n)} = \frac{v_{(n)} r_{(n)}}{\nu}$$

is too small to smash into smaller eddies, for the viscous will then dominate the processes and make the nth eddies stable. In the above equation, r_n is the characteristic scale of the smallest eddy and is called *the Kolmogorov scale* which is always denoted by η. The smashing processes as stated above are always called *cascade processes*.

During the cascade processes, the motions of the children eddies are randomly transferred by their mothers. Thus we can image that, when the length of the cascade processes is very large, the anisotropic and the inhomogeneous information of the big eddies will lost during the cascade processes and the small eddies will become statistically isotropic and homogeneous. For guaranteeing large length of the cascade processes, the Reynolds number of the turbulent flow

$$Re = \frac{UL}{\nu}$$

must be very large, where U is the characteristic velocity and L is the characteristic scale of the flow. The picture stated above forms the first hypothesis of K41 theory, as we shall see in Sec. 2. We emphasize here that the statistical isotropy and homogeneous is local. That is to say, the statistical isotropy and homogeneous is satisfied in a small space domain which scale is far smaller than L. **However, we should not recognize that the eddies of the whole flow field, which scales are far smaller than L, are statistically isotropic and homogeneous**.

From the energy point of view, the energy from the external forms the large eddies and maintains their motions, and then is passed successively to form the smaller eddies and maintain their motions by the cascade processes. During this processes, the viscosity is negligible by comparing with the inertia as the Reynolds numbers of the eddies are very large. Hence, there is almost no energy dissipated during the cascade processes. However, at the end of the energy transfer processes where the Reynolds number of the eddies is very small, the viscosity is effective in dissipating the kinetic energy. From above discussions, we can see that the range of scale ℓ of the eddies can be divided into three parts. When ℓ is on the order of $r_{(1)}$ which approximates to L, most external energy will enter the eddies of scales in this range which is called *energy-containing range*. The motions of the eddies in the energy-containing range are impacted by the external and may be anisotropic. When $\eta < \ell \ll L$, the cascade processes pass the energy inviscidly and we thus call this range *inertial subrange*. The range of $\ell < \eta$ is called *dissipation range* as the energy is mainly dissipated in this range. Both the inertial subrange and the dissipation range are called *universal equilibrium range*.

2 Hypothesis of K41

The Definitions and Hypothesis discussed as follows are refereed to Kolmogorov (1941a). The velocity field of the flow is denoted by $\mathbf{u}(\mathbf{r}, t)$. The differences between the velocities at two different locations are denoted by $w_i(\mathbf{r}, \mathbf{x}, t) = u_i(\mathbf{x} + \mathbf{r}, t) - u_i(\mathbf{x}, t)(i = 1, 2, 3)$. If we pick out N points in the space domain G, then we have N random variables of $\mathbf{w}^{(k)}(k = 1, 2, \cdots, N)$. The joint PDF of $\mathbf{w}^{(k)}$ are denoted by F_N. Why are the statistics of the velocity differences used in K41? The answers are refereed to Frisch (1995), where they discussed that the velocity differences can be seen as the characteristic velocities of the eddies.

Definitions of Local Homogeneous. The turbulence is called locally homogeneous in the space domain G, if for every fixed N and $r^{(k)}(k = 1, 2, \cdots, N)$, the N-point PDF F_N is independent of $\mathbf{x}$ and $\mathbf{u}(\mathbf{x}, t)$.

Definitions of Local Isotropy. The turbulence is called locally isotropic in the space domain G, if it is homogeneous and if, besides, the PDF F_N is invariant with respect to rotations and reflections of the coordinate axes of the original system.

The Hypothesis of the Finiteness of the Energy Dissipation. The energy dissipation $\langle \varepsilon \rangle = \nu \left\langle \dfrac{\partial u_i}{\partial x_j} \dfrac{\partial u_i}{\partial x_j} \right\rangle$. We suppose that $\left\langle \dfrac{\partial u_i}{\partial x_j} \dfrac{\partial u_k}{\partial x_l} \right\rangle$ is finite and the finiteness of the energy dissipation can be satisfied.

The Hypothesis of Local Isotropy. In an arbitrary turbulent flow with a sufficiently large Reynolds number, the turbulence is local isotropy with good approximation in sufficiently small time interval and domains G not lying near the boundary of the flow or its other special regions [The expression here has a little difference from Kolmogorov (1941a), see Tsinober (2001)]. By a 'sufficiently small time interval', we mensa here a time interval whose characteristic scale is small in comparison with $T = U/L$ (that is, the turbulence can be considered as a stationary flow field approximately). By a 'sufficiently small domains G', we mensa here a domain G whose characteristic scale is small in comparison with L or $r_{(1)}$ (that is, the scale of G is in the universal equilibrium range).

What is the results of the isotropy (including reflection)? For the scalar field $T(\mathbf{r})$ and the vector field $\mathbf{T}(\mathbf{r})$, the isotropy requires that

$$\begin{aligned} T(\mathbf{r}) &= T(\mathcal{O}\mathbf{r}) \\ T_i(\mathbf{r}) &= \mathcal{O}_{ij}^{-1} T_j(\mathcal{O}\mathbf{r}), \end{aligned}$$

where $\mathcal{O}$ is the rotation and reflection transform operator. Then the forms of T and $\mathbf{T}$ can be easily deduced from above transform rules,

$$T(\mathbf{r}) = A(r) \tag{1}$$

$$T_i(\mathbf{r}) = A(r) r_i. \tag{2}$$

The higher-order isotropic tensor can be decomposed by the 1-order isotropic tensor field $T_i(\mathbf{r})$ and the 2-order isotropic tensor δ_{ij}. For example, the 2-order isotropic tensor can be decomposed as follows,

$$\begin{aligned} T_{ij}(\mathbf{r}) &= f_1(r) T_i(\mathbf{r}) T_j(\mathbf{r}) + f_2(r)\delta_{ij} \\ &= A(r) r_i r_j + B(r)\delta_{ij}. \end{aligned}$$

For the 3-order tensor field, there are four kinds of isotropic compositions which are $T_i T_j T_k$, $T_i \delta_{jk}$, $T_j \delta_{ik}$ and $T_k \delta_{ij}$, respectively. Thus, the 3-order isotropic tensor can be written as

$$T_{ijk}(\mathbf{r}) = A(r) r_i r_j r_k + B(r) r_i \delta_{jk} + C(r) r_j \delta_{ik} + D(r) r_k \delta_{ij} + E(r)\varepsilon_{ijk},$$

where ε_{ijk} is Levi-Civita symbol and

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$$

We note that the higher-order isotropic tensor field $T_{ijk...}(\mathbf{r})$ constructed by above method satisfies the following transform rule

$$T_{ijk...}(\mathbf{r}) = \mathcal{O}_{i\ell}^{-1}\mathcal{O}_{jm}^{-1}\mathcal{O}_{kn}^{-1}\dots T_{\ell mn...}(\mathcal{O}\mathbf{r}). \tag{3}$$

We now use above results to obtain some useful results of locally isotropic turbulence. According to the definition of local isotropy, the distribution F_N can be regarded as a isotropy scalar functional with respect to r. Thus by Eq. (1), we have

$$F_N[\mathbf{w}(\mathbf{r})] = f(w_1, w_2, w_3; r),$$

that is, the shape of F_N does not depend on the direction of $\mathbf{r}$. If we reflect x axis with respect to y axis, the shape does not change and we thus have

$$f(-w_1, w_2, w_3, r) = f(w_1, w_2, w_3, r), \tag{4}$$

that is, F_N is a even function with respect to w_1. In fact, F_N is a even function with respect to any velocity component. If we exchange x axis and y axis, the shape also dose not change and we thus have

$$f(w_2, w_1, w_3, r) = f(w_1, w_2, w_3, r).$$

that is, F_N is a symmetric function with respect to w_1 and w_2. In fact, F_N is a symmetric function with respect to any two velocity components. Any non-random function defined on this isotropic turbulent field is isotropic. Thus the form of these function will be the same as the isotropic tensor which we have deduced above.

Why does the local isotropy require a large Reynolds number? For a large Reynolds number, there will be a wider scale range of cascade processes and at the end of this processes there would be a range independent of internal impacts, that is there would be a inertial subrange. This inference will be clearly seen after we define the Kolmogorov scale at Sec. 3.

The First Hypothesis of Similarity. For the locally isotropic turbulence the distributions F_N are uniquely determined by the quantities ν and $\langle\varepsilon\rangle$, where $\langle\varepsilon\rangle$ is the mean dissipation rate per unit mass.

The Second Hypothesis of Similarity. If $\eta \ll r \ll L$, then the distributions F_N is uniquely determined by the quantity $\langle\varepsilon\rangle$ and does not depend on ν, as the cascade processes are independent of the viscosity. However, why dose F_N depend on the dissipation rate? In the inertial subrange, we may naturally suppose that F_N depends on the energy transfer rate from large scale to small scale. Just as the cascade is independent of viscosity, the energy is mainly dissipate at Kolmogorov scale and the energy transfer rate approximates the dissipation rate.

3 Main Results

In this section, we derive some results from above hypothesis, which make the experimental verification of the physical picture introduced in Sec. 1 possible in particular cases. The following discussions are referred to Pope (2000).

3.1 Second-order statistics

Kolmogorov (1941a) defined the second-order structure function D_{ij} by

$$D_{ij}(\mathbf{r},\mathbf{x},t) = \langle [u_i(\mathbf{x}+\mathbf{r},t) - u_i(\mathbf{x},t)][u_j(\mathbf{x}+\mathbf{r},t) - u_j(\mathbf{x},t)] \rangle.$$

For the locally isotropic turbulence, the structure function D_{ij} is a 2-order isotropic tensor and only related to $\mathbf{r}$. Thus it can be written as

$$D_{ij}(\mathbf{r}) = D_{N^2}(r)\delta_{ij} + [D_{L^2}(r) - D_{N^2}(r)]\frac{r_i r_j}{r^2}, \tag{5}$$

where D_{L^2} and D_{N^2} are called, respectively, the longitudinal and transverse structure function. If the coordinate system is chosen so that $\mathbf{r}$ is in the x direction, then we obtain

$$\begin{aligned} D_{11} &\equiv D_{L^2}(r) \\ D_{22} &= D_{33} \equiv D_{N^2}(r) \\ D_{ij} &= 0, \quad \text{if } i \neq j \end{aligned} \tag{6}$$

Here we give an example, that is $D_{22} = D_{33}$, to test above results. First, the structure function of D_{22} is

$$D_{22} = \int d\Delta u_2 (\Delta u_2{}^2) \int d\Delta u_1 \int d\Delta u_3 F(\Delta u_1, \Delta u_2, \Delta u_3)$$

where $\Delta u_2 = u_2(\mathbf{x} + r\mathbf{e}_x) - u_2(\mathbf{x})$. Then we exchange y-axis and z-axis by the rotation and reflection of x-y axes. In the new system of coordinate axes, the structure function D'_{22} is

$$\begin{aligned} D'_{22} &= \int d\Delta u'_2 ({\Delta u'_2}^2) \int d\Delta u'_1 \int d\Delta u'_3 F'(\Delta u'_1, \Delta u'_2, \Delta u'_3) \\ &= \int d\Delta u'_2 ({\Delta u'_2}^2) \int d\Delta u'_1 \int d\Delta u'_3 F(\Delta u'_1, \Delta u'_3, \Delta u'_2) \\ &= D_{33}. \end{aligned}$$

For the isotropic flow, the distributions of $\mathbf{w}$ do not vary during the coordinate transformation, that is, $F'(\Delta u'_1, \Delta u'_2, \Delta u'_3) = F(\Delta u_1, \Delta u_2, \Delta u_3)$, and hence, $D'_{22} = D_{22}$. Finally, we have

$$D_{22} = D_{33}.$$

According to the continuity equation of the incompressible fluid

$$\partial u_i / \partial r_i = 0,$$

the differential of the structure function is

$$\begin{aligned} \frac{\partial D_{ij}}{\partial r_i} &= \left\langle [u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x})] \frac{\partial [u_j(\mathbf{x}+\mathbf{r}) - u_j(\mathbf{x})]}{\partial r_i} \right\rangle \\ &= \lim_{\Delta r \to 0} \left\langle [u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x})] \frac{[u_j(\mathbf{x}+\mathbf{r}+\Delta r \mathbf{e}_i) - u_j(\mathbf{x}+\mathbf{r})]}{\Delta r} \right\rangle \end{aligned} \quad (7)$$

For the locally isotropic turbulence (considering Eq. (4)), we then have

$$\frac{\partial D_{ij}}{\partial r_i} = 0. \quad (8)$$

According to Eqs. (5) and (8), we can obtain

$$\begin{aligned} \frac{\partial D_{ij}}{\partial r_i} &= \frac{\partial D_{N^2}}{\partial r} \frac{r_i}{r} \delta_{ij} + \left[\frac{\partial D_{L^2}}{\partial r_i} - \frac{\partial D_{N^2}}{\partial r_i} \right] \frac{r_i r_j}{r^2} + (D_{L^2} - D_{N^2}) \frac{\partial (r_i r_j / r^2)}{\partial r_i} \\ &= \frac{r_j}{r} \frac{\partial D_{L^2}}{\partial r} + (D_{L^2} - D_{N^2}) \left[2 \frac{r_j}{r^2} + \frac{1}{r^2} \frac{\partial {r_j}^2}{\partial r_j} + r_i r_j \frac{\partial (1/r^2)}{\partial r_i} \right] \\ &= \frac{r_j}{r^2} \left[r \frac{\partial D_{L^2}}{\partial r} + 2(D_{L^2} - D_{N^2}) \right] \end{aligned}$$

It then follows that D_{N^2} is uniquely determined by D_{L^2} according to

$$D_{N^2} = D_{L^2} + \frac{1}{2} r \frac{\partial}{\partial r} D_{L^2}. \quad (9)$$

When $r \ll L$, that is in the universal equilibrium range, D_{L^2} is only related to three variables, r, ν and $\langle\varepsilon\rangle$, according to the first similarity hypothesis. The units of the three variables are $\mathcal{L}$, $\mathcal{L}^2T^{-1}$ and $\mathcal{L}^2T^{-3}$. Thus, we can obtain one independent non-dimensional group from these variables, which can conveniently be taken to be $r\langle\varepsilon\rangle^{1/4}/\nu^{3/4} = r/\eta$. As we shall see, η is just the Kolmogorov scale. Then it can be obtained

$$\frac{D_{L^2}}{(\langle\varepsilon\rangle r)^{2/3}} = f\left(\frac{r}{\eta}\right), \tag{10}$$

where $f(r/\eta)$ is a non-dimensional universal function for any turbulence.

When $\eta \ll r \ll L$, that is in the inertial subrange, D_{L^2} is only related to two variables, r and $\langle\varepsilon\rangle$, according to the second similarity hypothesis. In this case there is no non-dimensional group that can be formed from r and $\langle\varepsilon\rangle$, so D_{L^2} is given by

$$D_{L^2} = C_2(\langle\varepsilon\rangle r)^{2/3}, \tag{11}$$

where C_2 is a universal constant. The transverse structure function is, from Eq. (9),

$$D_{N^2} = \frac{4}{3}D_{L^2} = \frac{4}{3}C_2(\langle\varepsilon\rangle r)^{2/3},$$

and hence, from Eq. (5), D_{ij} is given by

$$D_{ij} = C_2(\langle\varepsilon\rangle r)^{2/3}\left(\frac{4}{3}\delta_{ij} - \frac{1}{3}\frac{r_i r_j}{r^2}\right).$$

When r is very small, that is, r is in the dissipation range and is far less than the Kolmogorov scale η, the transverse and longitudinal structure functions are

$$D_{L^2} \approx r^2\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle$$

$$D_{N^2} \approx r^2\left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle. \tag{12}$$

As in the discussions of Eq. (7), the 4-order tensor $\langle(\partial u_i/\partial x_j)(\partial u_k/\partial x_l)\rangle$ is isotropic, and hence can be written

$$\left\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right\rangle = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk},$$

where α, β and γ are scales which are independent of space variables in the small domain G. In view of the continuity equation $\partial u_i/\partial x_i = 0$, it shows that

$$\left\langle \frac{\partial u_i}{\partial x_i}\frac{\partial u_k}{\partial x_k} \right\rangle = 9\alpha + 3\beta + 3\gamma = 0,$$

and hence,

$$3\alpha + \beta + \gamma = 0. \tag{13}$$

As the local isotropy, we have

$$\left\langle \frac{\partial u_i}{\partial x_i}\frac{\partial u_k}{\partial x_k} \right\rangle = 9\alpha + 9\beta + 27\gamma = 0,$$

and hence,

$$\alpha + \beta + 3\gamma = 0. \tag{14}$$

According to Eqs. (13) and (14), we have

$$\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} \right\rangle = \beta \left(\delta_{ik}\delta jl - \frac{1}{4}\delta_{ij}\delta_{kl} - \frac{1}{4}\delta_{il}\delta_{jk} \right).$$

By this equation, it shows that

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle = \frac{1}{2}\beta, \quad \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle = 2\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle, \tag{15}$$

and

$$\langle \varepsilon \rangle = \nu \left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \right\rangle = \beta\nu \left(9 - \frac{3}{4} - \frac{3}{4} \right) = \frac{15}{2}\nu\beta = 15\nu \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \tag{16}$$

From Eqs. (15), (16) and (12), we can obtain

$$\begin{aligned} D_{L^2} &\approx r^2 \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle = \frac{r^2\langle\varepsilon\rangle}{15\nu} \\ D_{N^2} &\approx r^2 \left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle = \frac{2r^2\langle\varepsilon\rangle}{15\nu} = 2D_{L^2}(r). \end{aligned}$$

By above equations, we can estimate the characteristic scale of the smallest eddies

$$\ell \sim \left(\frac{u_\ell \nu}{\varepsilon}\right)^{1/2}. \tag{17}$$

The energy dissipation rate approximates energy transfer rate and can be estimated by

$$\varepsilon \sim \frac{u_\ell^3}{\ell} \sim \frac{U^3}{L} \tag{18}$$

According to Eqs. (17) and (18), we have

$$\ell \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4}.$$

We thus define the Kolmogorov scale by convention, which is just the length scale η in Eq. (10)

$$\eta = \left(\frac{\nu^3}{\langle\epsilon\rangle}\right)^{1/4}.$$

The relationship of η/L and the Reynolds number Re is then

$$\frac{\eta}{L} \sim Re^{-3/4}. \tag{19}$$

and hence, we can see that the bigger is the Reynolds number the wider is the inertial subrange.

We also can use Eq. (19) to estimate the computational time by the Direct Numerical Stimulation(DNS). In the Fourier space, L and η can be regarded as the largest and smallest wavelength of the Fourier modes respectively. Thus the total number of the modes needed to describe 3-dimensional turbulent field is

$$N \sim \left(\frac{L}{\eta}\right)^3 \sim Re^{9/4}.$$

According to the Discrete Fourier Transformation theory(My Report , 2008), the number of the grid points equals to N and is thus proportional to $Re^{9/4}$. The integral time steps during the time on the order of the turbulent decay time(or large eddy turnover time, defined as $\langle u_i u_i\rangle/2\langle\varepsilon\rangle$) per grid point is(Pope , 2000)

$$M \sim \frac{L}{\eta} \sim Re^{3/4}.$$

We suppose that 1,000 floating point operations per grid point per time step are needed. Then the computational time in days, T, needed to perform a simulation at a computing rate of 1 gigaflop (10^9 floating point operation per second) is

$$T = \frac{10^3 NM}{10^9 \times 60 \times 60 \times 24} \sim \left(\frac{Re}{4400}\right)^3 . \tag{20}$$

According to Eq. (2) and by DNS, 33 years and a gigaflop computer are needed to simulate the turbulence with Reynolds number $Re \sim 10^5$ which is the largest value obtained in the laboratory of today. In the atmospheric turbulence, the Reynolds number is 2 or 3 orders of magnitude greater and the computational time can be as long as 30,000,000 years. Thus it impracticable to stimulate turbulence with gigaflop computers by DNS, especially the nature turbulence with high Reynolds number and complex boundary conditions.

There is also another method to obtain above results. As the non-dimensional function $f(r/\eta)$ in Eq. (10) is universal for any turbulence, we thus assume that the flow field is globally isotropic and homogeneous, that is, the distribution function of $\mathbf{u}(\mathbf{x})$ is isotropic and homogeneous, and then re-derive Eq. (8). We define the two-point correlation

$$R_{ij}(\mathbf{r}, \mathbf{x}) \equiv \langle u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r}) \rangle . \tag{21}$$

From the substitution $\mathbf{x}' = \mathbf{x} + \mathbf{r}$, we have

$$R_{ij}(\mathbf{r}, \mathbf{x}) = R_{ji}(-\mathbf{r}, \mathbf{x}'),$$

and hence, for a statistically homogeneous field,

$$R_{ij}(\mathbf{r}) = R_{ji}(-\mathbf{r}).$$

The relation of the structure function and the two-point correlation for homogeneous turbulence with $\langle u \rangle = 0$ (which is satisfied when the turbulence is globally isotropic) is

$$\begin{aligned} D_{ij}(\mathbf{r}) &= 2R_{ij}(0) - R_{ij}(\mathbf{r}) - R_{ji}(\mathbf{r}) \\ &= 2R_{ij}(0) - R_{ij}(\mathbf{r}) - R_{ij}(-\mathbf{r}), \end{aligned}$$

and hence,

$$\frac{\partial D_{ij}}{\partial r_i} = \frac{\partial D_{ij}}{\partial r_j} = 0.$$

3.2 Higher-order statistics

According to the second hypothesis of similarity and the dimensional analysis, the higher-order structure functions in the inertial subrange are

$$D_{L^n} = C_n(\langle \varepsilon \rangle r)^{n/3},$$

where $D_{L^n} = \langle [u_1(\mathbf{x}+\mathbf{r}) - u_1\mathbf{x}]^2 \rangle$ and C_n are universal constants. Kolmogorov (1941b) has computed the 3-order structure function by using the Navier-Stokes equation and obtain $C_3 = 4/5$, that is so called *4/5 law*. In the following discussion, we will assume that the flow is global isotropic and homogeneous and derive the 4/5 law(Pope , 2000; Davidson , 2004).

First, we derive the Karman-Howarth equation expressed in terms of two-point correlation for isotropic and homogeneous turbulence. By using the Novier-Stokes equation

$$\frac{\partial u_i}{\partial t} = -\frac{\partial (u_i u_k)}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k^2},$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\langle u_i u_j' \rangle &= -\langle u_i \frac{\partial u_j' u_k'}{\partial x_k'} + u_j' \frac{\partial u_i u_k}{\partial x_k} \rangle - \frac{1}{\rho}\langle u_i \frac{\partial p'}{\partial x_j'} + u_j' \frac{\partial p}{\partial x_i} \rangle \\ &\quad + \nu \langle u_i \frac{\partial^2 u_j'}{\partial {x_k'}^2} + u_j' \frac{\partial^2 u_i}{\partial x_k^2} \rangle, \end{aligned} \tag{22}$$

where $\mathbf{u}' = \mathbf{u}(\mathbf{x}+\mathbf{r})$ and $\mathbf{x}' = \mathbf{x}+\mathbf{r}$. Before simplifying above equation, we note that: (1) the ensemble average and the differentiation commute; (2) $\partial/\partial x_i = -\partial/\partial x_j' = -\partial/\partial r_i$, if they operate on averages; (3) u_i is independent of $\mathbf{x}'$ and u_j' is independent of $\mathbf{x}$. For isotropic and homogeneous turbulence (the flow is not stationary, for we use the Navier-Stokes equation of the free decaying flow), the isotropic pressure terms is $\langle p' u_i \rangle = A(r) r_i$. For incompressible flow, we have $\partial \langle p' u_i \rangle / \partial r_i = 0$, and hence, $\langle p' u_i \rangle = a r_i / r^3$. When $\mathbf{r} = r\mathbf{e}_1$, $\langle p u_i \rangle = a/r^2$. If the flow is isotropic, $\langle p u_i \rangle = 0$, and hence $a = 0$. Finally, the pressure terms equal to zero. Eq. (22) then simplifies to the more compact result,

$$\frac{\partial R_{ij}}{\partial t} = \frac{\partial}{\partial r_k}[S_{ijk} + S_{jki}] + 2\nu \nabla^2 R_{ij}, \tag{23}$$

where $S_{ijk} = \langle u_i u_j u'_k \rangle$ and the pressure terms is zero. As in Eq. (6), the two-point correlation in Eq. (23) is

$$R_{ij} = u^2 \left(\frac{f-g}{r^2} r_i r_j + g\delta_{ij} \right), \tag{24}$$

where $u = \langle u_1^2 \rangle^{1/2} = \langle u_2^2 \rangle^{1/2} = \langle u_3^2 \rangle^{1/2} = [(1/3)\langle |\mathbf{u}|^2 \rangle]^{1/2}$. Moreover the incompressible requires that

$$\frac{\partial R_{ij}}{\partial r_i} = 0,$$

and hence,

$$g(r,t) = f(r,t) + \frac{1}{2} r \frac{\partial}{\partial r} f(r,t). \tag{25}$$

The two-point correlation of triple velocities is isotropic and depends only on $\mathbf{r}$. We can write it as

$$S_{ijk} = A r_i r_j r_k + B r_i \delta_{jk} + C r_j \delta_{ik} + D r_k \delta_{ij}.$$

A similar line of argument of R_{ij} allows us to rewrite S_{ijk} as a function of $K(r) = S_{111}/u^3$ only,

$$S_{ijk} = u^3 \left[\frac{K - rK'}{2r^3} r_i r_j r_k + \frac{2K + rK'}{4r} (r_i \delta_{jk} + r_j \delta_{ik}) - \frac{K}{2r} r_k \delta_{ij} \right]. \tag{26}$$

Substituting Eqs. (24) and (26) into Eq. (23), we can obtain **the Karman-Howarth equation expressed in terms of two-point correlation**.

$$\frac{\partial}{\partial t}[u^2 r^4 f(r,t)] = u^3 \frac{\partial}{\partial r}[r^4 K(r)] + 2\nu^2 \frac{\partial}{\partial r} \left[r^4 \frac{\partial f(r)}{\partial r} \right]. \tag{27}$$

Next we derive the Karman-Howarth equation re-expressed in terms of structure functions. We note that Eq. (27) has two independent functions $f(r,t)$ and $K(r,t)$ and hence is not a closure equation. We here discuss the properties of $f(r,t)$ and $K(r,t)$. If $\mathbf{r} = r\mathbf{e}_1$, we reflect the x-axis and have

$$\begin{aligned} f(-r) &= f(r) \\ -K(-r) &= K(r). \end{aligned}$$

When r is very small, the Taylor expansion of f and K are

$$\begin{aligned} f(r) &= 1 + \frac{f''(0)}{2!} r^2 + \cdots \\ K(r) &= \frac{k'''}{3!} r^3 + \cdots, \end{aligned} \tag{28}$$

where $K'(0) = 0$ for the incompressibility. If $r = 0$, we can obtain many properties about f:

$$\left(\frac{1}{r}\frac{\partial f}{\partial r}\right)_{r=0} = f''(0,t) = \frac{\partial^2 f}{\partial r^2}, \tag{29}$$

$$\begin{aligned}
-u^2 f''(0,t) &= -u^2 \lim_{r\to 0} \frac{\partial^2}{\partial r^2} f(r,t) \\
&= -\lim_{r\to 0} \frac{\partial^2}{\partial r^2} \langle u_1(\mathbf{x} + r\mathbf{e}_1, t) u_1(\mathbf{x}, t)\rangle \\
&= -\lim_{r\to 0} \left\langle \left(\frac{\partial^2 u_1}{\partial x_1^2}\right)_{\mathbf{x}+r\mathbf{e}_1} u_1(\mathbf{x}, t)\right\rangle \\
&= -\left\langle \left(\frac{\partial^2 u_1}{\partial x_1^2}\right) u_1 \right\rangle \\
&= -\left\langle \frac{\partial}{\partial x_1}\left(u_1 \frac{\partial u_1}{\partial x_1}\right) - \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \\
&= \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle = \frac{\langle \varepsilon \rangle}{15\nu}.
\end{aligned} \tag{30}$$

According to Eqs. (30) and (29) and the terms of K in Eq. (27) vanishes at $r = 0$, we finally have

$$\frac{d}{dt} u(t)^2 = -\frac{2}{3}\langle \varepsilon \rangle. \tag{31}$$

By Eq. (21), we can see that the 2-order longitudinal structure function and f are related by

$$u(t)^2 f(r,t) = u(t)^2 - \frac{1}{2} D_{L^2}(r,t). \tag{32}$$

The 3-order longitudinal structure function can also be related to K by

$$\begin{aligned}
D_{L^3} &= \langle [u_1(\mathbf{x} + r\mathbf{e}_1) - u_1(\mathbf{x}, t)]^3 \rangle \\
&= \langle {u_1'}^3 \rangle + \langle u_1^3 \rangle - 3\langle {u_1'}^2 u_1 \rangle + 3\langle u_1^2 u_1' \rangle \\
&= -3K(-r) + 3K(r) = 6K(r).
\end{aligned} \tag{33}$$

Substituting Eqs. (32) and (33) into Eq. (31) into Eq. (27), we have the **Karman-Howarth equation expressed in terms of structure function**

$$\frac{\partial}{\partial t} D_{L^2} + \frac{1}{3r^4}\frac{\partial}{\partial r}(r^4 D_{L^3}) = \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial D_{L^2}}{\partial r}\right) - \frac{4}{3}\langle \varepsilon \rangle.$$

Integrate this equation to obtain

$$\frac{3}{r^4}\int_0^r s^4 \frac{\partial}{\partial t} D_{L^2}(s,t)ds = 6\nu \frac{\partial D_{L^2}}{\partial r} - D_{L^3} - \frac{4}{5}\langle\varepsilon\rangle r. \quad (34)$$

For isotropic turbulence in inertial subrange, we readily have **the 4/5 law** by Eq. (34)

$$D_{L^3} = -\frac{4}{5}\langle\varepsilon\rangle r.$$

We must note here that the derivation of the 4/5 law does not use the hypotheses of similarity, and hence, it shows the consistency between the Kolmogorov hypotheses and the Navier-Stokes equations. That is also probably why Frisch (1995) recognized that the 4/5 law 'constitutes a kind of 'boundary condition' on theories of turbulence: such theories, to be acceptable, must either satisfy the four-fifths law, or explicitly violate the assumptions made in deriving it'. According to dimensional analysis and the hypotheses of similarity, then the structure function skewness is

$$S \equiv \frac{D_{L^3}}{D_{L^2}^{2/3}} = \text{universal constant}.$$

Thus, we have

$$C_2 = \left(\frac{-4}{5S}\right)^{2/3}.$$

At last, we introduce a well-defined quantity, Taylor-scale Reynolds number, R_λ, that is often used,

$$R_\lambda \equiv \frac{u\lambda}{\nu},$$

where λ is Taylor's microscale,

$$\lambda \equiv \left[-\frac{1}{2}g''(0,t)\right]^{-1/2}.$$

According to Eq. (25), we have

$$g''(r,t) = 2f''(r,t) + \frac{1}{2}rf'''(r,t),$$

From above equation, the Taylor scale can be related to dissipation rate by

$$\begin{aligned}\lambda &= \frac{1}{\sqrt{2}}\left[-\frac{1}{2}f''(0,t)\right]^{-1/2} \\ &= u\left(\frac{15\nu}{\langle\varepsilon\rangle}\right)^{1/2}.\end{aligned}$$

Hence, for isotropic turbulence the Taylor-scale Reynolds number can be computed by

$$Re_\lambda = \frac{u^2}{\nu\sqrt{\left(\frac{\partial u_1}{\partial x_1}\right)^2}}$$

4 Conclusion

For any turbulence, we have following results:

- According to the hypothesis of isotropy, the transverse structure function $D_{i^n}(r) = \langle[u_i(\mathbf{x}+r\mathbf{e}_1) - u_i(\mathbf{x})]^n\rangle, (i = 2, 3; n = 1, 2, 3, \cdots)$ is

  $$D_{2^n}(r) = D_{3^n}(r)$$

  in the universal equilibrium range. If n is a odd number, we have

  $$D_{2^n}(r) = D_{3^n}(r) = 0.$$

- According to the dimensional analysis and the hypothesis of isotropy and similarity, the longitudinal structure function $D_{1^n}(n = 1, 2, 3, \cdots)$ is

  $$D_{1^n} = C_n(\langle\varepsilon\rangle r)^{n/3}.$$

- According to the Navier-Stokes equation and the hypothesis of isotropy, the 3-order longitudinal structure function is

  $$D_{111} = -\frac{4}{5}\langle\varepsilon\rangle r,$$

  that is, *the 4/5 law*. Note that the hypothesis of similarity is not used in the derivation of 4/5 law. However, this law is compatible with the hypothesis of similarity.

- According to the 4/5 law and the dimensional analysis, we have

$$C_2 = \left(\frac{-4}{5S}\right)^{2/3},$$

where $S = D_{L^3}/D_{L^2}^{2/3}$.

- According to the hypothesis of isotropy, the energy dissipation rate is

$$\langle\varepsilon\rangle = 15\nu\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle = \frac{15\nu}{2}\left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle,$$

if we assume the finiteness of energy dissipation.

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