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ORIGINAL ARTICLE

Chaotic behavior in accretion disks

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Abstract The eccentric luminosity variations of quasars are still a mystery. Analytic results of this behavior ranged from multi-periodic behavior to a purely random process. Recently, we have used the non-linear time-series analysis techniques to analyze the optical light curve of 3C 273 and found that its eccentric behavior may be a low-dimensional chaos. This result induces us to seek some non-linear mechanism for the eccentric luminosity variation. In this paper, we propose a simple non-linear accretion disk model and find that under some circumstances, the chaos appears both in the disk and in the light curve. Then we compute the outburst energy ΔI, defined by the difference between the maximum luminosity and the minimum luminosity, and the mean luminosity $\langle I \rangle$ and find that when chaos appears, $\Delta I \sim \langle I \rangle^{\alpha} \sim M^{0.5\alpha}$, where M is the mass of black hole and $\alpha \approx 1$. These results are confirmed by and/or compatible with the observational data analysis.

Keywords Quasar · Accretion disk · Chaos

1 Introduction

Since the discovery of quasar in 1963 (Smith and Hoffleit 1963), the luminosity variation has played an important role in our understanding of its nature. Although it has been subjected to extensive analysis, there is no generally accepted method of extracting the information in the light curve. Results of analysis of the light curve ranged from multi-periodic behavior (Kunkel 1967; Jurkevich 1971; Sillanpää et al. 1988; Lin 2001) to a purely random process (Manwell and Simon 1968; Terrell and Olsen 1970; Fahlman and Ulrych 1975). On the basis of these analysis, many theories which ranged from periodic mechanisms (Lin 2001; Abraham and Romero 1999; Romero et al. 2000) to superimposed random events such as the so-called Christmas tree model (Moore et al. 1982), have been proposed. Then whatever does this seemingly random light curve tell us? Could this seeming randomness be some behavior other than multi-periodic or purely random?

Also in 1963, Edward Lorenz published his monumental work entitled *Deterministic Nonperiodic Flow* (Lorenz 1963). In this paper, he found a strange behavior in a non-linear dissipative system which seems random and unpredictable, and is called Chaos. Chaotic behavior is not multi-periodic because it has a continuous spectrum. Useful information can not always be extracted from the power spectrum of a chaotic signal. On the other hand chaotic behavior is not random either because it can appear in a completely deterministic system. The concept of attractor is often used when describing chaotic behaviors. As the dissipative system evolves in time, the trajectory in state space may head for some final region called attractor. The attractor may be an ordinary Euclidean object or a fractal (Feder 1988) which has a non-integer dimension and often appears in the state space of a chaotic system. For many practical systems, we may not know in advance the required degrees of freedom and hence can not measure all the dynamic variables. How can we discern the nature of the attractor from the available

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experimental data? Packard et al. (1980) introduced a technique which can be used to reconstruct state-space attractor from the time series data of a single dynamical variable. Moreover, the correlation integral algorithm subsequently introduced by Grassberger and Procaccia (1983) can be used to determine the dimension of the attractor embedded in the new state space. These techniques constitute a useful diagnostic method of chaos in practical systems.

We have used the above-mentioned diagnostic methods to analyze the optical light curve of 3C 273, and found that its eccentric behavior may be a low-dimensional chaos (Liu 2006). This result tells us that non-linear may play a very important role in the nature of quasar. Then whatever role does the non-linear play? Could the non-linear mechanisms help us understand the eccentric luminosity variations of quasars? For seeking the keys, we in this paper propose a simple non-linear accretion disk model, by borrowing basic ideas from the standard accretion disk (Shakura and Sunyaev 1973; Pringle 1981). We find that under some circumstance the long-term behavior of the non-linear accretion disk and its light curve will be chaos.

There is few papers on the non-linear origin of the eccentric luminosity variations. Some authors (Kawaguchi et al. 1998; Dendy et al. 1998; Xiong et al. 2000) have tried to explain the variability by using the theory of Self-Organized Criticality (SOC) (Bak et al. 1987). However, SOC model is not a pure non-linear model but a complex model which always contains many degrees of freedom with strong non-linear interactions. A pure non-linear model dose not necessarily contain so many degrees of freedom, and its long-term behavior is simpler and is qualitatively different with SOC model (Bak 1995). In this paper, we specifically investigate the non-linear origin of the eccentric luminosity variations and thus only a few requisite degrees of freedom are maintained in our model. That is, for only investigating the effect of the non-linearity on the variability, we neglect many possible mechanisms, some of which may be very important, and maintain the simplest but requisite non-linear terms in our model.

The main simplifications in our model are discussed as follows. First, disks are assumed to be geometrically thin and optically thick, and are in thermal equilibrium. Second, except for the thermal radiation, other candidate radiation mechanisms, such as synchrotron radiation and braking radiation, are not considered in our model. Many authors (Czerny and Elvis 1987; Sun and Malkan 1989; Laor 1990) have found that the optical-UV continuum radiation is probably the thermal radiation of the accretion disk. Thus, our model in fact investigates the optical variability arising in the accretion disk. Additionally, accretion disks are generally magnetized and even very weak magnetic fields can completely alter the stability behavior of astrophysical gases, both rotationally and thermally, by engendering new degrees of freedom in the host fluid (Balbus 2001). Thus, magnetorotational instability may be an important non-linear mechanism for fluctuation and transportation in the accretion disk (Balbus and Hawley 1998; Balbus 2003; Ji et al. 2006). However, as the primary exploration, we do not include the magnetic field and MHD turbulence in our model. As a result of these simplifications, only the shear-induced chaos appears in our model. The shear-induced chaos is not the shear-induced turbulence. It is because that turbulence has many universal qualitative features, such as the extremely wide range of strongly and non-locally interacting degrees of freedom, which chaos does not have (Tsinober 2001). Therefore, our model could not effectively transport momentum, but it can effectively explain some observational features of optical variability, as we will investigate in the following sections.

2 Description of the chaotic accretion disk model

In the non-linear chaotic system, there are always two kinds of interacting factors and each affects the behavior of the systems in a contrary manner. For example, in the famous Logistic model (May 1976), one factor is related with the plenty food and space etc., and the other is related with the overpopulation and disease. The formal can make the population grow, but the latter makes the population decrease. The interaction of the two factors will lead to the chaotic variation of the population under some circumstances. We note that the standard accretion disk (SAD) also has two contrary factors. Due to the viscous stress, the tangential velocity will be reduced and the gas will spiral into the black hole. At the same time, the gravitational potential energy will be converted into the kinetic energy, which makes the tangential velocity increase. Thus the viscous stress and the gravity are just the contrary factors in the SAD. This is the start point of our non-linear accretion disk (NAD) model. By borrowing these basic ideas, we now construct the NAD model as follows.

We consider an accretion disk which has only two layers of gas, as in Fig. 1. It is supposed that (a) the thickness of each layer is very small that the tangential velocity in each

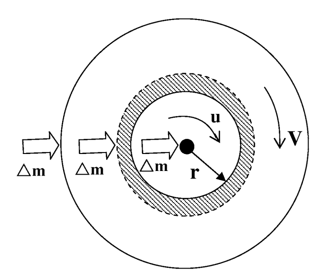

Fig. 1 The NAD model. The *black dot* at the centre of this figure represents a black hole and the shadow region is the boundary layer. Broad arrays represent the flow of mass towards black hole

layer is slightly different from its average tangential velocity. That is to say, we only need one velocity to describe the circular motion of one layer; (b) the height of disk H is very small that the motion is almost two-dimensional; (c) the average tangential velocity of outer layer is a constant and equals its average Kepler's velocity V. Due to the viscous stress, the average tangential velocity of inner layer u does not equal its average Kepler's velocity U, and is a time variable; (d) the inner layer interacts with outer layer by viscous stress, but it does not interact with the black hole; (e) during small interval $\Delta\tau$, the mass Δm flowing across any circle (all the centers of circles are on the center of black hole) is the same. If $U^2 > u^2$, the centrifugal force is less than the gravity and the gas will flow towards the black hole. Define $\Delta m > 0$ in such a case. If $U^2 < u^2$, the gas will flow far from the black hole and $\Delta m < 0$. When $U^2 = u^2$, $\Delta m = 0$. Thus, we simply suppose that $\Delta m \propto U^2 - u^2$; (f) there is a boundary layer between the outer layer and inner layer, and the effect of boundary is so strong that the velocity in this layer always tends to be uniform. We thus simply assume that when gas in the outer layer flows across the boundary layer, the average tangential velocity will change from V to U. Comparing with the radius r of the inner layer, the thickness of boundary layer h is very small that the velocity is thought to be suddenly changed at the boundary; (g) any relativistic effect is not considered here. All above are the basic assumptions of our NAD model and based on these assumptions, the equations of motion are derived as follows.

At the n^{th} moment, the viscous stress acting on the inner layer is,

$$f_{\mathrm{n}} = 2\pi r H\mu \frac{u_{\mathrm{n}} - V}{h},$$

where u_{n} is the average tangential velocity of inner layer at the n^{th} moment and μ is not the turbulent viscosity but the kinematic viscosity. It is necessary to emphasize that our model considers the viscous stress only to explain non-linear behavior in the accretion disk, not to explain the accretion process itself. Then we have,

$$2\pi r H\mu \frac{u_{\mathrm{n}} - V}{h} = m\frac{u_{\mathrm{n}} - u_{\mathrm{n+1}}{}^{-}}{\Delta\tau},$$

where m is the mass of inner layer, $u_{\mathrm{n+1}}{}^{-}$ is the average tangential velocity of inner layer at $(\mathrm{n}+1)^{\mathrm{th}}$ moment under the action of viscous stress, and $\Delta\tau$ is the lag between n^{th} and $(\mathrm{n}+1)^{\mathrm{th}}$ moments. Thus it can be obtained that,

$$u_{\mathrm{n+1}}{}^{-} = u_{\mathrm{n}} - \frac{2\pi r H\mu\Delta\tau}{hm}(u_{\mathrm{n}} - V). \tag{1}$$

When $u_{\mathrm{n}}{}^2 < U^2$, the mass Δm_{n} in inner layer will flow into the black hole, which causes equal mass in the outer layer flowing across the boundary layer into the inner layer. When the gas flows across the boundary layer, its velocity will change from V to U. According to the assumption (e),

$$\frac{\Delta m_{\mathrm{n}}}{m} = C\left(1 - \frac{u_{\mathrm{n}}{}^2}{U^2}\right) \tag{2}$$

where C is a dimensionless parameter. At the $(\mathrm{n}+1)^{\mathrm{th}}$ moment, the average tangential velocity under the action of the inflow or outflow is

$$u_{\mathrm{n+1}}{}^{+} = \frac{(m - \Delta m_{\mathrm{n}})u_{\mathrm{n}} + \Delta m_{\mathrm{n}} U}{m} = u_{\mathrm{n}} + (U - u_{\mathrm{n}})\frac{\Delta m_{\mathrm{n}}}{m}.$$

According to (2), we have

$$u_{\mathrm{n+1}}{}^{+} = u_{\mathrm{n}} + C\left(1 - \frac{u_{\mathrm{n}}{}^2}{U^2}\right)(U - u_{\mathrm{n}}). \tag{3}$$

When $u_{\mathrm{n}}{}^2 > U^2$, there is only mass flowing out of the inner layer, which will not change the average tangential velocity of inner layer. Thus,

$$u_{\mathrm{n+1}}{}^{+} = u_{\mathrm{n}}. \tag{4}$$

Finally, the average tangential velocity of inner layer at the $(\mathrm{n}+1)^{\mathrm{th}}$ moment is

$$u_{\mathrm{n+1}} = u_{\mathrm{n+1}}{}^{+} + u_{\mathrm{n+1}}{}^{-} - u_{\mathrm{n}}.$$

According to (1), (3) and (4), we have

$$u_{\mathrm{n+1}} = \begin{cases} u_{\mathrm{n}} + C(1 - u_{\mathrm{n}}{}^2/U^2)(U - u_{\mathrm{n}}) \\ \quad - (2\pi r H\mu\Delta\tau/hm)(u_{\mathrm{n}} - V) \\ \quad \text{when } u_{\mathrm{n}}{}^2 \le U^2 \\ u_{\mathrm{n}} - (2\pi r H\mu\Delta\tau/hm)(u_{\mathrm{n}} - V) \\ \quad \text{when } u_{\mathrm{n}}{}^2 > U^2. \end{cases}$$

Normalizing above equations, we have

$$\overline{u_{\mathrm{n+1}}} = \begin{cases} \overline{u_{\mathrm{n}}} + C(1 - \overline{u_{\mathrm{n}}}^2)(1 - \overline{u_{\mathrm{n}}}) - A(\overline{u_{\mathrm{n}}} - B) \\ \quad \text{when } \overline{u_{\mathrm{n}}}^2 \le 1 \\ \overline{u}_{\mathrm{n}} - A(\overline{u}_{\mathrm{n}} - B) \\ \quad \text{when } \overline{u_{\mathrm{n}}}^2 > 1 \end{cases} \tag{5}$$

where $\overline{u_{\mathrm{n}}} = u_{\mathrm{n}}/U$ and dimensionless parameters

$$A = \frac{2\pi r H\mu\Delta\tau}{hm}, \qquad B = \frac{V}{U}. \tag{6}$$

We assume that the whole heat produced by viscous stress was radiated out along the direction perpendicular to the accretion disk. Thus the luminosity at the n^{th} moment is

$$I_{\mathrm{n}} = f_{\mathrm{n}} \cdot (u_{\mathrm{n}} - V) = 2\pi r H\mu \frac{(u_{\mathrm{n}} - V)^2}{h}.$$

Normalizing above equation, we have

$$\overline{I_n} = A(\overline{u_n} - B)^2, \quad (7)$$

where the dimensionless luminosity is

$$\overline{I_n} = \frac{I_n \Delta\tau}{mU^2}.$$

From (6), we can see that A is related to the viscous and geometry properties of the NAD. For avoiding the singularities in NAD model, the inequality

$$\left|\frac{d\overline{u}_{n+1}}{d\overline{u}_n}\right| < 1$$

must be satisfied when $\overline{u_n}^2 > 1$. Then we can obtain that

$$0 < A < 2.$$

Under the assumption (a) and (6), we have

$$B = \sqrt{\frac{3r_g}{r}} = \sqrt{\frac{6GM}{c^2 r}}, \quad (8)$$

where r_g is the Schwarzschild radius, c is the velocity of light and M is the mass of the black hole. Clearly, B is related to the mass of black hole and the geometry properties of the NAD and

$$0 < B < 1.$$

The parameter C can be also related to some physical quantities. The distance that the mass Δm at the outer edge of inner layer travels during $\Delta\tau$ is,

$$\Delta s = \frac{U^2 - u^2}{2r}\Delta\tau^2.$$

Then we can obtain that

$$\frac{\Delta m}{m} = \frac{\pi\rho H U^2 \Delta\tau^2}{m}(1 - \overline{u}^2). \quad (9)$$

Comparing (9) with (2), we find that

$$C = \frac{\pi\rho H U^2 \Delta\tau^2}{m} = \frac{U^2 \Delta\tau^2}{r^2} = \frac{c^2 \Delta\tau^2}{6r^2}.$$

We can see that C is only related to the geometry properties of the NAD. Here we assume that the mass of inflow or outflow during Δt does not exceed the mass of the inner layer, that is,

$$-1 < \frac{\Delta m_n}{m} < 1. \quad (10)$$

As $\Delta m_n/m < 1$ is always satisfied when $\overline{u_n}^2 \leq 1$, we will choose

$$0 < C < 1$$

in the following discussions. Among these parameters, we can see that only A is related to the viscous property of NAD, and only B is related to the mass of black hole. By tuning the corresponding parameter, we can easily know what happens to the behavior of the NAD and its light curve when some physical property changes. Some results are given in the following sections.

3 Results

In this section, some properties of our model will be discussed by numerical computation. First, we will discuss the long-term chaotic behavior of NADs with different viscous properties. As discussed in the above section, the viscous property of NAD is only related to the parameter A. Thus in our following discussion both B and C are fixed but A is adjustable. Figures 2a and 2b are typical bifurcation diagrams when $B = 0.01$ and $C = 0.8$. They are very similar to the Logistic bifurcation diagram (Hilborn 1994). From these figures, we can see that the behavior of NAD will become more and more complex with the increase of viscosity. Specially, there is a piece of vague area on the right side of each bifurcation diagram. The behavior of u in Fig. 2a (or I in Fig. 2b) in this area will be chaotic. Figure 2c is the evolution of the average tangential velocity u of the inner layer and Fig. 2d is the light curve generated by NAD model. Both of them are plotted when $A = 1.95$, $B = 0.01$, $C = 0.8$, with which the system is chaotic. Here we must note that the choice of the parameters is not arbitrary. Any choice of the parameters should guarantee that (10) must be satisfied. By many numerical computations, we find that when A is greater than about 1.9 and C is greater than about 0.6, (10) is satisfied and the luminosity is chaotic.

The correlation integral (Grassberger and Procaccia 1983) is also used to analyze our model. Let $X_1, X_2, \ldots, X_N$ are samples of a physical variable (u or I in our model) at the $\mathrm{i^{th}}$ moment, $\mathrm{i} = 1, 2, \ldots, \mathrm{N}$. The correlation integral is defined as

$$\mathrm{Cor(R)} = \frac{1}{\mathrm{N(N-1)}}\sum_{\mathrm{i=1}}^{\mathrm{N}}\sum_{\mathrm{j=1, j\neq i}}^{\mathrm{N}}\theta(\mathrm{R} - |X_{\mathrm{i}} - X_{\mathrm{j}}|),$$

where $\theta(x)$ is the Heaviside function,

$$\theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Grassberger and Procaccia (1983) have studied many chaos models and found that

$$\mathrm{Cor(R)} \propto \mathrm{R}^D,$$

where D is called correlation dimension. Strictly D is not the dimension of the attractor, but is very close to it. Ac-

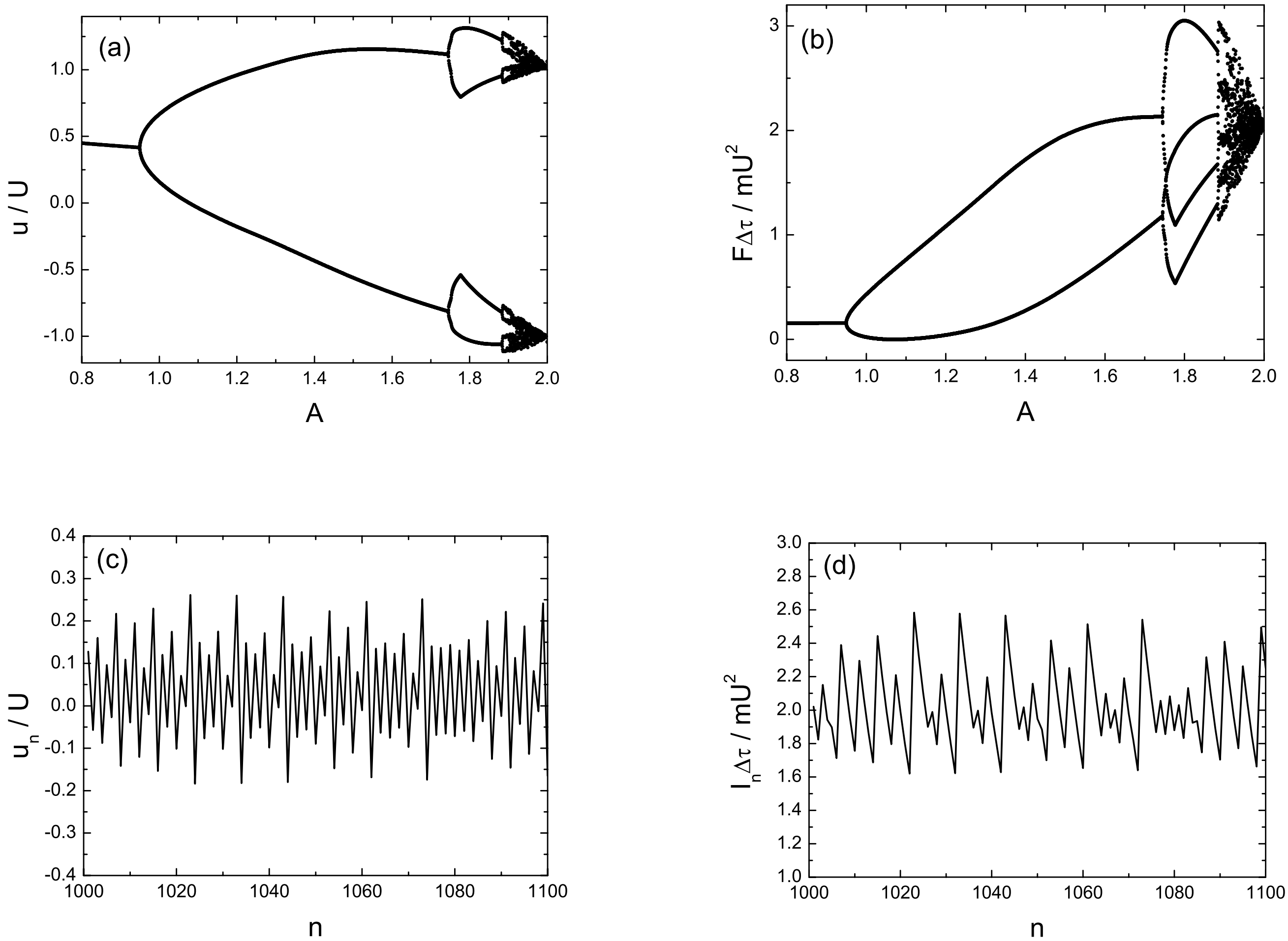

Fig. 2 (**a**) and (**b**) are bifurcation diagrams of NAD model when $B = 0.01$ and $C = 0.8$. (**a**) is computed by picking a value of A and from the initial point $u_0 = 1$, iterating (5) 1000 times to allow the trajectory to approach the attractor and plotting the next 5000 values of u. Then (**b**) is computed by (7). (**c**) is the average tangential velocity u_n of inner disk as a function of time and is computed by picking $A = 1.95$, $B = 0.01$ and $C = 0.8$, iterating (5) 1000 times from initial point $u_0 = 1$ and then plotting the next 100 values of u. For clearly showing its chaotic behavior, all the values of u_n are minus 0.9 if they are greater than zero; conversely, they are plus 0.9. (**d**) is the light curve computed by (7) also with $A = 1.95$, $B = 0.01$ and $C = 0.8$

cording to chaos and non-linear dynamics, if an attractor of a dissipative system has a non-integer dimension, then it is a chaotic attractor (Hilborn 1994). Therefore, the correlation integral is a useful diagnostic tool of chaos. However, itself does not have any physical meaning. Figures 3a and 3b are typical results of correlation integral when $A = 1.95$, $B = 0.01$, $C = 0.8$. From these results, we can see that attractors in the state space of u and I are indeed chaotic attractors.

We then discuss the relation between the outburst energy and the mean luminosity. According to Pica and Smith (1983), the outburst energy ΔI is defined by the difference between maximum luminosity $\overline{I}(\max)$ and minimum luminosity $\overline{I}(\min)$, that is,

$$\Delta I = \overline{I}(\max) - \overline{I}(\min).$$

For investigating the relation between the outburst energy and the mass of black hole, we here fix the parameters A and C and compute the luminosity with different B. In order to satisfy (10) and make chaos appear in the light curve, we choose $A > 1.9$ and $C > 0.6$. Then we compute the outburst energy and the mean luminosity $\langle I \rangle$, and find that

$$\Delta I \sim \langle I \rangle^{\alpha}, \tag{11}$$

where $\alpha \approx 1$, as in Fig. 4. This result is very similar with the observational facts found in the Fig. 7 of Pica and Smith (1983), where they found that for most sources it appears that the outburst energy scales with the mean luminosity. Additionally, we should note that B, the only parameter which is related to the mass of the black hole, does not appear in the power exponent of (11) and this result in fact suggests that the mass of black hole may not affect the na-

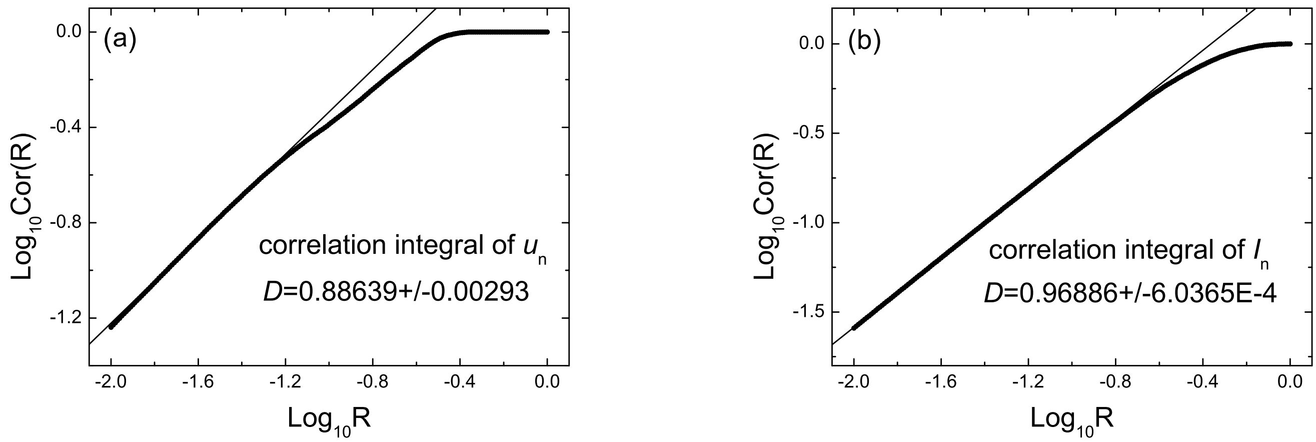

Fig. 3 (**a**) and (**b**) are correlation integrals when $A = 1.95$, $B = 0.01$ and $C = 0.8$. They are computed by using 5000 data after 1000 iterations as in Fig. 2(a)–(d), and the corresponding correlation dimension is given in each plot

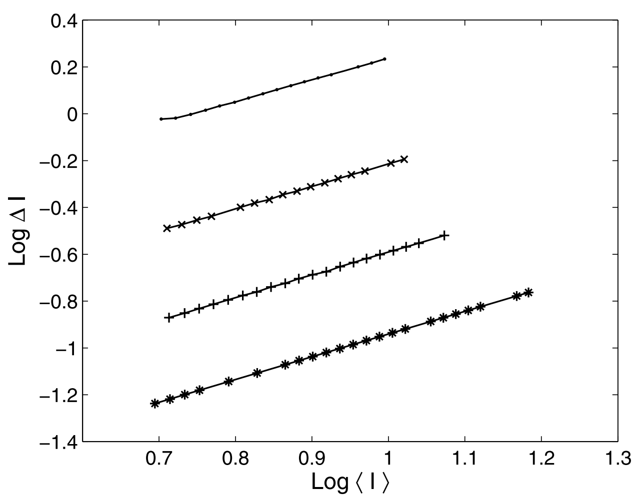

Fig. 4 The outburst energy ΔI as a function of the mean luminosity I_m. Here, we choose four groups of parameters. In each group, parameters A and C are fixed, whereas B varies in the interval (0, 1), so long as the light curve is chaotic and (10) is satisfied. The samples of the light curve are produced as in Fig. 2 and the results are plotted with *lines* and *different symbols*. The corresponding parameters and the slope of the line α are given as follows. *Point*, $A = 1.95, C = 0.8, \alpha \approx 0.93$; *cross*, $A = 1.97, C = 0.8, \alpha \approx 0.96$; *plus*, $A = 1.98, C = 0.8, \alpha \approx 0.97$; *star*, $A = 1.98, C = 0.6, \alpha \approx 0.97$

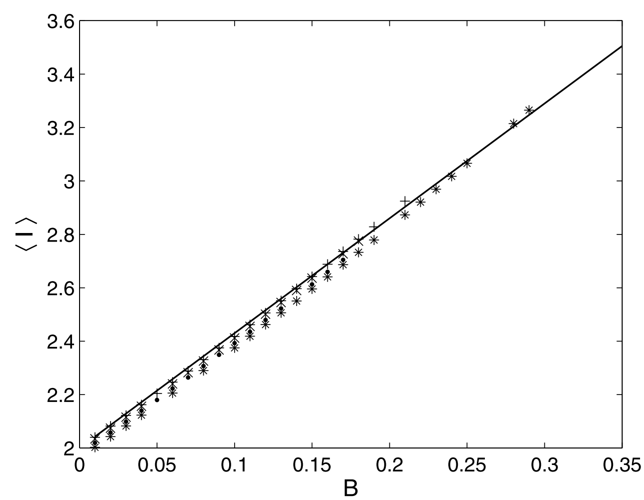

Fig. 5 The mean luminosity as a function of the parameter B. The parameters, the samples of the light curve and the corresponding symbols are the same as in Fig. 4. The *line* in the plot is $\langle I\rangle = 4.3B + 2$

ture of the properties of the luminosity variation. Quasars with big black holes will obey the similar rule of luminosity variation as the small ones. By fixing A and C, the relation between mean luminosity and the mass of black hole is also discussed. We find that the mean luminosity scales with B, as in Fig. 5. With (8), we conclude that,

$$\langle I\rangle \sim M^{1/2}. \tag{12}$$

Combining (11) with (12), we immediately obtain that,

$$\Delta I \sim M^{\beta}, \tag{13}$$

where $\beta \approx 1/2$. Wold et al. (2007) have reported that it is evident that the sources displaying largest variability amplitudes have, on average, higher black hole masses, although there is no linear relationship between them. Equation (13) is compatible with their results.

4 Discussion and conclusion

In this paper we propose a non-linear accretion disk model (NAD) which can be used to describe the chaotic behavior observed in the light curve of quasar 3C 273 (Liu 2006). We note that the tangential velocity of accretion disk is influenced by two factors. One is the viscous stress and the other is the flow of mass towards black hole. Under some circumstances, one of them may reduce the velocity and the other may increase it. Because of non-linear, the winner of gaining upper hand among them always varies with time. Then some irregular oscillation of the accretion disk would come out, which finally leads to the chaotic luminosity variation.

Although any complex multi-periodic mechanism and unnatural random event is not included, very simple non-linear terms in our model can also produce extremely complex behavior. In addition, our model shows that when the chaos appears, the outburst energy ΔI, defined by the difference between the maximum luminosity and the minimum luminosity, can be related to the mean luminosity $\langle I \rangle$ and the mass of black hole M by $\Delta I \sim \langle I \rangle^{\alpha} \sim M^{0.5\alpha}$, where $\alpha \approx 1$. There relations are confirmed by and/or compatible with the observational data analysis (Pica and Smith 1983; Wold et al. 2007).

Due to the production of extreme energy observed in the AGNs, the standard accretion disk model (SAD) is recognized as a standard picture of AGNs (Lynden-Bell 1969). The NAD model just borrows the basic idea of SAD, and thus it is expected that the chaotic behavior presents in this model can be found in other AGNs. Some authors have reported that various types of long-term variability exhibited by the black hole system GRS 1915 + 105 may be a low-dimensional chaos (Misra et al. 2006). However, we hope that there will be more evidences to support or oppose the assertions of our model, especially the quantitative relations predicted in this paper.

For understanding the underlying non-linear nature of the variability in the light curve, we ignore many physical details in our model, some of which may be important. For example, the effects of the thermodynamics (Narayan and Yi 1994), the interaction between the inner edge of the inner layer and the black hole (Krolik and Hawley 2002), and the details of the fluid dynamics happening in the boundary layer are not considered here. Additionally, we also ignore the MHD turbulence which may be an important mechanism for fluctuation and transportation in the accretion disks (Balbus and Hawley 1998; Balbus 2003; Ji et al. 2006). As a result, the shear-induced chaos, which is not the shear-induced turbulence, appears in our model (Tsinober 2001). Totally, our model is just a primary exploration and there are still many interesting problems remaining unsolved. For example, are there new features of the variability if we introduce the shear-induced turbulence in the accretion disk? Can the variability in other band be explained if more radiation mechanisms are introduced in our NAD model? What is the effect of MHD turbulence on the variability? How does the light curve behave if the MHD turbulence and shear-induced turbulence both exist in the accretion disk? Are they independent? If not, how are they related and how do they affect the behavior of the light curve? We believe that the nature of variability would be grasped if we solve these problems in the future work.

Overall, our results reveal that the non-linear plays a key role in the cause of the eccentric luminosity variation of quasars. The omnipresent non-linear is very important in our understanding of this complex world, as chaos and non-linear dynamics tells us. Thus it is nature to study the luminosity variation of quasars from the view of non-linear. And we hope chaos and non-linear dynamics may be helpful in the further study of quasars.

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