Comment on "Conjectures on exact solution of three-dimensional (3D) simple orthorhombic Ising lattices"*

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Abstract

It is shown that a recent article by Z.-D. Zhang is in error and violates well-known theorems.

After receiving an electronic reprint of Zhang's recent paper [1] some time ago, I have had an email exchange with the author pointing out a number of errors in the paper, which unfortunately invalidate all its main results. As now also follow-up papers [2, 3] have appeared using Zhang's erroneous results, I felt finally compelled to write down some of my criticism. The editor of the journal has supplied me with copies of a competing comment

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[4] and Zhang's reply to it. Here I shall bring up several other issues with [1] that are not discussed in [4].

Putative "exact solutions" of the 3d Ising model have been advocated before, see e.g. [5, 6, 7]. In 1952 J.R. Maddox (the later editor of Nature) showed his solution [5] of the 3d Ising model at the StatPhys 2 conference in Paris.[1] The error in his calculation was caught at the meeting and was the result of an incorrect application of the Jordan-Wigner transformation [8]. This error has been made also by Zhang in eqs. (15) and (16) [(3.3) and (3.4) on pages 12 and 13][2] of [1].

Using both Kaufman's original notations—see eq. (11) in [8]—and more modern notations, the Jordan–Wigner transformation is given by

$$\begin{aligned} s_j \equiv \quad & \sigma_j^x && = \left[\prod_{k=1}^{j-1} i\Gamma_{2k-1}\Gamma_{2k}\right]\Gamma_{2j-1}, \\ is_jC_j \equiv \quad & \sigma_j^y && = \left[\prod_{k=1}^{j-1} i\Gamma_{2k-1}\Gamma_{2k}\right]\Gamma_{2j}, \\ C_j \equiv \quad & -\sigma_j^z && = i\Gamma_{2j-1}\Gamma_{2j}, \quad \Gamma_{2j-1}\equiv P_j, \quad \Gamma_{2j}\equiv Q_j. \end{aligned} \tag{1}$$

It is essential that all Γ matrices anticommute ($\Gamma_k\Gamma_l = -\Gamma_l\Gamma_k$, $k \neq l$, but $\Gamma_k^2 = 1$), in order to be able to use the theory of spinor representations of the rotation group. In Zhang's paper [1], j runs from 1 to nl, corresponding to his $(\mathbf{r}, \mathbf{s})$ running from $(1, 1)$ to (n, l), or $j = (n-1)r + s$. Then one finds,[3]

$$\sum_j \sigma_j^x\sigma_{j+1}^x = \sum_j i\Gamma_{2j}\Gamma_{2j+1}, \quad -\sum_j \sigma_j^z = \sum_j i\Gamma_{2j-1}\Gamma_{2j}, \tag{2}$$

agreeing with (15a) and (15c) [(3.3a) and (3.3c)] of [1], but one should have

$$\sum_j \sigma_j^x\sigma_{j+n}^x = \sum_j i\Gamma_{2j}\left[\prod_{k=j+1}^{j+n-1} i\Gamma_{2k-1}\Gamma_{2k}\right]\Gamma_{2j+2n-1}, \tag{3}$$

whereas in (15b) [(3.3b)] of [1] one finds a quadratic form equivalent to

$$\sum_j \sigma_j^x\sigma_{j+n}^x = \sum_j i\Gamma_{2j}\Gamma_{2j+2n-1}. \tag{4}$$

[1] In the proceedings he is called M. Maddox, with M for Monsieur (Mister in French).

[2] References to equations and pages of the arXiv preprint are given within square brackets.

[3] I omit here the extra U-factors in the terms $j = nk$ (for $k = 1, \ldots, l$) resulting from periodic boundary conditions [8], which are also present in [1]. With open boundary conditions there are no such U's.

There is a corresponding error in (16) [(3.4)] of [1], as one can verify that the P's and the Q's there do not all anticommute as is required.[4]

This is the first major error in [1]. Therefore, Zhang has gotten a free fermion model in three dimensions in his formulae (15), just like Maddox [5]. The U factors in (15) are irrelevant, since Zhang could have started with open boundary conditions. In the thermodynamic limit for the bulk free energy per site the boundary terms have no effect because of the Bogoliubov inequality, as the surface to volume ratio vanishes for the infinite system. After this error there is no need to go to a fourth dimension as done in (17) on page 5317 [(3.5) on page 14] of [1]; Maddox's result follows more directly.

The high-order terms in (3) above and in the corresponding exponential factors of the transfer matrix render the 3d Ising model inaccessible to exact solution. Barry Cipra, Sorin Istrail and others have even claimed that the 3d Ising model is NP-complete [9]. This means that one should not expect a simple closed-form solution similar to the one of the 2d Ising model in zero field to exist.

One of the main results of [1] is formula (49) for the partition function per site on page 5325 [(3.37) on page 26], which has three parameters given in the appendix. On page 5399 [page 137] one finds eqs. (A.1), (A.2) and the following text, where these three parameters are expressed as $w_x = 1$, $w_y = w_z$ equal to an expression with the coefficients b_0 through b_{10} fitted such that the high-temperature series is recovered. Therefore, this expression for the free energy contains no more information than the known coefficients of the high-temperature series used.

On page 5400 [page 139] Zhang insists that $w_y = w_z \equiv 0$ as soon as the temperature is finite. This is discussed further in eqs. (A.11)–(A.13) on pages 5405–5406 [pages 145–146], with κ the usual high-temperature variable $\tanh K$. There is a marked difference between the “high-temperature limit” (A.11) and eq. (A.13) for more general temperature, as the author chooses $w_x = 1$, and $w_y = w_z = 0$, as soon as the temperature is finite, which is highly inconsistent with the earlier fit.

Indeed, the procedure is clearly wrong as the convergence of the high-temperature series has been rigorously proved in the 1960s [10, 11] and this proof has been quoted in many textbooks [12, 13, 14]. This proof is based on

[4]Lou and Wu [7] correctly apply an equation equivalent to (3) (using Z for σ^x and X for $-\sigma^z$). However, they go wrong after (77) in [7], as they assume too much commutativity for the factors of P. I thank Dr. Zhang for bringing [7] to my attention.

the proof of Gallavotti and Miracle-Solé [10] of the convergence of the fugacity expansion by a use of the Kirkwood–Salzburg equations for the lattice-gas, which is equivalent to the Ising model. Another theorem of Lebowitz and Penrose [11] is then used to establish a finite radius of convergence for the correlation functions and the free energy expressed as series in $1/T$. They are even real analytic up to a critical point [11, 13]. Paper [1] therefore violates well-established theorems. The statements on page 5376 [page 102] are, therefore, manifestly wrong.

Another criticism concerns the result for the spontaneous magnetization given in eqs. (102) and (103) on page 5342 [(4.28) and (4.29) on page 50]. This can be expanded as $I = 1 - 6x^8 + \ldots$, with $x = \exp(-2K)$, with $K = J/k_{\mathrm{B}}T$. However, in Table 2 on page 5380 [page 154] one finds $I = 1 - 2x^6 + \ldots$, taken from the well-known low-temperature series in the literature. That $I - 1$ starts with x^8, corresponds to the fact that each site in 4d has 8 neighbors; it should be x^6 for 6 neighbors in 3d. Zhang's result is analytic in the low-temperature variable x, up to his critical point and it also gives the exact value $I = 1$ at $T = 0$. It has a finite radius of convergence expanded as a series in x. Therefore, it must agree with the well-known series result in Table 2, which it does not.

It has also been established that in the ferromagnetic Ising model the thermodynamic bulk limit converges to a unique state, apart from $H = 0$, $T < T_{\mathrm{c}}$ where the state can be any convex combination of the states obtained by the infinite-volume limits with all boundary spins up or all down, see [14] and references quoted. One can then study an infinite hierarchy of a discrete version of the Schwinger–Dyson equations connecting the correlation functions with an odd number of spins in the thermodynamic limit with all spins up on the boundary. This way one can easily and rigorously establish the start of the low-temperature series for the spontaneous magnetization, in agreement with the old results in the literature. Hence, because of the discrepancy, Zhang's result is manifestly wrong.

Much more can be said about the correlation functions, susceptibility, and critical exponents in sections 5, 6, and 7. Again, all the main results are in error. I will not go into more detail as this should already be clear from the arguments above.

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