ABC inequality

Show that

$$a^2 + ac + c^2 + 3b(a + b + c) \geq 0, \ \forall \ a, b, c.$$

Proof

We need to show

$$f(b) = 3b^2 + 3(a + c)b + \left(a^2 + ac + c^2\right) \geq 0 .$$

Since $3 > 0$, we are led to

$$\begin{aligned} \Delta &= 9(a + c)^2 - 12\left(a^2 + ac + c^2\right) \\ &= 3\left[3a^2 + 6ac + c^2 - 4a^2 - 4ac - 4c^2\right] \\ &= -3\left[a^2 - 2ac + c^2\right] \\ &= -3(a - c)^2 \leq 0. \end{aligned}$$

The proof is complete if we check the graph of $f(b)$ as a function b, with a, c being constant.

Note The inequality was showed by Dr. Wu to me this morning at yaodisgrp.