Convexity

Let $f: I \to R$ be strictly convex, where $I \subset R$ is an interval. If x_0 is a minimal point of f, then x_0 is the unique minimal point of f.

Proof

- Since f is strictly convex,

  $$f(\lambda x + (1-\lambda)x_0) < \lambda f(x) + (1-\lambda) f(x_0)$$

  for all $x \in I$, $x \neq x_0$, $\lambda \in (0,1)$.

- Since x_0 is a minimal point of f,

  $$\exists \delta > 0 \text{, s.t. } |y - x| < \delta \Rightarrow f(y) \geq f(x).$$

- Thus

  $$\lambda \in \left(0, \min\left\{\frac{1}{|x - x_0|}, 1\right\}\right)$$

  $$\Rightarrow f(x_0) \leq f(\lambda x + (1-\lambda)x_0) < \lambda f(x) + (1-\lambda) f(x_0)$$

  $$\Rightarrow f(x_0) < f(x).$$

The proof is complete.