2009-11-17

Middle Test on Mathematical Analysis

For freshman

1. If $f'(x_0)$ exists, find the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + 3\Delta x) - f(x_0 - 2\Delta x)}{\Delta x}.$$

Answer: $5f'(x_0)$.

2. Let ϕ and ψ be twice differentiable on $[\alpha, \beta]$,

and

$$\begin{cases} x = \phi(t), \\ y = \psi(t). \end{cases}$$

Find the second derivative $\frac{d^2y}{dx^2}$.

Answer: $\frac{d^2y}{dx^2} = \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3}$.

3. Find the following two limits:

1) $\lim_{n \to \infty} \frac{n^{\frac{2}{3}} \sin n^2}{n + 1}$;

2) $\lim_{n \to \infty} \left[\frac{1}{\sqrt{n^2 + 1}} + \cdots + \frac{1}{\sqrt{n^2 + n}} \right]$.

Answer: 1) 0 ; 2) 1.

4. Assume that $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$, show that $\{a_n\}$ converge.

5. Let $a_n = \frac{c^n}{n\,!}\ (c > 0)$, $n = 1, 2, \cdots$. Show that $\lim_{n\to\infty} a_n$ exist and find its value.

Answer: 0.

6. Find the following limits:

1) $\lim_{x\to\frac{\pi}{2}} (1 + \cos x)^{\frac{3}{\cos x}}$;

2) $\lim_{x\to 0^+} \frac{x}{\sqrt{1 - \cos x}}$.

Answer: 1) e^3 ; 2) $\sqrt{2}$.

7. Let

$$f(x) = \begin{cases} \frac{\ln(1+x)}{x}, & if\ x > 0; \\ 1, & if\ x = 0; \\ \frac{\sqrt{1+x} - \sqrt{1-x}}{x}, & if\ -1 \le x < 0. \end{cases}$$

If f continuous at 0? Show your answer.

Answer: Indeed, f is continuous at 0.

8. If $f(x)$ is continuous on $[a, b]$, and $f > 0$. Show that $\dfrac{1}{f(x)}$ is continuous on $[a, b]$ using $\varepsilon - \delta$ language.

9. Assume that f is continuous on $[a, b]$, and has $n + 1\,(n \geq 1)$ different zeros. If f is n times differentiable on (a, b). Show that $\exists\ \xi \in (a, b)$, such that $f^{(n)}(\xi) = 0$.

Answer: Invoking Rolle's theorem n times.

10. Let f be differentiable on $[a, b]$. Show that $\exists\ \xi \in (a, b)$, such that

$$3\xi^2 \left[f(b) - f(a)\right] = \left(b^3 - a^3\right) f'(\xi).$$

Answer: Applying Cauchy's intermediate derivative theorem.