# A CONSERVATIVE STABILIZATION SCHEME OF SPECTRAL / FINITE ELEMENT METHOD FOR SOLVING RADIATIVE TRANSFER EQUATION

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## ABSTRACT

Based on a recently proposed stabilization scheme of spectral element method (SEM) for solving radiative transfer equation, namely, adaptive isotropical diffusion (AISO) scheme, which can effectively mitigate the ray effects in the solution of radiative transfer equation, a conservative adaptive isotropical diffusion (CAISO) scheme that satisfies radiation energy equation is developed, as such, overcomes the drawback of AISO scheme which can not strictly preserve the radiative energy per unit volume. In the CAISO scheme, besides an artificially added isotropical diffusion term, a new compensate term is designed and added to recover the energy loss caused by the artifical diffusion term, hence the energy equation is exactly satisfied. A SEM based on the CAISO scheme (CAISO-SEM) is presented. The performances of the CAISO-SEM for solving radiative transfer equation are verified by benchmark problems. The CAISO scheme inherited the advantages of AISO scheme, such as, easy and efficient to be implemented under the spectral or finite element method framework, and very good performance in mitigating the 'wiggles' in both low and high order spatial approximation. Numerical experiments show that the CAISO-SEM is stable, high order accurate and effective to solve radiative transfer in simple and complex geometry, and also robust to mitigate ray effects of different origins.

## NOMENCLATURE

| | |
|---|---|
| $C$ | Adjustment parameter |
| $h_a, h_s$ | Local angular and spatial discretization scale |
| $G$ | Incident radiation, W/m$^2$ |
| $\mathbf{H}$ | Matrix defined in Eq. (13b) |
| $I$ | Radiative intensity, W/(m$^2$sr) |
| $I_b$ | Black body radiative intensity, W/(m$^2$sr) |
| $\mathbf{K}$ | Matrix defined in Eq. (13a) |
| $L$ | Side length of rectangular medium, m |
| $\mathbf{n}_w$ | Unit normal vector of wall |
| $N_{sol}$ | Total number of solution nodes |
| $N_\theta, N_\varphi$ | Discretization number of polar and azimuth angle |
| $\mathbf{q}$ | Radiative heat flux, W/m$^2$ |
| $\mathbf{r}$ | Spatial coordinates vector |
| $S$ | Source function defined in Eq. (2) |
| $V$ | Solution domain |
| $w$ | Weight for angular quadrature |
| $\alpha$ | Artificial diffusion coefficient |
| $\beta$ | Extinction coefficient, m$^{-1}$ |
| $\varepsilon_w$ | wall emissivity |
| $\phi$ | Shape function |
| $\Phi$ | Scattering phase function |
| $\kappa_a$ | Absorption coefficient, m$^{-1}$ |
| $\kappa_s$ | Scattering coefficient, m$^{-1}$ |
| $\sigma$ | Stefan-Boltzmann constant, W/m$^2$K$^4$ |
| $\tau_L$ | Optical thickness |
| $\omega$ | Single scattering albedo |
| $\mathbf{\Omega}, \mathbf{\Omega}'$ | Vector of radiation direction |
| $\Omega$ | Solid angle |

*Subscripts and Superscripts*

| | |
|---|---|
| $i, j$ | Spatial solution node index |
| $m$ | The $m$th angular direction |
| $w$ | Value at wall |

## INTRODUCTION

In recent years, many methods have been developed and investigated for solving the radiative transfer equation (RTE) in absorbing, emitting, and/or scattering medium. Among them, the methods based on direct discretization of RTE have received considerable attention owing to their good compromise between accuracy, flexibility and computational efforts. This group of methods include the traditional discrete ordinate method (DOM) (Fiveland, 1988), finite volume method (FVM) (Raithby and Chui, 1990), and some new contributions, such as spectral or *hp* finite element methods (Pontaza and Reddy, 2005; Zhao and Liu, 2006) and discontinuous Galerkin methods (Cui and Li, 2005;

Liu and Liu, 2007), which have shown very good performance. However, two kinds of numerical error exist in these methods as shown by former studies, namely, ray effects and false scattering, which set critical limitation in the application of these methods. The ray effects are attributed to the angular discretization, while the false scattering is attributed to spatial discretization. Furthermore, these two kinds of errors are not independent. The interaction process of these errors has not been well understood.

The ray effects are often considered a dominant error by comparison with false scattering in many cases for it will cause unrealistic 'wiggles' in the results and may even totally spoil the solution. Therefore, some former works were especially focused on the mitigation of ray effects. Ramankutty and Crosbie (1997) developed a modified DOM (MDOM) to solve the radiative transfer in semitransparent media. Based on the same principle as MDOM, Coelho (2002, 2004) proposed a new improved version of MDOM (NMDOM). In the MDOM or NMDOM, the radiative intensity have to be decomposed into two parts, namely, direct component and a diffuse component, in which the former is solved analytically, while the diffuse component is solved by the DOM. As a result, the solution process of the MDOM is rather complex and difficult to be implemented as compared to the original DOM or FVM. Recently, by taken a completely different approach used in the MDOM and NDOM, Zhao and Liu (2008) developed an adaptive isotropical diffusion (AISO) scheme, which can successfully mitigate the ray effects and can be implemented without excessive additional effort in the spectral or finite element method framework. Though the AISO scheme is effective in mitigates the ray effects, one drawback is that it does not exactly satisfy the radiation energy equation, hence can not strictly preserve the radiative energy per unit volume.

In this paper, based on the AISO scheme, a conservative adaptive isotropical diffusion scheme (CAISO) is developed, which strictly satisfies the radiative energy equation, as such overcomes the drawback of the AISO scheme. In the CAISO scheme, besides an artificially added isotropical diffusion term, a new compensate term is designed and added to recover the energy loss caused by the artifical diffusion term, hence the energy equation is totally satisfied. The artificial diffusion coefficient is determined from both local angular discretization scale and local spatial discretization scale as in the AISO scheme. The performances of CAISO scheme combined with SEM for solving radiative transfer equation is verified by benchmark problems.

## FORMULATION OF THE CAISO SCHEME

### Radiative Transfer Equation

The radiative transfer equation in absorbing, emitting, and scattering gray media can be written as

$$\mathbf{\Omega}\bullet\nabla I(\mathbf{r},\mathbf{\Omega})+\beta I(\mathbf{r},\mathbf{\Omega})=S(\mathbf{r},\mathbf{\Omega}) \tag{1}$$

where the source term $S(\mathbf{r})$ is given as

$$S(\mathbf{r},\mathbf{\Omega})=\kappa_a I_b(\mathbf{r}) + \frac{\kappa_s}{4\pi}\int_{4\pi} I(\mathbf{r},\mathbf{\Omega}')\Phi(\mathbf{\Omega},\mathbf{\Omega}')\,\mathrm{d}\Omega' \tag{2}$$

Here $\mathbf{r}$ is spatial coordinate vector, $\mathbf{\Omega}$ is radiation direction vector, $\beta$ is the extinction coefficient, $\kappa_a$ and $\kappa_s$ are the absorption and scattering coefficients, respectively, $\Phi$ is the scattering phase function.

As mentioned in the introduction, the ray effects and false scattering encounters in the numerical solution of the RTE, of which the ray effects may cause unrealistic 'wiggles' in the results and even totally spoil the solution, thus stabilization techniques are necessary to be taken to overcome this problem. In the following section, we will introduce a new stabilization scheme, namely, the CAISO scheme for the solution of RTE.

### CAISO Scheme for Solving RTE

The CAISO scheme can be considered as an improved version of AISO scheme developed in Zhao and Liu (2008). First, we give a brief introduction to the AISO scheme. In the AISO scheme, an isotropical artificial diffusion term is added to the original RTE, which results in a modified RTE:

$$\mathbf{\Omega}\bullet\nabla I+\beta I-\alpha\nabla^2 I=S \tag{3}$$

where $\alpha$ is a small parameter known as artificial diffusion coefficient, which is determined adaptively from local spatial and angular discretization scale. Though the added artificial diffusion term was demonstrated to be able to successfully mitigate the ray effects, it will bring the imbalance of energy per unit volume shown following.

By integrating Eq. (3) over whole angular space, the corresponding radiation energy conservation equation for the modified RTE is obtained as

$$\nabla\bullet\mathbf{q}=\kappa_a\left(4\pi I_b-G\right)+\alpha\nabla^2 G \tag{4}$$

where $\mathbf{q}$ is the radiative heat flux and $G$ is the incident radiation. As can be seen, Eq. (4) is not exactly the same as the radiation energy conservation equation for the existence of the additional last term, namely, $\alpha\nabla^2 G$, which is introduced by artificial diffusion. The additional term appeared in energy equation is considered to cause physically energy imbalance over per unit volume. As a result, if the artificial diffusion coefficient is not zero, the AISO scheme cannot strictly conserve the radiative energy per unit volume.

In order to ensure sufficient numerical stability and also exactly satisfy the radiation energy equation at the same time, an energy compensation term is added to Eq. (3), which yields the following CAISO scheme:

$$\mathbf{\Omega}\bullet\nabla I+\beta I-\alpha\nabla^2 I=S-\alpha\frac{1}{4\pi}\nabla^2 G \tag{5}$$

The corresponding radiation energy conservation equation for the modified RTE of CAISO scheme can be obtained by integrate Eq. (5) over angular space,

$$\nabla\bullet\mathbf{q}=\kappa_a\left(4\pi I_b-G\right) \tag{6}$$

As can be seen, Eq. (6) is exactly the radiation energy equation and does not depend on the artificial diffusion coefficient, that is, with adding the compensatory term, the energy diffusion term appeared in AISO scheme (Eq. (4)) is completely compensated. Hence the CAISO scheme strictly preserves the radiation energy equation.

**Discretization by Spectral Element Method**

The angular space discretization is conducted by discrete ordinates approach, which results in a set of discrete ordinates equation of the CAISO scheme:

$$\mathbf{\Omega}^m \bullet \nabla I^m + \beta I^m - \alpha \nabla^2 I^m = S^m - \alpha \frac{1}{4\pi} \nabla^2 G \quad (7)$$

in which

$$S^m = \kappa_a I_b + \frac{\kappa_s}{4\pi} \sum_{m'=1}^{M} I^{m'} \Phi(\mathbf{\Omega}^m, \mathbf{\Omega}^{m'})\, w^{m'} \quad (8)$$

$$G = \sum_{m'=1}^{M} I^{m'}(\mathbf{r})\; w^{m'} \quad (9)$$

The boundary conditions are given for opaque and diffuse upwind walls ( $\mathbf{\Omega}^m \bullet \mathbf{n}_w < 0$ ) as

$$I_w^m = \varepsilon_w I_{bw} + \frac{1-\varepsilon_w}{\pi} \sum_{\mathbf{n}_w \bullet \mathbf{\Omega}^{m'} > 0} I_w^{m'} \left| \mathbf{n}_w \bullet \mathbf{\Omega}^{m'} \right| w^{m'} \quad (10)$$

where $\varepsilon_w$ is the wall emissivity and $w^{m'}$ is the weight of direction $\mathbf{\Omega}^{m'}$ for angular quadrature.

The spatial discretization is conducted by spectral element method (SEM), which can be considered as a special kind of finite element method. The feature of SEM is that nodal basis functions $\phi_i(\mathbf{r})$ are constructed on each element by orthogonal polynomial expansion. In this paper, Chebyshev polynomial expansion is employed. Details on building global nodal basis function were described in Zhao and Liu (2006). The unknown radiative intensity can be approximated by nodal basis function as

$$I^m(\mathbf{r}) \simeq \sum_{i=1}^{N_{sol}} I_i^m \phi_i(\mathbf{r}) \quad (11)$$

where $\phi_i$ is the nodal basis function, $I_i^m$ denotes radiative intensity of direction $\mathbf{\Omega}^m$ at solution nodes $i$, and $N_{sol}$ is the total number of solution nodes.

Assuming the artificially diffusion diminishes along the boundary, substitute Eq. (11) into Eq. (7) and apply the standard Galerkin approach, the discretization of CAISO scheme can be written in matrix form as

$$K_{ji}^m I_i^m = H_j^m \quad (12)$$

where

$$K_{ji}^m = \int_V \left( \mathbf{\Omega}^m \bullet \nabla \phi_i + \beta \phi_i \right) \phi_j \, \mathrm{d}V + \left[ C h_a + (1-C) h_{s,j}^{p+1} \right] \int_V \nabla \phi_i \bullet \nabla \phi_j \, \mathrm{d}V \quad (13a)$$

$$H_j^m = \int_V S^m \phi_j \, \mathrm{d}V + \left[ C h_a + (1-C) h_{s,j}^{p+1} \right] \sum_{i=1}^{N_{sol}} \frac{G_i}{4\pi} \int_V \nabla \phi_i \bullet \nabla \phi_j \, \mathrm{d}V \quad (13b)$$

Here, the similar method as in AISO scheme is used for the determination of artificial diffusion coefficient $\alpha$, which fully account for the coupled manner of ray effects and false scattering, namely,

$$\alpha_j = \alpha(\mathbf{r}_j) = C h_a + (1-C) h_{s,j}^{p+1} \quad (14)$$

$$h_a = \max_m (h_a^m) \quad (15)$$

where, $C \in [0,1]$ is a balancing parameter for angular and spatial discretization error, $p$ is the order of polynomial approximation for SEM, $h_a^m$ is the local angular scale of direction $\mathbf{\Omega}^m$ defined as

$$h_a^m = w^m \quad (16)$$

and $h_{s,j}$ is the local spatial length scale at the node $j$ defined as

$$h_{s,j} = h_s(\mathbf{r}_j) = \sqrt[d]{\tilde{w}_j}, \quad \tilde{w}_j = \int_V \phi_j(\mathbf{r}) \mathrm{d}V \quad (17)$$

where $d$ denotes spatial dimension, $\tilde{w}_j$ can be considered a virtual volume around node $j$. It is noted that the sum of $\tilde{w}_j$ through each node equals the volume of solution domain. The value of the balancing parameter $C$ is determined through numerical experiment.

**RESULTS AND DISCUSSION**

The CAISO scheme discretized by SEM (CAISO-SEM) presented above are implemented using a procedure given in Zhao and Liu (2006). The matrix equation of each direction given by Eq. (12) is solved by Gaussian elimination method. The maximum relative error $10^{-4}$ of incident radiation ($\left| G_{new} - G_{old} \right| / G_{new}$) is taken as stop criterion for global iteration. Two benchmark test cases are selected to verify the performance of the presented method.

**Case 1: Square enclosure filled with isotropical scattering media**

Radiative heat transfer in a square enclosure filled with isotropical scattering gray media is considered. The configuration of the enclosure and problem definition is shown in Fig. 1. The scattering albedo of the media is $\omega$ =1.0. The optical thickness based on the side length $L$ of the square is $\tau_L$ =1.0. The temperature of left wall is kept hot ( $T_{w1}$ =1000K), while all other walls and the media are kept cold (0K). This case was studied by several researchers (Crosbie and Schrenker, 1984; Larsen and Howell, 1985; Ramankutty and Crosbie, 1997; Zhao and Liu, 2007) and serves a good test case to verify the performance of the numerical method.

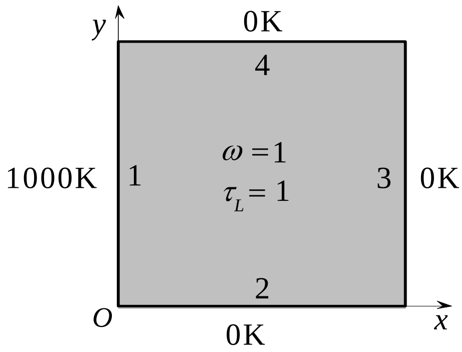



Fig. 1 Square enclosure and problem definition.

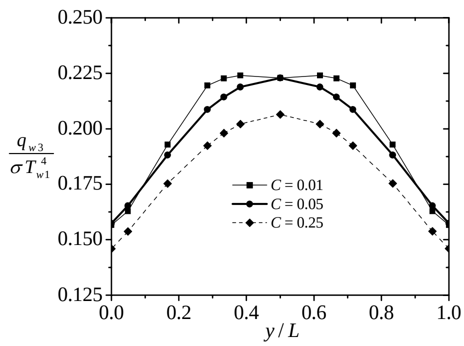
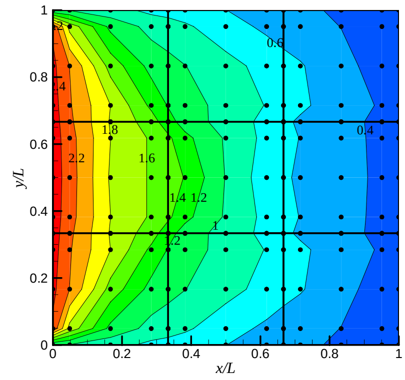


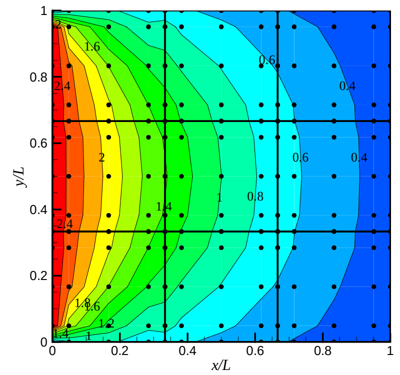

Fig. 2 The effect of balancing parameter $C$ on stabilization performance of CAISO scheme.

The CAISO-SEM is applied to solve the radiative heat flux distribution along the right wall. First the balancing parameter $C$ is determined from numerical experiment. Figure 2 shows results obtained by the CAISO-SEM under different values of $C$, namely, $C$ =0.01, 0.05 and 0.25. Here the square enclosure is uniformly decomposed into 9 elements and 4th order polynomial approximation is used (shown in Fig. 3), the angular discretization takes $S_8$. It can be seen that very small value of $C$ result in bad stability, while very large value of $C$ result in large false scattering. Here, the value of $C$ takes 0.05 works very well. This is the same value confirmed and used in the AISO scheme based SEM (AISO-SEM) (Zhao and Liu, 2008). In following analysis, $C$ is taken as 0.05 for more general verification.

Figure 3 (a) and (b) shows the contour plot of the dimensionless incident radiation ( $G/\sigma T_{w1}^4$ ) field obtained by CAISO-SEM and the conventional Galerkin approach based SEM (Galerkin-SEM) under the same spatial and angular discretization, respectively. It is clear that severe 'wiggles' exists in the contour plot of incident radiation field obtained by Galerkin-SEM, while the results obtained by CAISO-SEM is smooth and free of 'wiggles' caused by ray effects.




(a)




(b)

Fig. 3 Computational grid of SEM ( $M \times p = 3\times 4$ ) and comparison of dimensionless incident radiation field obtained by different methods: (a) SEM with Galerkin approach; (b) SEM based on CAISO scheme.

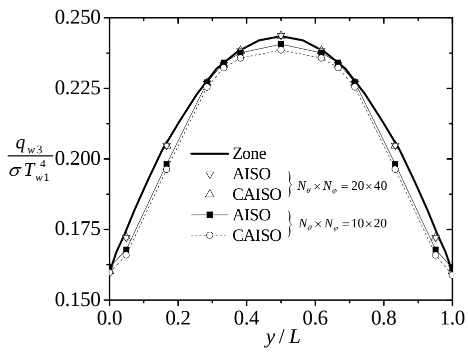



Fig. 4 Comparison of the SEM based on CAISO scheme and AISO scheme.

A comparison of the performance of the CAISO-SEM and AISO-SEM is shown in Fig. 4 for solving dimensionless radiative heat flux along the right wall. Here two angular decomposition schemes are used, namely PCA approach with $N_\theta \times N_\varphi = 10\times 20$ and $N_\theta \times N_\varphi = 20\times 40$ . For different angular discretization, the results obtained by CAISO-SEM and AISO-SEM are free of 'wiggles'. Though the CAISO-SEM and the AISO-SEM give comparable results, the former is considered to be more reliable than the latter from theoretical considering.

Angular convergence test of the CAISO-SEM is shown in Fig. 5 for different angular decomposition schemes, and the result obtained by Galerkin-SEM is also shown for comparison. The result obtained by Zone method (Crosbie and Schrenker, 1984) is selected as a benchmark. The spatial decomposition is the same as former analysis. The CAISO-SEM is stable for different angular decomposition schemes and stably converged to the benchmark solution with angular refinement.

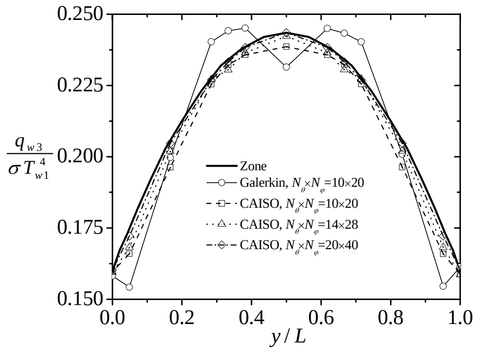



Fig. 5 Angular convergence test of the SEM based on CAISO scheme and comparison with Galerkin approach.

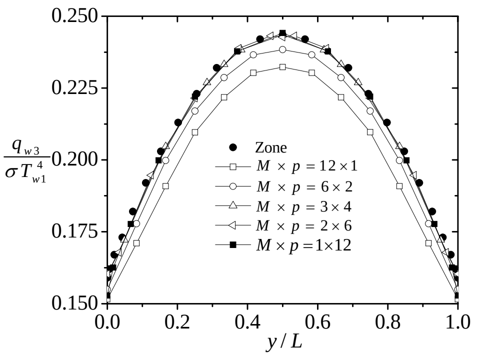



Fig. 6 Spatial *p*-convergence test of the SEM based on CAISO scheme.

Spatial *p*-convergence test of the CAISO-SEM is shown in Fig. 6 for solving the radiative heat flux distribution along the right wall of square enclosure obtained under 5 different spatial decomposition, namely $M \times p$ =$12\times1$, $6\times2$, $3\times4$, $2\times6$ and $1\times12$. Here, the square enclosure is decomposed uniformly into quadrilateral elements and the spatial decompositions are denote as $M \times p$, where $M$ is the number of elements per side of square enclosure and $p$ is the order of polynomial approximation. In this notation, the total number of elements is $N_{el} = M \times M$ and total number of solution nodes is $N_{sol} = \left(M \times p + 1\right)^2$. The spatial decomposition schemes are selected to have the same number of solution nodes. Here, angular discretization takes PCA approach with $N_\theta \times N_\varphi = 20 \times 40$ for all computation. With increasing the order of polynomial approximation, the accuracy of the result of CAISO-SEM increase rapidly as compared to reference result and no 'wiggles' exist in the solutions, which demonstrate the robustness of the CAISO-SEM in both low and high order polynomial approximation and higher order approximation gives better accuracy, which demonstrate the virtues of AISO scheme is inherited by CAISO scheme.

**Case 2: Semicircular enclosure with a circular hole**

The ray effects encountered in the former case is induced by the discontinuity of boundary thermal loading, which is well interpreted in Zhao and Liu (2007). As a further verification, in this case, we consider radiative heat transfer in a semicircular enclosure with a circular hole filled with nonscattering gray media as shown in Fig. 7. The optical thickness of the media is $\tau_L = \beta R$ =0.1. The media is kept hot (1000K), while all other walls are black and kept cold (0K). In this case, the circular hole plays a role as an obstacle. The shielding effect of the obstacle will cause discontinuity in angular distribution of radiative intensity along the bottom wall, which will induce ray effects (Zhao and Liu, 2007).

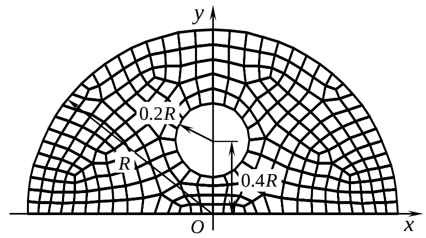



Fig. 7 Configuration of the semicircular enclosure and mesh used for computation (272 elements).

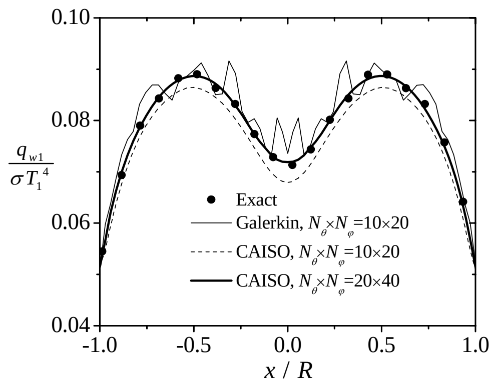



Fig. 8 Comparison of the SEM based on CAISO scheme and AISO scheme.

The CAISO-SEM is applied to obtain the radiative heat flux along the bottom wall of the enclosure. The enclosure is decomposed into 272 elements (shown in Fig. 7). Figure 8 shows the results obtained by the CAISO-SEM with $p = 2$ and two different angular discretization schemes, namely, PCA approach with $N_\theta \times N_\varphi = 10 \times 20$ and $N_\theta \times N_\varphi = 20 \times 40$. The exact solution obtained by Kim et al. (2001) is taken here as a benchmark. The result obtained by Galerkin-SEM under the same spatial discretization is also shown as a comparison. It is clear that strong 'wiggles' exists in the results obtained by the Galerkin-SEM. However, the results of CAISO-SEM are free of 'wiggles' for different angular decomposition. With the refinement of angular discretization, the result of CAISO-SEM stably approaches the exact result. Though the origin that induces ray effects is different than the former discussed case, the CAISO-SEM is demonstrated to be very robust to solve this kind of problem.

## CONCLUSIONS

A new stabilization scheme for mitigate the ray effects encountered in the solution of radiative transfer equation, named conservative adaptive isotropical diffusion (CAISO) scheme, is developed. The CAISO scheme is based on the adaptive isotropical diffusion (AISO) scheme. The CAISO scheme strictly satisfies radiation energy equation, thus overcomes the drawback of AISO scheme which theoretically can not preserve the radiative energy per unit volume.

In the CAISO scheme, besides an artificially added isotropical diffusion term, a new compensate term is designed and added to recover the energy loss caused by the artifical diffusion term and the energy equation is exactly satisfied. The artificial diffusion coefficient is determined heuristically from both local angular discretization scale and local spatial discretization scale as in AISO scheme.

A Spectral element method based on the CAISO (CAISO-SEM) scheme is presented to solve the radiative heat transfer. The CAISO scheme is demonstrate to inherit the advantages of AISO scheme, such as, easy and efficient to be implemented under the spectral or finite element method framework, and very good performance in mitigating the 'wiggles' in both low and high order spatial approximation. Numerical experiments show that the CAISO-SEM is stable, high order accurate and effective to solve radiative transfer in simple and complex geometry, and robust to mitigate ray effects of different origins.

## ACKNOWLEDGEMENTS

The support of this work by the National Nature Science Foundation of China (50425619, 50336010) is gratefully acknowledged.

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