

# On Diameter and Average Distance of Graphs\*

Zhou Tao<sup>†</sup> Xu Junming<sup>†</sup> Liu Jun<sup>†</sup>

## Abstract

The diameter and average distance of a graph are two important parameters to measure the efficiency of interconnection networks. Ore gave an upper bound of the number of edges of an undirected graph in terms of order and diameter of the graph. Entringer *et al* gave a lower bound of the average distance of an undirected graph and, respectively, a digraph in terms of order and the number of edges of the graph. The present paper provides short proofs of these two results and gives a counterpart of Ore's result for a digraph, and improves Entringer *et al*'s results in term of diameter of the graph. Combining our results with Ore's will yield a new lower bound on  $\sigma(G)$  better than that given by Plesnik.

**Key words:** diameter, distance, average distance, extremal problem

**Subject Classification** (GB/T13745-92): 110.74

## 关于图的直径和平均距离

周涛<sup>†</sup> 徐俊明<sup>†</sup> 刘隽<sup>†</sup>

**摘要:** 本图的直径和平均距离是度量网络有效性的两个重要参数。Ore 通过图的顶点数和直径给出无向图的最大边数。Entringer, Jakson, Slater 和 Ng, Teh 通过图的顶点数和边数分别给出无向图和有向图平均距离的下界。该文提供这两个结果的简单证明, 给出有向图类似 Ore 的结果, 并通过图的直径改进 Entringer 等人的结果到更一般的情形。结合本文和 Ore 的结果, 可以得到一个无向图和有向图平均距离的下界, 它比 Plesnik 得到的下界更好。

**关键词:** 运筹学, 图论, 网络, 直径, 平均距离

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## 1 Introduction

We follow Bondy and Murty<sup>[1]</sup> for graph-theoretical terminology and notation not defined here. In the present paper, a graph  $G = (V, E)$  always means a simple graph with vertex-set  $V = V(G)$  and edge-set  $E = E(G)$ . The cardinality  $|V(G)|$  is called the order of  $G$ , and  $\varepsilon(G) = |E(G)|$  is the size or the number of edges of  $G$ . The distance from a vertex  $x$  to another vertex  $y$ , denoted by  $d_G(x, y)$ , is defined as the length of a shortest path from  $x$  to  $y$  in  $G$ . The diameter of  $G$ , denoted by  $d_G(G)$ , is defined as the maximum distance between any pair of vertices in  $G$ . The average distance of  $G$  with order  $v (\geq 2)$ , denoted by  $\mu(G)$ ,

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<sup>†</sup>Department of Math. University of Science and Technology of China, Hefei, 230026, China; 中国科学技术大学数学系, 合肥市, 230026.

is defined as

$$\mu(G) = \frac{\sigma(G)}{v(v-1)},$$

where  $\sigma(G) = \sum_{x,y \in V} d_G(G; x, y)$ .

The diameter and average distance are two important parameters to measure the efficiency of interconnection networks. It is the reason why the two concepts have received considerable attention in the literature. There are several excellent surveys of earlier results on diameter and average distance of graphs, see, for example, Chung<sup>[2]</sup> and Plesnik<sup>[3]</sup>.

Of the known results, two are considered in the present paper. The one is well-known Ore's theorem<sup>[4]</sup> that gives an upper of  $\varepsilon(G)$  for an undirected graph  $G$  in terms of order and diameter of  $G$ . The other, due to Entringer, Jakson, Slater<sup>[5]</sup> and Ng, Teh<sup>[6]</sup>, gives a lower bound of  $\sigma(G)$  for an undirected graph and, respectively, a digraph in terms of order and size of the graph. Ore's theorem has been used in the proofs of many extremal problems (see, for example, [7] and [8]). The original proof, however, is somewhat cumbersome. Apart from average distance,  $\sigma(G)$  also occurs in the computation of other graph-theoretical parameters, such as the forwarding index of a routing [9] and [10].

We, in the present paper, provides a short proof of Ore's theorem and a counterpart of a digraph. We also give a short proof of Entringert et al's results. It can be easily found from our proof that Entringert et al's lower bound on  $\sigma(G)$  is obtained subject to the diameter of  $G$  being at most two tacitly. We will improve this lower bound in term of diameters of the graph subject to the diameter more than two. Combining our results with Ore's will yield a new lower bound on  $\sigma(G)$  better than that given by Plesnik (see, Theorem 2 and Theorem 3 in [3]).

## 2 On Ore's theorem

A graph is called a  $(v, k)$ -graph if it has order  $v$  and diameter  $k$ . It is clear that any (undirected or directed) graph of order  $v$  can be obtained by removing some edges from a complete graph  $K_v$  or a complete digraph  $K_v^*$  of order  $v$ . It is this simple observation that can be used to give a short proof of Ore's theorem.

**Theorem 1** (Ore<sup>[4]</sup>) *For any connected undirected  $(v, k)$ -graph  $G$ ,*

$$\varepsilon(G) \leq k + \frac{1}{2}(v - k + 4)(v - k - 1).$$

**Proof** Use the symbol  $\eta(v, k)$  to denote the minimum  $\eta$  for which  $\eta$  edges must be removed from  $K_v$  to obtain a  $(v, k)$ -graph. Thus, for any  $(v, k)$ -graph  $G$ , we have that

$$\varepsilon(G) \leq \varepsilon(K_v) - \eta(v, k). \tag{1}$$

Assume that  $G'$  is a  $(v, k)$ -graph obtained by removing  $\eta(v, k)$  edges from  $K_v$ . Clearly,  $\varepsilon(G) \leq \varepsilon(G')$ . Let  $P = (x_0, x_1, \dots, x_k)$  be a shortest path of length  $k$  in  $G'$ . Obviously, any two vertices  $x_i$  and  $x_j$  in  $P$  with  $j - i > 1$  are nonadjacent in  $G'$  because of the shortness of  $P$ . This implies that at least  $\frac{1}{2}k(k-1)$  edges should be removed from  $K_v$  to obtained  $G'$ .

On the other hand, because of the shortness of  $P$  for any vertex  $x$  in  $G'$  not in  $P$ , if exists, it can not be adjacent to two vertices  $x_i$  and  $x_j$  in  $P$  with  $|x_i - x_j| > 2$  at the same time. Thus,  $x$  can be adjacent to at most 3 vertices in  $P$ . In other words, there are at least

$(k-2)$  vertices in  $P$  not adjacent to  $x$ . As a result,  $(v-k-1)(k-2)$  extra edges in  $K_v$  must be also removed to obtained  $G'$ . It follows that

$$\eta(v, k) \geq \frac{1}{2}k(k-1) + (v-k-1)(k-2). \quad (2)$$

Combining (1) and (2), we have that

$$\begin{aligned} \varepsilon(G) \leq \varepsilon(G') &\leq \frac{1}{2}v(v-1) - \frac{1}{2}k(k-1) - (v-k-1)(k-2) \\ &= k + \frac{1}{2}(v-k+4)(v-k-1), \end{aligned}$$

as desired and the theorem follows.

Using the same way, we can easily obtain an upper bound on the number of edges of  $(v, k)$ -digraphs.

**Theorem 2** *For any strongly connected  $(v, k)$ -digraph  $D$ ,*

$$\varepsilon(D) \leq v(v-k+1) + \frac{1}{2}(k^2 - k - 4).$$

**Proof** Assume that  $D'$  is a  $(v, k)$ -digraph obtained by removing  $\eta(v, k)$  edges from a complete digraph  $K_v^*$ . Clearly,  $\varepsilon(D) \leq \varepsilon(D')$ . Let  $P = (x_0, x_1, \dots, x_k)$  be a shortest directed path of length  $k$  in  $D'$  from  $x_0$  to  $x_k$ . Clearly, for any two vertices  $x_i$  and  $x_j$  in  $P$  with  $j-i > 1$ , the edge  $(x_i, x_j) \notin E(D')$ . Thus, at least  $\frac{1}{2}k(k-1)$  edges in  $K_v^*$  have to be removed to obtain  $D'$ . Also because of the shortness of  $P$ , for any vertex  $y$  in  $D'$  not in  $P$ , if exists, the two edges  $(x_i, y)$  and  $(y, x_{i+3})$  can not appear in  $D'$  simultaneously for each  $i = 0, 1, \dots, k-3$ . This fact implies that at least  $(v-k-1)(k-2)$  extra edges must be removed from  $K_v^*$  to obtain  $D'$ . Thus, we have

$$\begin{aligned} \varepsilon(D) \leq \varepsilon(D') &\leq \varepsilon(K_v^*) - \frac{1}{2}k(k-1) - (v-k-1)(k-2) \\ &= v(v-k+1) + \frac{1}{2}(k^2 - k - 4) \end{aligned}$$

as desired and the theorem follows.

From our proofs, it is easy to construct a  $(v, k)$ -graph and a  $(v, k)$ -digraph such that the upper bounds given in Theorem 1 and Theorem 2 can be attained, respectively.

### 3 On average distance

The following lower bounds on  $\sigma(G)$  are obtained by Entringer, Jackson and Slater [5] for an undirected graph and Ng and Teh [6] for a digraph. We here provide a short proof by using the same way as one used in the above section.

**Theorem 3** *Let  $G$  be a graph with  $v$  vertices and  $\varepsilon$  edges, then*

- (a)  $\sigma(G) \geq 2v(v-1) - 2\varepsilon$  if  $G$  is an undirected graph;
- (b)  $\sigma(G) \geq 2v(v-1) - \varepsilon$  if  $G$  is a digraph.

Moreover, the equality occurs if  $d(G) \leq 2$ .

**Proof** We consider an undirected graph  $G$  obtained by removed  $p$  edges from a complete graph  $K_v$ . Then

$$\varepsilon(G) = \frac{1}{2}v(v-1) - p. \quad (3)$$

Suppose that these  $p$  edges are  $x_1y_1, x_2y_2, \dots, x_py_p$ . Then

$$\sigma(G) = \sum_{xy \in E(G)} d_G(x, y) + \sum_{i=1}^p [d_G(x_i, y_i) + d_G(y_i, x_i)]. \quad (4)$$

Obviously,  $d_G(x, y) = d_G(y, x) = 1$  if  $xy \in E(G)$ , and  $d_G(x, y) = d_G(y, x) \geq 2$  otherwise. It follows from (4) that

$$\sigma(G) \geq 2\varepsilon + 4p. \quad (5)$$

Combining (3) and (5) yields the conclusion (a).

The same consideration to a digraph  $D$ , we have  $\varepsilon(D) = \varepsilon(K_v^*) - p = v(v-1) - p$ , and so

$$\begin{aligned} \sigma(D) &= \sum_{(x,y) \in E(D)} d_D(x, y) + \sum_{i=1}^p d_D(x_i, y_i) \\ &\geq 2\varepsilon + 2p = 2v(v-1) - \varepsilon. \end{aligned}$$

The last assertion that the equality occurs if  $d(G) \leq 2$  is obviously from our proof. The theorem follows.

It can be found from our proof that the lower bound on  $\sigma(G)$  in Theorem 3 is obtained subject to  $d(G) \leq 2$  tacitly. Note that if  $d(G) = 1$ , then  $G$  is a complete graph and so there is nothing to do for  $\sigma(G)$ . We can suppose that  $d(G) \geq 2$  in our discussion below and improve the lower bounds of  $\sigma(G)$  which depends on  $d(G)$  strongly.

**Theorem 4** *Let  $G$  be an undirected graph with  $v$  vertices and  $\varepsilon$  edges. If  $d(G) = k \geq 2$ , then,*

$$\sigma(G) \geq \begin{cases} 2v(v-1) - 2\varepsilon + \frac{1}{3}k(k-1)(k-2) + \frac{1}{2}(v-k-1)(k-2)(k-4), & \text{if } k \text{ is even;} \\ 2v(v-1) - 2\varepsilon + \frac{1}{3}k(k-1)(k-2) + \frac{1}{2}(v-k-1)(k-3)^2, & \text{if } k \text{ is odd.} \end{cases}$$

**Proof** For each  $j = 1, 2, \dots, k$ , let  $c_j$  be the number of ordered pairs of vertices with distance  $j$ . Clearly,

$$\sigma(G) = \sum_{j=1}^k jc_j, \quad c_1 = 2\varepsilon, \quad \sum_{j=2}^k c_j = v(v-1) - 2\varepsilon.$$

Thus, we have that

$$\begin{aligned}
\sigma(G) &= c_1 + \sum_{j=2}^k jc_j \\
&= c_1 + 2 \sum_{j=2}^k c_j + \sum_{j=2}^k (j-2)c_j \\
&= 2v(v-1) - 2\varepsilon + \sum_{j=2}^k (j-2)c_j \\
&= 2v(v-1) - 2\varepsilon S,
\end{aligned} \tag{6}$$

where  $S = \sum_{j=2}^k (j-2)c_j$ .

So we need to only consider pairs of vertices with distance larger than 2. Let  $P = (x_0, x_1, \dots, x_k)$  be a shortest path of length  $k$  in  $G$  and let  $c_P^l$  be the number of pairs of vertices in the set  $V_P = \{x_0, x_1, \dots, x_k\}$  with distance  $l$  ( $1 \leq l \leq k$ ) in  $P$ . Then

$$d_P(x_i, x_j) = |i - j|, \quad \text{and} \quad c_P^l = k + 1 - l.$$

It follows that the contribution of  $V_P$  to  $S$  is

$$S_P = 2 \sum_{l=3}^k (l-2)c_P^l = 2 \sum_{l=3}^k (l-2)(k+1-l) = \frac{1}{3}k(k-1)(k-2). \tag{7}$$

For any vertex  $y \in V_O = V(G) \setminus V_P$ , we have

$$d_G(x_i, y) + d_G(x_j, y) = d_G(y, x_i) + d_G(y, x_j) \geq |i - j|, \quad x_i, x_j \in V_P.$$

We consider the pairs of vertices  $x_i$  and  $x_{k-i}$  in  $p$ . Note that  $d_G(x_i, y) + d_G(x_{k-i}, y)$  will do some contribution to  $S$  if  $|k - 2i| > 4$ . Thus, the contribution of a single vertex  $y$  to  $S$  is

$$S_y \geq \begin{cases} 2[2 + 4 + \dots + (k-4)] = \frac{1}{2}(k-2)(k-4), & \text{if } k \text{ is even;} \\ 2[1 + 3 + \dots + (k-4)] = \frac{1}{2}(k-3)^2, & \text{if } k \text{ is odd.} \end{cases}$$

It follows that the contribution of  $V_O$  to  $S$  is

$$S_O \geq |V_O| \cdot S_y \geq \begin{cases} \frac{1}{2}(v-k-1)(k-2)(k-4), & \text{if } k \text{ is even;} \\ \frac{1}{2}(v-k-1)(k-3)^2, & \text{if } k \text{ is odd.} \end{cases} \tag{8}$$

Noting  $S \geq S_P + S_O$  and combining (6) with (7) and (8), we complete the theorem immediately.

By a similar consideration, we can obtain the following result, the proof is omitted.

**Theorem 5** *Let  $D$  be a digraph with  $v$  vertices and  $\varepsilon$  edges. If the diameter of  $G$  is  $k \geq 2$ , then:*

$$\sigma(D) \geq \begin{cases} 2v(v-1) - \varepsilon + \frac{1}{6}k(k-1)(k-2) + \frac{1}{4}(v-k-1)(k-2)(k-4), & \text{if } k \text{ is even;} \\ 2v(v-1) - \varepsilon + \frac{1}{6}k(k-1)(k-2) + \frac{1}{4}(v-k-1)(k-3)^2, & \text{if } k \text{ is odd.} \end{cases}$$

As a direct consequence of Theorem 4 and Theorem 5, we can immediately obtain Theorem 3 and the equality occurs if and only if  $d(D) \leq 2$ .

Substituting the upper bounds on  $\varepsilon(G)$  in Theorem 1 and  $\varepsilon(D)$  in Theorem 2 into Theorem 4 and Theorem 5, respectively, yield new lower bounds of  $\sigma(G)$  and  $\sigma(D)$  only in terms of their order and diameter, which is better than that given by Plesnik (see, Theorem 2 and Theorem 3 in [3]). The details are omitted here and left to the reader.

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