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Brief paper

Consensus of discrete-time multi-agent systems with transmission nonlinearity☆

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ABSTRACT

A multi-agent system (MAS) consists of multiple agents, each under the influence of a local rule that represents its interaction with other agents. Most recent research on discrete-time MASs concentrates on local rules that are linear and does not deal with communication constraints on the information exchange among agents. However, local interactions between agents in the real world are more likely governed by nonlinear rules and are in the presence of time-varying delays. This paper aims to investigate the consensus of a discrete-time MAS with transmission nonlinearity and time-varying delays. In particular, based on a representative general nonlinear model, we obtain several basic criteria for the consensus of the MAS. These results cover several existing results as their special cases. Moreover, the model we consider does not satisfy the convexity assumption which was commonly taken as an important condition for the consensus of discrete-time MASs. The assumptions we make on the nonlinear transmission function are necessary in the sense that, if they are not satisfied, a connected topology can be constructed that does not guarantee consensus.

1. Introduction

Over the past decade, consensus (or synchronization, coordination) of multi-agent systems (MASs) has received increasing attention in various disciplines, including control theory, computer science, physics, mathematics, and artificial intelligence (Jadbabaie, Lin, & Morse, 2003; Li & Wang, 2004; Liu & Guo, 2007; Vicsek, Czirok, Ben-Jacob, Cohen, & Sochet, 1995). It is well known that consensus or synchronization is a typical collective behavior in an MAS and is a fundamental nature phenomenon with a very long history (Lü & Chen, 2005). By consensus we mean a general agreement among all members of a group or community, each exercising some discretion in its decision making and in its interactions with other members. A typical example is the heading synchronization of the Vicsek model (Vicsek et al., 1995). To reveal the inherent mechanism of consensus in an MAS, many mathematical or physical models have been introduced, including the classical Vicsek model (Vicsek et al., 1995) and the Couzin–Levin model and its variants (Lü, Liu, Couzin, & Levin, 2008). Based on these models, many interesting results have been obtained on the consensus of MASs (Cao, Morse, & Anderson, 2008a; Jadbabaie et al., 2003; Lin & Jia, 2009; Liu & Liu, 2011; Olfati-Saber, Fax, & Murray, 2007).

There are several well-developed approaches to dealing with the consensus of continuous-time MASs, such as the Lyapunov method (Bauso, Giarré, & Pesenti, 2006; Cortes, 2008; Cucker & Smale, 2007; Hui & Haddad, 2008; Munz, Papachristodoulou, & Allgower, 2008; Olfati-Saber, 2006). However, the analysis of discrete-time MASs is quite different from that of the continuous-time MASs. Olshevsky and Tsitsiklis have recently proven in Olshevsky and Tsitsiklis (2008) that it is impossible to construct a common quadratic Lyapunov function for a discrete-time MAS with a switching topology, even for the basic case of

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t) x_j(t). \tag{1}$$

Therefore, it is necessary to develop some new methods for dealing with discrete-time MASs based on, for example, graph theory (Cao et al., 2008a; Cao, Morse, & Anderson, 2008b), stochastic matrix theory (Wolfowitz, 1963), convex analysis (Blondel, Hendrickx,

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Olshevsky, & Tsitsiklis, 2005; Gazi, 2008; Li & Wang, 2004; Savkin, 2004), and the set-valued Lyapunov function method (Goebel, 2011; Moreau, 2005).

Recently, there have been some results on the consensus of discrete-time MASs (see, e.g., Blondel et al., 2005; Chen, Lü, & Lin, 2009; Jadbabaie et al., 2003; Li & Wang, 2004; Ren & Beard, 2005; Sarlette, Sepulchre, & Leonard, 2006; Xiao & Wang, 2008). The matrix product approach in Wolfowitz (1963) lays a foundation for the theoretical analysis of consensus of discrete-time MASs in many works (Cao et al., 2008a; Jadbabaie et al., 2003; Ren & Beard, 2005; Xiao & Wang, 2008). Most of these results pertain to the situation where the interaction among agents is governed by linear local rules (i.e., modified versions of (1)) and communication constraints are seldom considered (see, e.g., Cao et al., 2008a; Jadbabaie et al., 2003; Li & Wang, 2004; Ren & Beard, 2005). Even though models that are more general than (1) have been considered, these models are required to satisfy the convexity conditions proposed in Moreau (2005) (see Cortes, 2008).

Moreover, in most real-world applications, the information transmitted among agents is often subject to various constraints (Tsitsiklis, Bertsekas, & Athans, 1986). In this paper, inspired by Blondel et al. (2005); Li and Wang (2004), we aim to further investigate the consensus in a discrete-time MAS (1) with nonlinear transmission and time-varying delays. On the basis of a representative nonlinear model, several basic criteria for consensus are established. These results include several existing results as special cases. Furthermore, we discover that the above model does not satisfy the convexity condition in Moreau (2005), which was often taken as a necessary condition for the consensus of discrete-time MASs. The assumptions we make on the nonlinear transmission function are necessary in the sense that, if they are not satisfied, a connected topology can be constructed that does guarantee consensus.

This paper is organized as follows. Section 2 describes the consensus problem for discrete-time MASs with nonlinear transmission and time-varying delays. Several consensus criteria are proposed in Section 3. In Section 4, an example is given to illustrate the proposed consensus criteria. Finally, Section 5 concludes the paper.

2. Description of the problem

Consider a discrete-time MAS of n agents, labeled from 1 to n. Denote all the agents by the set $V = \{1, 2, \dots, n\}$. Let the state of agent $i \in V$ be denoted by $x_i(t) \in \mathbf{R}^m$ $(t \geq 0)$. There exist some communication connections among these n agents. If agent i has access to the information of agent j, then j is said to be a neighbor of agent i and the set of all neighbors of agent i at time t is denoted by $N_i(t)$. Consequently, $i \in N_i(t)$ if and only if agent i has access to the information of itself. Let graph $G(t) = (V, E(t))$ be the communication topology at time t, where $E(t) : \mathbf{Z}^+ \to V \times V$ is the set of edges and $(i, j) \in E(t)$ if and only if $j \in N_i(t)$. A graph $G = (V, E)$ is undirected if $(i, j) \in E \Rightarrow (j, i) \in E$. For several different graphs $G_k = (V, E_k)$ with $1 \leq k \leq K$, their union is $\bigcup_{k=1}^{K} G_k = (V, \bigcup_{k=1}^{K} E_k)$. A graph G can also be represented by a matrix $A = (a_{ij})_{n\times n}$, where a_{ij} is the weight of the edge (j, i) and $a_{ij} > 0$ if and only if $(j, i) \in E$.

For any two nodes i and j in a graph $G = (V, E)$, if there exist k different nodes $i_s \in V$ $(1 \leq s \leq k)$ such that (i, i_1), (i_1, i_2), $\dots$ (i_{k-1}, i_k), $(i_k, j) \in V$, then there is a path from i to j. If there are paths from a node $i \in V$ to any other node $j \in V$ $(i \neq j)$, then this graph contains a spanning tree rooted at node i. If there are paths from any node $i \in V$ to any other node $j \in V$, the graph is said to be strongly connected.

For an $\varepsilon > 0$, denote $O_\varepsilon = \{x \in \mathbf{R}^m : \|x\| \leq \varepsilon\}$. Given a set $\mathcal{B} \subseteq \mathbf{R}^m$, let $\mathrm{bd}(\mathcal{B})$ be the set of boundary points of $\mathcal{B}$. For $x \in \mathbf{R}^m$, let $\mathrm{d}(x, \mathcal{B}) = \inf_{y\in\mathcal{B}} \|x - y\|$ be the distance from x to $\mathcal{B}$. For $\mathcal{A}, \mathcal{B} \subseteq \mathbf{R}^m$, define $\mathcal{A} + \mathcal{B} = \{x + y : x \in \mathcal{A}, y \in \mathcal{B}\}$.

Assume that the transmission function is nonlinear and there exist time-varying delays in the communication channels among agents. In this case, the evolution of agent i complies with

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t) f_{ij}\left(x_j(t - \tau_j^i(t))\right), \tag{2}$$

where $a_{ij}(t) \in \mathbf{R}, x_i(t) \in \mathbf{R}^m$, and $a_{ij}(t)$ are the weights in the graph $G(t)$ representing the coupling strength from agent j to agent i at time t. Here, the term $f_{ij}\left(x_j(t - \tau_j^i(t))\right)$ represents the nonlinearity and time delays in the transmission of information from agent j to agent i. The overall updating rule (2) describes the process of transmitting the information of the neighboring agents into the state of agent i in the next time instant after collecting the information from all its neighbors.

For the local interactions (2), the question of interest is what kinds of functions f_{ij} and communication topology $G(t)$ will guarantee the consensus of all the agents, that is,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall i, j \in V.$$

3. Consensus criteria

Let the MAS considered in this paper be denoted by $(V, G(t), (2))$ and make the following definition.

Definition 1. A function f belongs to $\mathcal{F}$ if the following conditions are satisfied:

(a) $f : \mathbf{R}^m \to \mathbf{R}^m$ is continuous.
(b) There exists a nonempty compact convex set $\mathcal{B} \subset R^m$ and a nonempty bounded convex set $\mathcal{U} \subset \mathcal{B}$ such that: (i) $f(x) \in \mathcal{B}$ for all $x \in \mathcal{B}$; (ii) $f(x) = x$ for all $x \in \mathcal{U}$, and $\mathrm{d}(f(x), \mathcal{U}) < \mathrm{d}(x, \mathcal{U})$ for all $x \in \mathcal{B}$ such that $f(x) \neq x$.

For example, let $f(x) = x^3$ with $\mathcal{B} = [-1, 1]$ and $\mathcal{U} = \{0\}$. Then, it is obvious that $f \in \mathcal{F}$.

3.1. Consensus when $G(\infty)$ is connected

Let $G(\infty) = \lim_{k\to\infty} \bigcup_{t\geq k} G(t)$. We make the following assumptions.

A1 $f_{ij} \in \mathcal{F}$ for any $i, j \in V$, and $\{f_{ij}\}_{i,j=1}^{n}$ share two common sets $\mathcal{B}$ and $\mathcal{U}$.
A2 There exists an integer $\Gamma \geq 0$ such that, whenever $(i, j) \in E(\infty)$ and $(i, j) \in E(t)$, there exists a t' satisfying $t \leq t' \leq t + \Gamma$ and $(j, i) \in E(t')$.
A3 $G(\infty)$ is undirected and connected.
A4 There exists an integer $B > 0$ such that, for any $t \geq 0$, $0 \leq \tau_j^i(t) < B$ for any $i \neq j$ and $\tau_i^i(t) = 0$.
A5 $a_{ij}(t) \geq 0, a_{ii}(t) > 0, \sum_{j=1}^{n} a_{ij}(t) = 1$ for any $i, j \in V$ and $\inf_{a_{ij}(t)>0,\, t\geq 0} a_{ij}(t) \geq \alpha$ for some $\alpha \in (0, \frac{1}{2}]$.

Assumption A1 defines conditions on the nonlinear functions, A2 and A3 pose conditions on the topology structure, A4 states that the time delay should be bounded, and A5 is an assumption on the weights of the topology.

For the state $x_i(t)$ in (2), denote $d_i(t) = \mathrm{d}(x_i(t), \mathcal{U})$ and,

$$M_i(t) = \max_{0\leq\tau<B} d_i(t-\tau), \qquad M_i = \limsup_{t\to\infty} M_i(t),$$
$$M(t) = \max_{1\leq i\leq n} M_i(t), \qquad M = \max_{1\leq i\leq n} M_i,$$
$$m_i(t) = \min_{0\leq\tau<B} d_i(t-\tau), \qquad m_i = \liminf_{t\to\infty} m_i(t).$$

For $0 < a' < b'$, denote $V_t(a') = \{i \in V : M_i(t) \le a'\}$ and $\Lambda_t(a', b') = \{i \in V : a' \le M_i(t) \le b'\}$. For any $V' \subseteq V$ and $G(t) = (V, E(t))$, let $\partial^t_{\text{in}}(V')$ and $\partial^t_{\text{out}}(V')$ be the in-degree and out-degree of V' with respect to $G(t)$, respectively. Furthermore, let $\partial^t(V') = \partial^t_{\text{in}}(V') + \partial^t_{\text{out}}(V')$.

In what follows, we will state several lemmas. The proofs of the first two lemmas are obvious and omitted.

Lemma 2. *For a convex set $\mathcal{U} \subseteq \mathbf{R}^m$, $x, y \in \mathbf{R}^m$, and $\alpha \in [0, 1]$, one has*

$$\mathrm{d}(\alpha x + (1-\alpha)y, \mathcal{U}) \le \alpha \mathrm{d}(x, \mathcal{U}) + (1-\alpha)\mathrm{d}(y, \mathcal{U}).$$

Furthermore, $\mathrm{d}(x, \mathcal{U})$ is continuous in x.

Lemma 3. *Let Assumptions A1, A4 and A5 hold for MAS $(V, G(t), (2))$. Then, for any initial states $x_i(t_0) \in \mathcal{B}$ $(i \in V, -B < t_0 \le 0)$, $x_i(t) \in \mathcal{B}$ for any $t > 0$ and $i \in V$.*

The proofs of the following three lemmas can be found in the (Appendix).

Lemma 4. *Let Assumptions A1, A4 and A5 hold for MAS $(V, G(t), (2))$. Then, $\lim_{t\to\infty} M(t) = M$.*

Lemma 5. *Given any $0 \le l < M$ and any positive integer N, there exist an $\varepsilon > 0$ and a sequence $\{l_p\}$, with $l_0 = l$, $l_{p+1} > l_p$, such that, for any $0 \le p \le N$,*

$$l_{p+1} + \varepsilon > (1-\alpha)(M+\varepsilon) + \alpha(l_p + \varepsilon), \tag{3}$$

$$M - \varepsilon > l_p + \varepsilon. \tag{4}$$

Lemma 6. *There exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ of natural numbers satisfying $\lim_{t_k\to\infty} M_i(t_k) = r_i$ and $M = \max_{i\in V} M_i = \max_{i\in V} r_i$.*

Theorem 7. *Let Assumptions A1–A5 hold for MAS $(V, G(t), (2))$. Then, for any given initial states $x_i(t_0) \in \mathcal{B}$ $(i \in V, -B < t_0 \le 0)$, one of the following three cases holds:*

(a) *The MAS $(V, G(t), (2))$ will reach consensus.*
(b) *There exists $H > 0$ such that*

$$\lim_{t\to\infty} \mathrm{d}(x_i(t), \mathcal{U}) = H$$

for any $i \in V$. Furthermore,

$$\lim_{t\in T_{ij}, t\to\infty} \mathrm{d}(x_j(t), \{x : f_{ij}(x) = x\}) = 0$$

for any $j \in V$, where

$$T_{ij} = \{t - \tau^i_j(t) : a_{ij}(t) > 0, (j, i) \in E(\infty)\}.$$

(c) *$\lim_{t\to\infty} \mathrm{d}(x_i(t), \mathrm{bd}(\mathcal{U})) = 0$ for any $i \in V$.*

For the above three cases of Theorem 7, Case (a) means that MAS (2) will reach consensus; Case (b) means that the states of the MAS will approach a surface on which each point has equal distance to $\mathcal{U}$; and Case (c) means that the states of agents will approach the surface of $\mathcal{U}$. The intuitive meanings of the above cases are illustrated in Fig. 1.

To illustrate the basic idea of the following proof of Theorem 7, denote $m(t) = \min_{1\le i\le n} m_i(t)$ and consider the following two sets:

$$\overline{\mathcal{C}}(t) = \{x \in \mathbf{R}^m : \mathrm{d}(x, \mathcal{U}) = M(t)\},$$

$$\underline{\mathcal{C}}(t) = \{x \in \mathbf{R}^m : \mathrm{d}(x, \mathcal{U}) = m(t)\}.$$

The aim of the proof is to show that the gap between $\overline{\mathcal{C}}(t)$ and $\underline{\mathcal{C}}(t)$ will gradually approach zero, as illustrated in Fig. 2.

Proof of Theorem 7. By Lemma 6, there exists a subsequence $\{t_k\}$ of natural numbers satisfying $\lim_{k\to\infty} M_i(t_k) = r_i$ and $M = \max_{i\in V} r_i$. Denote $R = \min_{i\in V} r_i$. In what follows, the proof is separated into three cases, as illustrated by the roadmap of the technical analysis shown in Fig. 4 in the Appendix.

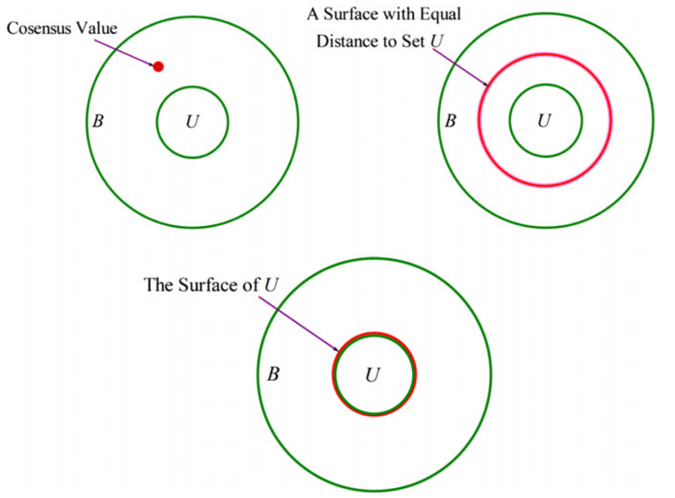

Fig. 1. Illustration of the three cases of Theorem 7. Upper left plot: Case (a); upper right plot: Case (b); lower plot: Case (c).

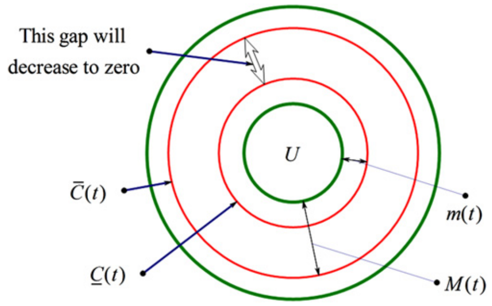

Fig. 2. Illustration of the proof of Theorem 7.

CASE 1: If $R < M \ne 0$, let $l = \max\{r_i \mid r_i < M\}$ and $N = n(B + \Gamma)$. By Lemma 5, there exist an $\varepsilon > 0$ and a sequence $\{l_p\}$ satisfying inequalities (3) and (4).

For a sufficiently large t_k in $\{t_k\}$, one has

$$V_{t_k}(l+\varepsilon) = V_{t_{k+s}}(l+\varepsilon) \ne V,$$

$$\Lambda_{t_k}(M-\varepsilon, M+\varepsilon) = \Lambda_{t_{k+s}}(M-\varepsilon, M+\varepsilon) \ne \emptyset,$$

$$V_{t_{k+s}}(l+\varepsilon) \bigcap \Lambda_{t_{k+s}}(M-\varepsilon, M+\varepsilon) = \emptyset,$$

for $\forall s > 0$. Given the following iteration

$$\#(t_k + p + 1)$$

$$= \begin{cases} 0, & p < p_1 - 1, \\ \#(t_k + p) + 1, & p \in [p_i - 1, p_i + B + \Gamma - 1), \\ \#(t_k + p), & p \in [p_i + B + \Gamma - 1, p_{i+1} - 1), \\ \#(t_k + p) + 1, & p = p_n - 1, \end{cases}$$

$$A_{p+1} = V_{t_k+p+1}(l_{\#(t_k+p)} + \varepsilon), \tag{5}$$

$$C_p = V - A_p, \tag{6}$$

$$p_{j+1} = \min\{p : p \ge p_j + B + \Gamma, \partial^{t_k+p}(A_p) \ne 0\},$$

where $0 \le p \le p_n$ and the initial states are given by

$$A_0 = V_{t_k}(l+\varepsilon), \qquad p_1 = \min\{p : \partial^{t_k+p}(A_p) \ne 0, p \ge 0\}.$$

Intuitively speaking, $\#(t_k + p)$ denotes the number of times when A_t is affected by the nodes of C_t for $t \in [t_k, t_k + p]$. Fig. 3 illustrates the meaning of $\#(t_k + p)$.

It should be noted that the definition of A_p and p_{j+1} are interdependent. Initially, one can get the value of A_0, then one

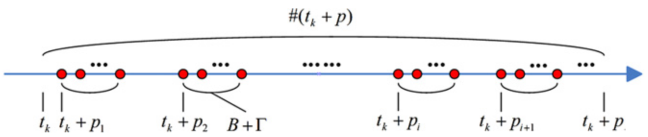

Fig. 3. $\#(t_k+p)$ denotes the number of small circles in the interval $[t_k, t_k+p]$, and there are $B+\Gamma$ consecutive small circles in each subinterval.

checks whether $p_1 = 0$ or not. If $p_1 = 0$, then one gets the values of $\#(t_k), \#(t_k+1), \ldots, \#(t_k+B+\Gamma-1)$, and next check whether $p_2 = t_k+B+\Gamma$ or not; If $p_1 \neq 0$, one gets the value of $\#(t_k) = 0$ and consequently the value of A_1, then repeat the process and check $p_1 = 1$ or not.

Here, the connectivity of $G(\infty)$ can naturally guarantee the existence of p_1. According to the above definition, there is $\#(t_k+p_j) = \#(t_k+p_{j-1})+B+\Gamma$. Consequently, $\#(t_k+p_n) = (n-1)(B+\Gamma)+1 < N$.

According to Appendix A.4 in the Appendix, one gets $A_p \subseteq A_{p+1}$. By the definition of p_1 and A_p, one has $\partial_{\mathrm{out}}^{t_k+p_1}(A_{p_1}) \neq 0$ or $\partial_{\mathrm{in}}^{t_k+p_1}(A_{p_1}) \neq 0$. Then, we have the following detailed discussion.

(i) If $\partial_{\mathrm{out}}^{t_k+p_1}(A_{p_1}) \neq 0$, then there exists an $i \in C_{p_1}$ satisfying $N_i(t_k+p_1) \cap A_{p_1} \neq \emptyset$. According to Appendix A.5 in the Appendix, one obtains $i \in A_{p_1+B}$ and $A_{p_1} \subsetneq A_{p_1+B} \subseteq A_{p_1+B+\Gamma}$.

(ii) If $\partial_{\mathrm{in}}^{t_k+p_1}(A_{p_1}) \neq 0$, then there exists an $i \in A_{p_1}$ satisfying $N_i(t_k+p_1) \cap C_{p_1} \neq \emptyset$, then one can select some $j \in C_{p_1}$ satisfying $(i, j) \in E(t_k+p_1)$. In view of Assumption A2, if $A_{p_1} = A_{p_1+B+\Gamma}$, then there exists some τ with $0 \le \tau \le \Gamma$ satisfying $(j, i) \in E(t_k+p_1+\tau)$. Hence, similarly to the reasoning of Appendix A.5 in the Appendix, there is $A_{p_1+\tau} \subsetneq A_{p_1+\tau+B}$, which, in view of $A_p \subseteq A_{p+1}$, is in contradiction with the assumption of $A_{p_1} = A_{p_1+B+\Gamma}$.

Based on the above discussion, one obtains $A_{p_1} \subsetneq A_{p_1+B+\Gamma}$. By the definition of p_i, one has $A_{p_1} \subsetneq A_{p_2}$. Since A_{p_1} has at least one element, by inductive reasoning, there exists some $\triangle$, $1 \le \triangle \le n-1$, such that $A_{p_1} \subsetneq A_{p_2} \subsetneq \cdots \subsetneq A_{p_\triangle} = V$. Consequently, according to the definition of Λ_t, there exists some $t_{k+s} > t_k + p_\triangle$ satisfying $\Lambda_{t_{k+s}}(M-\varepsilon, M+\varepsilon) = \emptyset$, which contradicts with $\Lambda_{t_{k+s}}(M-\varepsilon, M+\varepsilon) \neq \emptyset$.

CASE 2: If $M = R \neq 0$, then one has $r_i = M_i = M$ for any $i \in V$. The detailed analysis is given in the following.

(i) If $m_i = M_i$ for any $i \in V$, then $\lim_{t\to\infty} \mathrm{d}(x_i(t), \mathcal{U}) = M$. For the T_{ij} given in Case (b) of the theorem, denote $S_{ij} = \{x_j(t) : t \in T_{ij}\}$. Moreover, if there exist a pair of indices $\hat{i}, \hat{j}$ and an accumulation $\hat{\xi}$ of $S_{\hat{i}\hat{j}}$ satisfying $f_{\hat{i}\hat{j}}(\hat{\xi}) \neq \hat{\xi}$, according to the definition of $\mathcal{F}$, it holds that

$$\mathrm{d}(f_{\hat{i}\hat{j}}(\hat{\xi}), \mathcal{U}) < \mathrm{d}(\hat{\xi}, \mathcal{U}) = M.$$

Let $\varepsilon > 0$ be such that

$$M - \varepsilon > (1-\alpha)(M+\varepsilon) + \alpha(\mathrm{d}(f_{\hat{i}\hat{j}}(\hat{\xi}), \mathcal{U}) + \varepsilon).$$

According to the definition of $\hat{\xi}$, there exists a $T \in T_{\hat{i}\hat{j}}$ satisfying $M - \varepsilon < d_{\hat{i}}(t)$ for $t \ge T$ and

$$\mathrm{d}\left(f_{\hat{i}\hat{j}}(x_{\hat{j}}(t - \tau_{\hat{j}}^{\hat{i}}(t))), \mathcal{U}\right) < \mathrm{d}\left(f_{\hat{i}\hat{j}}(\hat{\xi}), \mathcal{U}\right) + \varepsilon$$

for any $t - \tau_{\hat{j}}^{\hat{i}}(t) \in T_{\hat{i}\hat{j}}$ and $t - \tau_{\hat{j}}^{\hat{i}}(t) \ge T$. Consequently, according to Lemma 2, it holds that

$$\begin{aligned} d_{\hat{i}}(t+1) &\le \sum_{j=1}^{n} a_{\hat{i}j}(t)\mathrm{d}(f_{\hat{i}j}(x_j(t - \tau_j^{\hat{i}}(t))), \mathcal{U}) \\ &\le (1-\alpha)(M+\varepsilon) + \alpha \cdot (\mathrm{d}(f_{\hat{i}\hat{j}}(\hat{\xi}), \mathcal{U}) + \varepsilon), \end{aligned}$$

for any $t - \tau_{\hat{j}}^{\hat{i}}(t) \in T_{ij}$ and $t - \tau_{\hat{j}}^{\hat{i}}(t) \ge T$. In view of $M - \varepsilon < d_{\hat{i}}(t+1)$, this leads to a contradiction with the definition of ε.

Therefore, $\lim_{t\in T_{ij}, t\to\infty} \mathrm{d}(x_j(t), \{x : f_{ij}(x) = x\}) = 0$ for any $j \in V$ and it becomes Case (b) of the theorem.

(ii) If there exists an $i \in V$ satisfying $\liminf_{t\to\infty} M_i(t) \neq M$, then there exists a sequence $\{t_k'\}$ satisfying $M_i(t_k') \to r_i \neq M$ and $M_j(t_k') \to r_j$ for $\forall j \in V$. Similar to the proof of CASE 1, the proof of this case can be easily deduced and hence omitted here.

(iii) Beside the above two cases, one only needs to consider the following CASE $\mathscr{A}$.

CASE $\mathscr{A}$: There exists an $i \in V$ satisfying $m_i \neq M_i$ and $\lim_{t\to\infty} M_j(t) = M$ for any $j \in V$.

For some i satisfying $m_i \neq M_i$, one can select an accumulation point x^* of $\{x_i(t)\}$ with $\mathrm{d}(x^*, \mathcal{U}) = \eta_0 < M$. By Lemma 5, there exists a sequence $\{\eta_p\}_{p=0}^{B+1}$ and an $\varepsilon > 0$ such that $(1-\alpha)(M+\varepsilon) + \alpha(\eta_p+\varepsilon) < \eta_{p+1}+\varepsilon$, $\eta_p + \varepsilon < M - \varepsilon$, and $\eta_p < \eta_{p+1} < M$ for $0 \le p \le B$.

Therefore, for the above ε, there exists a positive integer T satisfying $\mathrm{d}(x_i(T), \mathcal{U}) < \eta_0 + \varepsilon$ and $M - \varepsilon < M_j(t) < M + \varepsilon$ for $\forall j \in V$ and $t \ge T$. Since $\tau_i^i = 0$, by the definition of ε, one gets $d_i(T+1) \le (1-\alpha)(M+\varepsilon) + \alpha(\eta_0+\varepsilon) < \eta_1 + \varepsilon$.

Furthermore, $d_i(T+\tau) < \eta_\tau + \varepsilon < M - \varepsilon$ for any $1 \le \tau \le B$. Consequently, one obtains

$$M_i(T+B) < \max\{\eta_1, \eta_2, \ldots, \eta_B\} + \varepsilon = \eta_B + \varepsilon < M - \varepsilon.$$

Obviously, this is contradictory with the fact that $M - \varepsilon < M_j(t) < M + \varepsilon$ holds for any $j \in V$ and $t \ge T$.

CASE 3: If $M = 0$, denote $\mathcal{U}_\varepsilon^* = \bigcap_{(j,i)\in E(\infty)} \{x : \|f_{ij}(x) - x\| \le \varepsilon\}$. Then one has $\mathcal{U} \subseteq \mathcal{U}_\varepsilon^*$. By the uniform continuity of f_{ij} on $\mathcal{U}_\varepsilon^*$, for this ε, one can choose a suitable $0 < \tilde{\varepsilon} < \varepsilon$ satisfying $\mathcal{U}_{\tilde{\varepsilon}} = \{x \in \mathbf{R}^m : \mathrm{d}(x, \mathcal{U}) \le \tilde{\varepsilon}\} \subseteq \mathcal{U}_\varepsilon^*$. Thus, for any $\varepsilon > 0$, there exists some $\tilde{N} > 0$ such that $x_i(t) \in \mathcal{U}_{\tilde{\varepsilon}}$ for any $t \ge \tilde{N} - B$. Consequently, there is $f_{ij}(x_j(t-\tau)) = x_j(t-\tau) + \varepsilon_{ij}(t-\tau)$ with $\|\varepsilon_{ij}(t-\tau)\| \le \varepsilon$ for any $i, j \in V$ and $t \ge \tilde{N}$.

In detail, one has the following three cases:

(i) If there exists an integer $T > 0$ satisfying $\{x_i(T-\tau)\}_{i\in V, 0\le\tau<B} \subseteq \mathcal{U}$, then $x_i(t) \in \mathcal{U}$ for any $i \in V$ and $t \ge T$. Construct two sets $\tilde{\mathcal{B}} = \mathcal{U}$ and $\tilde{\mathcal{U}}_1 = \{\hat{x}_1\}$, where $\hat{x}_1$ is any element of $\mathcal{U}$. Repeating the reasoning of CASES 1 and 2, one obtains that there exists an $h_1 \ge 0$ satisfying

$$\lim_{t\to\infty} \mathrm{d}(x_i(t), \mathrm{bd}\{\{\hat{x}_1\} + O_{h_1}\}) = 0$$

for any $i \in V$. Iteratively, for any $1 \le k \le m$, let $\hat{x}_{k+1} \in \bigcap_{r=1}^{k} \mathrm{bd}\{\{\hat{x}_r\} + O_{h_r}\}$, there exists an $h_{k+1} \ge 0$ satisfying

$$\lim_{t\to\infty} \mathrm{d}(x_i(t), \mathrm{bd}\{\{\hat{x}_{k+1}\} + O_{h_{k+1}}\}) = 0$$

for any $i \in V$. Therefore, one has

$$\lim_{t\to\infty} \mathrm{d}\left(x_i(t), \bigcap_{k=1}^{m+1} \mathrm{bd}\{\{\hat{x}_k\} + O_{h_k}\}\right) = 0.$$

According to the construction of $\hat{x}_k$, one knows that $\bigcap_{k=1}^{m+1} \mathrm{bd}\{\{\hat{x}_k\} + O_{h_k}\}$ must be a singleton. Therefore, MAS $(V, G(t), (2))$ will reach consensus.

(ii) If $\limsup_{t\to\infty} \mathrm{d}(x_i(t), \mathrm{bd}(\mathcal{U})) = 0$ for any $i \in V$, it becomes Case (c) of the theorem.

(iii) Excluding the above two cases, one has

$$V_{bd}^- = \{i \in V : \liminf_{t\to\infty} \mathrm{d}(x_i(t), \mathrm{bd}(\mathcal{U})) = 0\} \neq \emptyset,$$

$$V_{bd}^+ = \{i \in V : \limsup_{t\to\infty} \mathrm{d}(x_i(t), \mathrm{bd}(\mathcal{U})) = 0\} \neq V.$$

Then, there exists an $i \in V$ such that $\{x_i(t)\}_{t\geq 0}$ has some accumulation point $x_i^* \in \mathcal{U} - \mathrm{bd}(\mathcal{U})$. For any ε, there exist $0 < \tilde{\varepsilon} < \varepsilon$ and $t_0 > 0$ satisfying $x_i(t_0) \in \{x_i^*\} + O_{\tilde{\varepsilon}}$ and $x_j(t) \in \mathcal{U}_{\tilde{\varepsilon}}$ for any $j \in V$ and $t \geq t_0$. Consequently, for $t \geq t_0$,

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_j^i(t)) + \varepsilon_i'(t), \tag{7}$$

where $\varepsilon_i'(t) = \sum_{j=1}^{n} a_{ij}(t)\varepsilon_{ij}(t - \tau_j^i(t))$ satisfies $\|\varepsilon_i'(t)\| \leq \varepsilon$.

Denote $P_1 = \{x_i^*\} + O_{\tilde{\varepsilon}}$ and recursively $P_{k+1} = \bigcup_{\beta\in[\alpha,1-\alpha]} (\beta P_k + (1-\beta)\mathcal{U}_{\tilde{\varepsilon}})$. Then, for sufficiently small ε, one has $P_{n(B+\Gamma)} \subsetneq \mathcal{U}$. Similarly to the proof in CASE 1, one can prove that there exists some $T > 0$ satisfying $x_i(t) \in P_{n(B+\Gamma)}$ for any $i \in V$ and $t > T$. As a result, it leads to a contradiction with the fact that $V_{bd}^- \neq \emptyset$.

Therefore, summarizing the above reasoning, the proof is completed. $\square$

Although Theorem 7 cannot tell us that consensus will be reached once $G(\infty)$ is connected, one gets the following result for $m = 1$, whose proof is omitted due to space limitation.

Corollary 8. *Suppose that Assumptions* A1–A5 *hold for MAS* $(V, G(t), (2))$ *and* $m = 1$. *Then, for any given initial states* $x_i(t_0) \in \mathcal{B}$ $(i \in V, -B < t_0 \leq 0)$, *MAS* $(V, G(t), (2))$ *will reach consensus.*

3.2. Consensus when $\{G(t)\}_{t\geq 0}$ is jointly connected

Consider the following assumptions:

A3.1 Each $\bigcup_{t=t_k}^{t_{k+1}-1} G(t)$ is strongly connected, where $1 \leq t_{k+1} - t_k \leq \Gamma$ for some $\Gamma \geq 1$.

A3.2 Each $\bigcup_{t=t_k}^{t_{k+1}-1} G(t)$ contains a spanning tree rooted at some common node $\tilde{v} \in V$ where $\tilde{v}$ has zero in-degree in each $G(t)$, and $1 \leq t_{k+1} - t_k \leq \Gamma$ for some $\Gamma \geq 1$.

When Assumption A3.1 is satisfied, we say that $\{G(t)\}_{t\geq 0}$ is jointly connected. We recall the following lemma.

Lemma 9 (Chen, Lü, & Yu, 2011). *Suppose that Assumptions* A3.1, A4 *and* A5 *hold for the MAS*

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_j^i(t)) + \omega_i(t), \tag{8}$$

with $\|\omega_i(t)\| \leq \varepsilon$. *Then, there exists a nonnegative continuous function* $h : \mathbf{R} \to \mathbf{R}$ *with* $h(0) = 0$ *such that* $\limsup_{t\to\infty} \max_{i,j\in V} \|x_i(t) - x_j(t)\| \leq h(\varepsilon)$.

Let $C1 = \{A1, A3.1, A4, A5\}$ and $C2 = \{A1, A3.2, A4, A5\}$. The following results can be obtained.

Theorem 10. *If the set of conditions* $C1$ *are satisfied for MAS* $(V, G(t), (2))$, *then the MAS will reach consensus for any initial states* $x_i(t_0) \in \mathcal{B}$ $(i \in V, -B < t_0 \leq 0)$.

Proof. Similarly to the proof of Theorem 7, one can get that one of Cases (a)–(c) will hold. In fact, for Case (b) or (c), similarly to the reasoning in (iii) of CASE 3 in the proof of Theorem 7, one can obtain formula (7). Then, by Lemma 9 and in view of the arbitrariness of ε and the continuity of $h(\cdot)$,

$$\limsup_{t\to\infty} \left(\max_{i,j\in V} \|x_i(t) - x_j(t)\| \right) = 0,$$

and the consensus is reached. $\square$

The following result can be viewed as the counterpart of the leader–follower case in Ren and Beard (2005).

Theorem 11. *If the set of conditions* $C2$ *are satisfied for MAS* $(V, G(t), (2))$ *and* $x_{\tilde{v}}(0) \in \mathcal{U}$, *then the MAS will reach consensus at* $x_{\tilde{v}}(0)$ *for any initial states* $x_i(t_0) \in \mathcal{B}$ $(i \in V, -B < t_0 \leq 0)$.

Proof. The main idea of the proof is similar to that of Theorem 10 and the proof is thus omitted. $\square$

Remark 12. The set of conditions $C2$ alone cannot guarantee the consensus of MAS $(V, G(t), (2))$. It can be easily verified that $(x_1(t), x_2(t))^T = (0, \frac{3}{2})^T$ for any $t \geq 0$, when

$$f(x) = \begin{cases} 3x, & 0 \leq x \leq 1, \\ 3, & 1 \leq x \leq 3, \end{cases} \tag{9}$$

and $A(t) = ((1\ \ 0.5)^T\ \ (0\ \ 0.5)^T)$ with initial states $(x_1(0), x_2(0))^T = (0, \frac{3}{2})^T$.

3.3. Necessity and comparison with the existing results

The condition $f \in \mathcal{F}$ is necessary for the consensus of MAS $(V, G(t), (2))$ when $m = 1$ in the sense of the following theorem.

Theorem 13. *Let* f *be a continuous and monotonically increasing function defined on some bounded closed interval* $\mathcal{B} \subsetneq \mathbf{R}$ *and* $f(x) \in \mathcal{B}$ *for any* $x \in \mathcal{B}$. *If the MAS*

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij} f(x_j(t)) \tag{10}$$

will reach consensus for any initial states $x_i(t_0) \in \mathcal{B}$ $(i \in V, -B < t_0 \leq 0)$, *and any fixed stochastic matrix* A *with* $a_{ii} > 0$ *and strongly connected* $\mathcal{G}(A)$, *then* $f \in \mathcal{F}$.

Proof. Let $\mathcal{B} = [a, b]$.

For $f : \mathcal{B} \to \mathcal{B}$, one knows that the set $\{x : f(x) = x\}$ must be nonempty. Denote

$$a' = \inf\{x : f(x) = x\}, \qquad b' = \sup\{x : f(x) = x\}.$$

Without loss of generality, suppose that $a = a'$ and $b = b'$. Indeed, if, for example, $b' < b$ (the case of $a < a'$ can be considered in a similar way), one knows that there is $f(x) > x$ or $f(x) < x$ on the interval $(b', b]$. If $f(x) > x$ for $x > b'$, then $f(b) > b$, which contradicts with $f(x) \in \mathcal{B}$ for $x \in \mathcal{B}$. Therefore, $f(x) < x$ for $x > b'$. Consider each $x \in (b', b]$, since f is monotonously increasing, there is $\mathrm{d}(f(x), \mathcal{U}') \leq \mathrm{d}(x, \mathcal{U}')$ for any $\mathcal{U}' \subseteq [a', b']$. Consequently, one just needs to construct the set $\mathcal{U}$ which belongs to the set $[a', b']$.

If there does not exist $\hat{x} \in [a, b]$ satisfying $f(\hat{x}) \neq \hat{x}$, let $\mathcal{U} = [a, b]$, then $f \in \mathcal{F}$.

If there exists an $\hat{x} \in [a, b]$ satisfying $f(\hat{x}) \neq \hat{x}$, then, for any such $\hat{x}$, by the continuity of f, there exists an interval $(a_{i(\hat{x})}, b_{i(\hat{x})})$ satisfying $\hat{x} \in (a_i, b_i), f(a_i) = a_i, f(b_i) = b_i$, and $f(x) \neq x$ for any $x \in (a_i, b_i)$. Let the set of all these intervals (a_i, b_i) be $\mathcal{H} = \{(a_i, b_i) : i \in \mathcal{I}\}$. Then, for any $(a_i, b_i) \in \mathcal{H}, f(x) - x$ will not change its sign on it because of the continuity of f. Let

$$\mathcal{H}^+ = \{(a_i, b_i) \in \mathcal{H} : f(x) > x \text{ for } x \in (a_i, b_i)\},$$
$$\mathcal{H}^- = \{(a_i, b_i) \in \mathcal{H} : f(x) < x \text{ for } x \in (a_i, b_i)\},$$
$$b^* = \sup\{b_s : (a_s, b_s) \in \mathcal{H}^+\}, \quad \text{when } \mathcal{H}^+ \neq \emptyset,$$
$$a^* = \inf\{a_s : (a_s, b_s) \in \mathcal{H}^-\}, \quad \text{when } \mathcal{H}^- \neq \emptyset.$$

If $\mathcal{H}^+$ is empty but $\mathcal{H}^-$ is not, let $\mathcal{U} = [a, a^*]$, then $f \in \mathcal{F}$.
If $\mathcal{H}^-$ is empty but $\mathcal{H}^+$ is not, let $\mathcal{U} = [b^*, b]$, then $f \in \mathcal{F}$.
If neither $\mathcal{H}^+$ nor $\mathcal{H}^-$ is empty, one will prove that $a^* \geq b^*$. If $a^* < b^*$, by the definitions of a^* and b^*, there exist c_1, c_2 and c_3 satisfying $a^* < c_1 < c_2 < c_3 < b^*, f(c_1) < c_1, f(c_2) = c_2$, and $f(c_3) > c_3$.

Consider an MAS (10) with 3 agents. Let the initial states be $x_1(0) = c_1, x_2(0) = c_2, x_3(0) = c_3$, and

$$A = \begin{pmatrix} \frac{c_1 - c_2}{f(c_1) - c_2} & \frac{f(c_1) - c_1}{f(c_1) - c_2} & 0 \\ \frac{c_2 - f(c_3)}{2(f(c_1) - f(c_3))} & \frac{1}{2} & \frac{f(c_1) - c_2}{2(f(c_1) - f(c_3))} \\ 0 & \frac{c_3 - f(c_3)}{c_2 - f(c_3)} & \frac{c_2 - c_3}{c_2 - f(c_3)} \end{pmatrix}.$$

Then, one gets that A is stochastic and $\mathcal{G}(A)$ is strongly connected. It is easy to verify that $x_i(t) = x_i(0)$ for $i = 1, 2, 3$ and $t \geq 0$. That is, the MAS (10) does not reach consensus. Similarly, one can construct counterexamples for the MAS with n agents. Therefore, $b^* \leq a^*$.

Hence, let $\mathcal{U} = [b^*, a^*]$, then $f \in \mathcal{F}$. $\square$

In fact, the conditions on f in Theorem 13 can be reduced to: $f : \mathcal{B} \to \mathcal{B}$ is continuous and defined on some bounded interval $\mathcal{B} \subsetneq \mathbf{R}$, $D_+f(x) \geq -1$ and $D_-f(x) \geq -1$[2] for any $x \in \mathcal{B}$. Because of space limitations, the related proof is omitted here.

Let $m = 1$ and $f_{ij}(x) = x$. It is obvious that $f_{ij}(x) = x \in \mathcal{F}$ since we can take $\mathcal{B} = \mathrm{conv}\{x_i(t_0) : i \in V, -B < t_0 \leq 0\}$[3] and $\mathcal{U} = \{\hat{x}\}$ with $\hat{x} \in \mathcal{B}$. When $\tau_j^i(t) = 0$, Theorem 11 covers Theorem 2 of Jadbabaie et al. (2003). When $\tau_j^i(t) \neq 0$, Theorem 11 covers Theorem 3 of Blondel et al. (2005). When $\tau_j^i(t) \neq 0$, Corollary 8 covers Theorem 5 of Blondel et al. (2005).

Moreau (2005) investigated the general nonlinear MAS

$$x_i(t+1) = f_i(t, x_1(t), \ldots, x_n(t)). \tag{11}$$

It should be pointed out that our MAS $(V, G(t), (2))$ does not always satisfy the convexity condition proposed in Moreau (2005). From Assumption 1 in Moreau (2005), convexity implies that

$$x_i(t+1) \in \mathrm{conv}_{i \in V}\{x_i(t)\} \tag{12}$$

hold for any $i \in V$ and $t \geq 0$. This is equivalent to the existence of nonnegative coefficients $a_{ij}(t)$ satisfying $\sum_{j=1}^n a_{ij}(t) = 1$ and $x_i(t+1) = \sum_{j=1}^n a_{ij}(t)x_j(t)$.

Consider the case of $m = 1$. Let $\tau_j^i(t) = 0$ for all $i, j \in V$, and $f(x)$ be given by Eq. (9). It is easy to verify that $\mathcal{B} = [0, 3]$, $\mathcal{U} = \{3\}$ and $f \in \mathcal{F}$.

Given a simple MAS of 2 agents with initial states $x_1(0) = 0$ and $x_2(0) = 1$. Let its topological structure be fixed: $a_{ij}(t) = \frac{1}{2}$ for any $i, j \in \{1, 2\}$ and $t \geq 0$. Then, one has $x_1(1) = x_2(1) = \frac{1}{2}(f(x_1(0)) + f(x_2(0))) = \frac{3}{2}$. Therefore, one gets $x_i(1) = \frac{3}{2} \notin \mathrm{conv}_{i \in V}\{x_i(0)\} = [0, 1]$, which contradicts with (12). In fact, based on this idea, similar counterexamples can be constructed for $m > 1$.

Angeli and Bliman (2006) generalized the result of Moreau (2005). According to Assumption A in Angeli and Bliman (2006), one deduces that

$$x_i(t+1) = x^* \quad \text{if } \{x_i(\tau)\}_{i \in V, t-B < \tau \leq t} = \{x^*\}, \tag{13}$$

which does not hold for our model (2). In fact, let $B = 1, n = 2$, and $f(x)$ be given by Eq. (9), $a_{ij}(t) = 0.5$, $x_i(0) = 0.1$ for model (2), then $x_i(1) = 0.3 \neq 0.1$. This contradicts with (13).

Lorenz and Lorenz (2010) further investigated the nonlinear MAS (11) by introducing a novel concept of y-averaging. Our main results are different from the main results of Theorem 2.4 in Lorenz and Lorenz (2010). Connectivity for each iteration is not required in our model (2), while for the model in Lorenz and Lorenz (2010), each iteration contains the connectivity information.

[2] $D_+f(x) = \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$, $D_-f(x) = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$.

[3] If $\mathcal{B} = \mathrm{conv}_{i \in V, -B < \tau \leq 0}\{x_i(\tau)\}$ is a singleton, then MAS reaches consensus and it is not necessary to make the following deduction.

4. An illustrative example

Consider the following nonlinear MAS

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases} \tag{14}$$

where

$$u_i(t) = -\gamma_0 v_i(t_k) + \gamma_1 \sum_{j=1, j \neq i}^{n} a_{ij}(t_k)(f_{ij}(x_j(t_k)) - x_i(t_k)), \tag{15}$$

for $t \in [t_k, t_{k+1})$. Denote $A_k = (a_{ij}(t_k))_{i,j=1}^n$, $x_i(t_k) = x_k^i$, $v_i(t_k) = v_k^i$ and $u_i(t_k) = u_k^i$.

One obtains

$$\begin{cases} x_{k+1}^i = x_k^i + h_k v_k^i + \frac{h_k^2}{2} u_k^i, \\ v_{k+1}^i = v_k^i + h_k u_k^i. \end{cases}$$

For brevity, let $h = h_k = t_{k+1} - t_k$ be the fixed sampling interval. However, Theorem 14 below can be easily extended to the case of time-varying $h_k = t_{k+1} - t_k$.

The consensus in (14) is defined by $\lim_{t \to \infty} x_i(t) = x^*$ and $\lim_{t \to \infty} v_i(t) = 0$ for any $i \in V$. This consensus is equivalent to $\lim_{k \to \infty} x_k^i = x^*$ and $\lim_{k \to \infty} v_k^i = 0$ for any $i \in V$.

Theorem 14. *The MAS (14) and (15) will reach consensus under the following three conditions:*

(i) $f_{ij} \in \mathcal{F}$ *for any* $i, j \in V$ *and* $i \neq j$.
(ii) $\{A_k\}_{k \geq 0}$ *are stochastic matrices with zero diagonal entries;* $\{\mathcal{G}(A_k)\}_{k \geq 0}$ *is jointly connected and* $\inf_{i,j \in V, k \geq 0} a_{ij}(t_k) \geq \alpha$ *for some* $\alpha \in (0, \frac{1}{2}]$.
(iii) $\gamma_0 > 2\sqrt{\gamma_1}$ *and*

$$h < \min\left\{1, \frac{2}{\gamma_0}, \frac{1}{\gamma_0 + \frac{1}{2}\gamma_0\sqrt{\gamma_1} - \sqrt{\gamma_1}}\right\}.$$

Proof. Define $\xi_k^i = x_k^i + \gamma v_k^i$. The dynamics of (14) and (15) is transformed to one characterized by the topology matrix H_k as given in (16), shown in Box I on the top of the next page. The rest of the proof can be derived easily from Theorem 10 and hence omitted. $\square$

5. Concluding remarks

In this paper, we have further investigated the consensus of discrete-time MASs with nonlinear transmission and time-varying delays. The obtained results generalized several well-known results. Detailed comparisons with some existing results, especially with respect to the commonly assumed convexity condition, were given in this paper.

Appendix

A.1. Proof of Lemma 4

By Lemma 2, it holds that $d_i(t+1) \leq \sum_{j=1}^n a_{ij}(t) d_j(t - \tau_j^i(t)) \leq M(t)$. Hence,

$$\begin{aligned} M(t+1) &= \max_{i \in V, 0 \leq \tau < B} d_i(t+1-\tau) \\ &\leq \max\{\max_{i \in V, 0 < \tau < B} d_i(t+1-\tau), M(t)\} \\ &\leq \max\{\max_{i \in V, 0 \leq \tau < B} d_i(t-\tau), M(t)\} \\ &= M(t). \end{aligned}$$

$$H_k = \begin{pmatrix} \left(1 - \dfrac{h}{\gamma} + \dfrac{h^2\gamma_0}{2\gamma} - \dfrac{h^2\gamma_1}{2}\right) I + \dfrac{h^2\gamma_1}{2} A_k & \left(\dfrac{h}{\gamma} - \dfrac{h^2\gamma_0}{2\gamma}\right) I \\ \left(\left(\dfrac{h^2}{2} + \gamma\, h\right)\left(\dfrac{\gamma_0}{\gamma} - \gamma_1\right) - \dfrac{h}{\gamma}\right) I + \left(\dfrac{h^2}{2} + \gamma\, h\right) A_k & \left(1 - \dfrac{\gamma_0}{\gamma}\left(\dfrac{h^2}{2} + \gamma\, h\right) + \dfrac{h}{\gamma}\right) I \end{pmatrix}. \tag{16}$$

Box I.

$$\begin{tikzcd}
M = 0\,? \arrow[r, "YES"] \arrow[d, "NO"'] & \text{CASE 3} \arrow[r] & \exists T, \{x_i(T-\tau)\}_{i\in V, 0\le\tau<B} \subseteq \mathcal{U}\,? \arrow[r, "NO"] \arrow[d, "YES"'] & \begin{array}{c}\limsup_{t\to\infty} \hat{d}_i(t) = 0 \text{ for } \forall i \in V?\\ \hat{d}_i(t) = \mathrm{d}(x_i(t), \mathrm{bd}(\mathcal{U}))\end{array} \arrow[r, "NO"] \arrow[d, "YES"'] & \text{Impossible} \\
R = M\,? \arrow[r, "YES"] \arrow[d, "NO"'] & \text{CASE 2} \arrow[d] & \text{Theorem 7: a)} & \text{Theorem 7: c)} & \\
\text{CASE 1} \arrow[d] & m_i = M_i, \forall i? \arrow[r, "NO"] \arrow[d, "YES"'] & \exists i, \liminf_{t\to\infty} M_i(t) \neq M? \arrow[r, "YES"] \arrow[d, "NO"'] & \text{Attributed to CASE 1} & \\
\text{Impossible} & \text{Theorem 7: b)} & \text{CASE } \mathscr{A} \arrow[r] & \text{Impossible} &
\end{tikzcd}$$

Fig. 4. Roadmap of technical analysis in the proof of Theorem 7.

Consequently, $M(t)$ is decreasing and $M(t) \geq 0$. Thus there is an M' satisfying $\lim_{t\to\infty} M(t) = M'$.

By the definition of $M(t)$, there exists some $j \in V$ such that the set $S = \{t : M(t) = M_j(t)\}$ is an infinite set. Select a subsequence $\{t_p\} \subseteq S$. One has $\limsup_{t_p\to\infty} M(t_p) = \limsup_{t_p\to\infty} M_j(t_p) \leq \limsup_{t\to\infty} M_j(t)$. That is, $M' \leq M_j \leq M$.

On the other hand, since $M = \max_{i\in V} M_i$, for any given infinitesimal $\epsilon > 0$, there exists a sequence $\{t_q\}$ and some $i \in V$ satisfying $M_i(t_q) \geq M - \epsilon$. By the definition of $M(t)$, one gets $M(t_q) \geq M - \epsilon$. Thus, one obtains $M' = \lim_{t_q\to\infty} M(t_q) \geq M - \epsilon$. Since ϵ is infinitesimal, one deduces $M' \geq M$.

It then follows that $M' = M$. □

A.2. Proof of Lemma 5

One can find a sequence $\{l_p\}$ with $l_0 = l$, $l_{p+1} > l_p$ such that $l_{p+1} > (1-\alpha)M + \alpha l_p$ and $M > l_p$ hold for any $0 \leq p \leq N$. Consequently, this result holds once ε is chosen sufficiently small. □

A.3. Proof of Lemma 6

Since $M_i(t) \leq M(0)$ for any $i \in V$ and $t \geq 0$, there exists a subsequence $\{t_k\}$ of $\{t\}$ satisfying $\lim_{t_k\to\infty} M_i(t_k) = r_i$. In what follows, one only needs to show that $M = \max_{i\in V} M_i = \max_{i\in V} r_i$.

On the one hand, since $M_i \geq r_i$, one has $M = \max_{i\in V} M_i \geq \max_{i\in V} r_i$. On the other hand, by the definition of $M(t)$, there exists some $j \in V$ satisfying that the set $S = \{t_k : M_j(t_k) = M(t_k)\}$ is an infinite set. Taking upper limits of $M_j(t_k)$ and $M(t_k)$ along S, one obtains from Lemma 4 that $M \leq r_j \leq \max_{i\in V} r_i$.

Therefore, $M = \max_{i\in V} r_i$. □

A.4. Proof of $A_p \subseteq A_{p+1}$

For $i \in A_p$, one has $M_i(t_k + p) \leq l_{\#(t_k+p-1)} + \varepsilon$. In the following, the proof is separated into two cases.

(i) If $\partial_{\text{in}}^{t_k+p}(A_p) \neq 0$, one gets $\#(t_k+p) = \#(t_k+p-1)+1$. Then one deduces

$$\begin{aligned} d_i(t_k+p+1) &\leq \alpha(l_{\#(t_k+p-1)} + \varepsilon) + (1-\alpha)(M+\varepsilon) \\ &< l_{\#(t_k+p-1)+1} + \varepsilon \\ &= l_{\#(t_k+p)} + \varepsilon. \end{aligned}$$

Hence,

$$\begin{aligned} M_i(t_k+p+1) &\leq \max\{M_i(t_k+p),\ d_i(t_k+p+1)\} \\ &< l_{\#(t_k+p)} + \varepsilon \end{aligned}$$

and $i \in A_{p+1}$.

(ii) If $\partial_{\text{in}}^{t_k+p}(A_p) = 0$, one obtains $\#(t_k+p) = \#(t_k+p-1)+1$ or $\#(t_k+p) = \#(t_k+p-1)$. Thus one has $d_i(t_k+p+1) \leq l_{\#(t_k+p)} + \varepsilon$. Therefore, one gets

$$\begin{aligned} M_i(t_k+p+1) &\leq \max\{l_{\#(t_k+p-1)} + \varepsilon, d_i(t_k+p+1)\} \\ &\leq l_{\#(t_k+p)} + \varepsilon \end{aligned}$$

and $i \in A_{p+1}$. □

A.5. Proof of $A_{p_1} \subsetneq A_{p_1+B}$

Since $i \in C_{p_1}$ and $N_i(t_k+p_1) \cap A_{p_1} \neq \emptyset$, one gets

$$\begin{aligned} d_i(t_k+p_1+1) &\leq \alpha(l_{\#(t_k+p_1-1)} + \varepsilon) + (1-\alpha)(M+\varepsilon) \\ &< l_{\#(t_k+p_1-1)+1} + \varepsilon \\ &= l_{\#(t_k+p_1)} + \varepsilon. \end{aligned}$$

Similarly, by the definition of $\#(t)$, for any $1 \leq \tau \leq B$, one obtains $d_i(t_k+p_1+\tau) < l_{\#(t_k+p_1+\tau-1)} + \varepsilon$. Thus, one has $M_i(t_k+p_1+B) < l_{\#(t_k+p_1+B-1)} + \varepsilon$ and $i \in A_{p_1+B}$. Therefore, $A_{p_1} \subsetneq A_{p_1+B}$. □

A.6. Roadmap of the proof of Theorem 7

See Fig. 4.

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