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Applications of Malliavin Calculus to Stochastic Partial Differential Equations

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1 Introduction

Nowadays, Malliavin calculus is underpinning important developments in stochastic analysis and its applications. In particular, research on SPDEs is benefiting from the ideas and tools of this calculus. Unexpectedly, this hard machinery is successfully used in financial engineering for the computation of Greeks, and in numerical approximations of SPDEs. The analysis of the dependence of the Malliavin matrix on its structural parameters is used in problems of potential theory involving SPDEs, like obtaining the optimal size of some hitting probabilities. The study of such questions, but also of some classical issues like the absolute continuity of measures derived from probability laws of SPDEs, is still an underdeveloped field.

These notes are a brief introduction to the basic elements of Malliavin calculus and to some of its applications to SPDEs. They have been prepared for a series of six lectures at the LMS-EPSRC Short Course on Stochastic Partial Differential Equations.

The first three sections are devoted to introduce the calculus: its motivations, the main operators and rules, and the criteria for existence and smoothness of densities of probabilities laws. The last three ones deal with applications to SPDEs. To be self-contained, we provide some ingredients of the SPDE framework we are using. Then we study differentiability in the Malliavin sense, and non-degeneracy of the Malliavin matrix. The last section is devoted to sketch a method to analyze the asymptotic behaviour of densities of small perturbations of SPDEs. Altogether, this is a short, very short, journey through a deep and fascinating subject.

To close this short presentation, I would like to express my gratitude to Professor Dan Crisan, the scientific organizer of the course, for a wonderful and efficient job, to the London Mathematical Society for the financial support, and to the students whose interest and enthusiasm has been a source of motivation and satisfaction.

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2 Integration by parts and absolute continuity of probability laws

This lecture is devoted to present the classical sufficient conditions for existence and regularity of density of finite measures on $\mathbb{R}^n$ and therefore for the densities of probability laws. The results go back to Malliavin (see [35], but also [74], [79] and [46]). To check these conditions, Malliavin developed a differential calculus on the Wiener space, which in particular allows to prove an integration by parts formula. The essentials on this calculus will be given in the next lecture.

2.1 Properties derived from an integration by parts formula

The integration by parts formula of Malliavin calculus is a simple but extremely useful tool underpinning many of the sometimes unexpected applications of this calculus. To illustrate its role and give a motivation, we start by showing how an abstract integration by parts formula leads to explicit expressions for the densities and their derivatives.

Let us introduce some notation. Multi-indices of dimension r are denoted by $\alpha = (\alpha_1, \dots, \alpha_r) \in \{1, \dots, n\}^r$, and we set $|\alpha| = \sum_{i=1}^r \alpha_i$. For any differentiable real valued function φ defined on $\mathbb{R}^n$, we denote by $\partial_\alpha \varphi$ the partial derivative $\partial^{|\alpha|}_{\alpha_1,\dots,\alpha_r}\varphi$. If $|\alpha| = 0$, we set $\partial_\alpha \varphi = \varphi$, by convention.

Definition 2.1 *Let F be a $\mathbb{R}^n$-valued random vector and G be an integrable random variable defined on some probability space $(\Omega, \mathcal{F}, P)$. Let α be a multi-index. The pair F, G satisfies an integration by parts formula of degree $|\alpha|$ if there exists a random variable $H_\alpha(F, G) \in L^1(\Omega)$ such that*

$$E\Big((\partial_\alpha \varphi)(F)G\Big) = E\Big(\varphi(F)H_\alpha(F, G)\Big), \tag{2.1}$$

for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$.

The property expressed in (2.1) is recursive in the following sense. Let $\alpha = (\beta, \gamma)$, with $\beta = (\beta_1, \dots, \beta_a)$, $\gamma = (\gamma_1, \dots, \gamma_b)$. Then

$$\begin{aligned} E\Big((\partial_\alpha \varphi)(F)G\Big) &= E\Big((\partial_\gamma \varphi)(F)H_\beta(F, G)\Big) \\ &= E\Big(\varphi(F)H_\gamma(F, H_\beta(F, G))\Big) \\ &= E\Big(\varphi(F)H_\alpha(F, G)\Big). \end{aligned}$$

The interest of this definition in connection with the study of probability laws can be deduced from the next result.

Proposition 2.1 *1. Assume that (2.1) holds for $\alpha = (1, \ldots, 1)$ and $G = 1$. Then the probability law of F has a density $p(x)$ with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover,*

$$p(x) = E\Big(1_{(x<F)} H_{(1,\ldots,1)}(F, 1)\Big). \tag{2.2}$$

In particular, p is continuous and bounded.

2. Assume that for any multi-index α the formula (2.1) holds true with $G = 1$. Then $p \in \mathcal{C}^{|\alpha|}(\mathbb{R}^n)$ and

$$\partial_\alpha p(x) = (-1)^{|\alpha|} E\Big(1_{(x<F)} H_{\alpha+1}(F, 1)\Big), \tag{2.3}$$

where $\alpha + 1 := (\alpha_1 + 1, \ldots, \alpha_d + 1)$.

Proof: We start by giving a non rigorous argument for part 1. By (2.1) we have

$$E\Big((\partial_{1,\ldots,1} 1_{[0,\infty)})(F - x)\Big) = E\Big(1_{[0,\infty)}(F - x) H_{(1,\ldots,1)}(F, 1)\Big),$$

But $\partial_{1,\ldots,1} 1_{[0,\infty)} = \delta_0$, where the latter stands for the delta Dirac function at zero, and the equality is understood in the sense of distributions. Moreover, at least at a heuristically level, $p(x) = E\Big(\delta_0(F - x)\Big)$ (see [79] for a proof), consequently

$$p(x) = E\Big(1_{[0,\infty)}(F - x) H_{(1,\ldots,1)}(F, 1)\Big),$$

Let us be more rigorous. Fix $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and set $\varphi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y) dy$. Fubini's theorem yields

$$\begin{aligned} E\Big(f(F)\Big) &= E\left((\partial_{1,\ldots,1}\varphi)(F)\right) = E\left(\varphi(F) H_{(1,\ldots,1)}(F, 1)\right) \\ &= E\left(\int_{\mathbb{R}^n} 1_{(x \le F)} f(x) dx) H_{(1,\ldots,1)}(F, 1) dx\right) \\ &= \int_{\mathbb{R}^n} f(x) E\left(1_{(x \le F)} H_{(1,\ldots,1)}(F, 1)\right) dx. \end{aligned}$$

Let B be a bounded Borel set of $\mathbb{R}^n$. Consider a sequence of functions $f_n \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ converging pointwise to 1_B. Owing to the previous identities (applied to f_n) and Lebesgue bounded convergence we obtain

$$E\Big(1_B(F)\Big) = \int_{\mathbb{R}^n} 1_B(x) E\Big(1_{(x \le F)} H_{(1,\ldots,1)}(F, 1)\Big) dx. \tag{2.4}$$

Hence the law of F is absolutely continuous and its density is given by (2.2). Since $H_{(1,\dots,1)}(F,1)$ is assumed to be in $L^1(\Omega)$, formula (2.2) implies the continuity of p, by bounded convergence. This finishes the proof of part 1.
The proof of part 2 is done recursively. For the sake of simplicity, we shall only give the details of the first iteration for the multi-index $\alpha = (1,\dots,1)$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\Phi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y)dy$, $\Psi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \Phi(y)dy$. By assumption,

$$\begin{aligned} E\Big(f(F)\Big) &= E\Big(\Phi(F)H_{(1,\dots,1)}(F,1)\Big) \\ &= E\Big(\Psi(F)H_{(1,\cdots,1)}(F, H_{(1,\dots,1)}(F,1))\Big) \\ &= E\Big(\Psi(F)H_{(2,\dots,2)}(F,1)\Big). \end{aligned}$$

Fubini's Theorem yields

$$\begin{aligned} & E\Big(\Psi(F)H_{(2,\dots,2)}(F,1)\Big) \\ &= E\left(\int_{-\infty}^{F_1} dy_1 \cdots \int_{-\infty}^{F_n} dy_n \left(\int_{-\infty}^{y_1} dz_1 \cdots \int_{-\infty}^{y_n} dz_n f(z)\right) H_{(2,\dots,2)}(F,1)\right) \\ &= E\left(\int_{-\infty}^{F_1} dz_1 \cdots \int_{-\infty}^{F_n} dz_n f(z) \int_{z_1}^{F_1} dy_1 \cdots \int_{z_n}^{F_n} dy_n H_{(2,\dots,2)}(F,1)\right) \\ &= \int_{\mathbb{R}^n} dz f(z) E\Big(\Pi_{i=1}^n (F_i - z_i)^+ H_{(2,\dots,2)}(F,1)\Big). \end{aligned}$$

This shows that the density of F is given by

$$p(x) = E\Big(\Pi_{i=1}^n (F_i - x_i)^+ H_{(2,\dots,2)}(F,1)\Big),$$

using a limit argument, as in the first part of the proof. The function $x \mapsto \Pi_{i=1}^n (F^i - x_i)^+$ is differentiable, except when $x_i = F_i$ for some $i = 1,\dots,n$, which happens with probability zero, since F is absolutely continuous. Therefore by bounded convergence

$$\partial_{(1,\dots,1)} p(x) = (-1)^n E\Big(1_{[x,\infty)}(F) H_{(2,\dots,2)}(F,1)).$$

□

Remark 2.1 *The conclusion in part 2 of the preceding Proposition is quite easy to understand by formal arguments. Indeed, roughly speaking the function φ in (2.1) should be such that its derivative ∂_α is the delta Dirac function δ_0. Since taking primitives makes functions smoother, the higher $|\alpha|$ is, the smoother φ should be. Thus, having (2.1) for any multi-index α yields infinite differentiability for $p(x) = E\Big(\delta_0(F - x)\Big)$.*

Remark 2.2 *Assume that (2.1) holds for* $\alpha = (1, \ldots, 1)$ *and a positive, integrable random variable* G. *By considering the measure* $dQ = GdP$, *and with a similar proof as for the first statement of Proposition 2.1, we conclude that the measure* $Q^{-1} \circ F$ *is absolutely continuous with respect to the Lebesgue measure and its density* $\tilde{p}$ *is given by*

$$\tilde{p}(x) = E\Big(1_{(x \leq F)} H_{(1,\ldots,1)}(F, G)\Big).$$

2.2 Malliavin's results

We now give Malliavin's criteria for the existence of density (see [35]). To better understand the assumption, let us explore first the one-dimensional case.
Consider a finite measure μ on $\mathbb{R}$. Assume that for every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ there exists a positive constants C, not depending on φ, such that

$$\left| \int_{\mathbb{R}} \varphi' d\mu \right| \leq C ||\varphi||_\infty.$$

Define

$$\varphi_{a,b}(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b, \end{cases} \tag{2.5}$$

$-\infty < a < b < +\infty$. By approximating $\varphi_{a,b}$ by a sequence of functions in $\mathcal{C}_0^\infty(\mathbb{R})$ we obtain

$$\mu\left([a, b]\right) \leq C(b - a).$$

Since this holds for any such $a < b$, it follows that μ is absolutely continuous with respect to the Lebesgue measure.
Malliavin proved that the same result holds true in dimension $n > 1$ as is stated in the next proposition

Proposition 2.2 *Let* μ *be a finite measure on* $\mathbb{R}^n$. *Assume that for any* $i \in \{1, 2, \ldots, n\}$ *and every function* $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, *there exist positive constants* C_i, *not depending on* φ, *such that*

$$\left| \int_{\mathbb{R}} \partial_i \varphi d\mu \right| \leq C_i ||\varphi||_\infty. \tag{2.6}$$

Then μ *is absolutely continuous with respect to the Lebesgue measure and the density belongs to* $L^{\frac{n}{n-1}}$.

When applying this proposition to the law of a random vector F, we have the following particular statement:

Proposition 2.3 *Assume that for any $i \in \{1, 2, \ldots, n\}$ and every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, there exist positive constants C_i, not depending on φ, such that*

$$|E((\partial_i \varphi)(F))| \le C_i ||\varphi||_\infty. \tag{2.7}$$

Then the law of F has a density.

In [35], the density obtained in the preceding theorem is proved to be in L^1; however in a remark the improvement to $L^{\frac{n}{n-1}}$ is mentioned and a hint for the proof is provided. We prove Proposition 2.2 following [46] which takes into account Malliavin's remark.

Proof: Consider an approximation of the identity on $\mathbb{R}^n$, for example

$$\psi_\epsilon(x) = (2\pi\epsilon)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{2\epsilon}\right).$$

Consider also functions c_M, $M \ge 1$, belonging to $\mathcal{C}_0^\infty(\mathbb{R}^n)$, $0 \le c_M \le 1$, such that

$$c_M(x) = \begin{cases} 1 & \text{if } |x| \le M \\ 0 & \text{if } |x| \ge M+1 \end{cases}$$

and with partial derivatives uniformly bounded, independently on M. The functions $c_M \times (\psi_\epsilon * \mu)$ clearly belong to $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and give an *approximation* of μ. Then, by Gagliardo-Nirenberg inequality (see a note at the end of this lecture)

$$\|c_M \times (\psi_\epsilon * \mu)\|_{L^{\frac{n}{n-1}}} \le \prod_{i=1}^n \|\partial_i(c_M \times (\psi_\epsilon * \mu))\|_{L^1}^{\frac{1}{n}}.$$

We next prove that he right-hand side of this inequality is bounded. For this, we notice that assumption (2.6) implies that the functional

$$\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} \partial_i \varphi d\mu$$

is linear and continuous and therefore it defines a signed measure with finite total mass (see for instance [32], page 82). We shall denote by ν_i, $i = 1, \ldots, n$

this measure. Then,

$$\begin{aligned}
\|\partial_i(c_M \times (\psi_\epsilon * \mu))\|_{L^1} &\le \int_{\mathbb{R}^n} c_M(x) \left| \int_{\mathbb{R}^n} \partial_i \psi_\epsilon(x-y)\mu(dy) \right| dx \\
&+ \int_{\mathbb{R}^n} |\partial_i c_M(x)| \left| \int_{\mathbb{R}^n} \psi_\epsilon(x-y)\mu(dy) \right| dx \\
&\le \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi_\epsilon(x-y)\nu_i(dy) \right| dx \\
&+ \int_{\mathbb{R}^n} |\partial_i c_M(x)| \left| \int_{\mathbb{R}^n} \psi_\epsilon(x-y)\mu(dy) \right| dx.
\end{aligned}$$

By applying Fubini's theorem, and because of the choice of ψ_ϵ, it is easy to check that each one of the two last terms is bounded by a finite constant, independent of M and ϵ. As a consequence, the set of functions $\{c_M \times (\psi_\epsilon * \mu), M \ge 1, \epsilon > 0\}$ is bounded in $L^{\frac{n}{n-1}}$. By using the weak compactness of the unit ball of $L^{\frac{n}{n-1}}$ (Alouglu's theorem), we obtain that μ has a density and it belongs to $L^{\frac{n}{n-1}}$.

□

The next result (see [74]) gives sufficient conditions on μ ensuring smoothness of the density with respect to the Lebesgue measure.

Proposition 2.4 *Let μ be a finite measure on $\mathbb{R}^n$. Assume that for any multi-index α and every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ there exist positive constants C_α not depending on φ such that*

$$\left| \int_{\mathbb{R}^n} \partial_\alpha \varphi d\mu \right| \le C_\alpha \|\varphi\|_\infty. \tag{2.8}$$

Then μ possesses a density which is a $\mathcal{C}^\infty$ function.

When particularising μ to the law of a random vector F, condition (2.8) clearly reads

$$|E\left((\partial_\alpha)(F)\right)| \le C_\alpha \|\varphi\|_\infty. \tag{2.9}$$

Remark 2.3 *When checking (2.6), (2.8), we have to get rid of the derivatives ∂_i, ∂_α and thus one naturally thinks of an integration by parts procedure.*

Some comments:

1. Let $n = 1$. The assumption in part 1) of Proposition 2.1 implies (2.6). However, for $n > 1$, both hypotheses are not comparable. The conclusion in the former Proposition gives more information on the density than in Proposition 2.4.

2. Let $n > 1$. Assume that (2.1) holds for any multi-index α with $|\alpha| = 1$. Then, by the recursivity of the integration by parts formula, we obtain the validity of (2.1) for $\alpha = (1, \cdots, 1)$.

3. Since the random variable $H_\alpha(F, G)$ in (2.1) belongs to $L^1(\Omega)$, the identity (2.1) with $G = 1$ clearly implies (2.9). Therefore the assumption in part 2 of Proposition 2.1 is stronger than in Proposition 2.4 but the conclusion more precise too.

Annex
Gagliardo-Nirenberg inequality

Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then

$$\|f\|_{L^{\frac{n}{n-1}}} \le \prod_{i=1}^{n} \|\partial_i f\|_{L^1}^{\frac{1}{n}}.$$

For a proof, we refer the reader to [73], page 129.

3 Stochastic calculus of variations on an abstract Wiener space

This lecture is devoted to introduce the main ingredients of Malliavin calculus: the derivative, divergence and Ornstein Uhlenbeck operators, and rules of calculus for them.

3.1 Finite dimensional Gaussian calculus

To start with, we shall consider a very particular situation. Let μ_m be the standard Gaussian measure on $\mathbb{R}^m$:

$$\mu_m(dx) = (2\pi)^{-\frac{m}{2}} \exp\Big(-\frac{|x|^2}{2}\Big)\, dx.$$

Consider the probability space $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \mu_m)$. Here n–dimensional random vectors are functions $F : \mathbb{R}^m \to \mathbb{R}^n$. We shall denote by E_m the expectation with respect to the measure μ_m.
The purpose is to find sufficient conditions ensuring absolute continuity with respect to the Lebesgue measure on $\mathbb{R}^n$ of the probability law of F, and the smoothness of the density. More precisely, we would like to obtain expressions such as (2.1). This will be done in a quite sophisticated way, as a prelude to the methodology we shall apply in the infinite dimensional case. For the sake of simplicity, we will only deal with multi-indices α of order one. Hence, we shall only address the problem of existence of density for the random vector F. As references of this section we mention [35], [74], [54].

The Ornstein-Uhlenbeck operator

Let $(B_t, t \geq 0)$ be a standard $\mathbb{R}^m$-valued Brownian motion. Consider the linear stochastic differential equation

$$dX_t(x) = \sqrt{2} dB_t - X_t(x)dt, \tag{3.1}$$

with initial condition $x \in \mathbb{R}^m$. Using Itô's formula, it is immediate to check that the solution to (3.1) is given by

$$X_t(x) = \exp(-t)x + \sqrt{2}\int_0^t \exp(-(t-s))dB_s. \tag{3.2}$$

The operator semigroup associated with the Markov process solution to (3.1) is defined by $P_t f(x) = E_m f(X_t(x))$, for a suitable class of functions f. Notice that the law of $Z_t(x) = \sqrt{2}\int_0^t \exp(-(t-s))dB_s$ is Gaussian, mean zero and with covariance given by $(1-\exp(-2t))I$. This fact, together with (3.2), yields

$$P_t f(x) = \int_{\mathbb{R}^m} f(\exp(-t)x + \sqrt{1-\exp(-2t)}y)\mu_m(dy). \tag{3.3}$$

We are going to identify the class of functions f for which the right hand-side of (3.3) makes sense, and we will also compute the infinitesimal generator of the semigroup. This is the Ornstein-Uhlenbeck operator in finite dimension.

Lemma 3.1 *The semigroup generated by $(X_t, t \geq 0)$ satisfies the following:*

1. *$(P_t, t \geq 0)$ is a contraction semigroup on $L^p(\mathbb{R}^m; \mu_m)$, for all $p \geq 1$.*
2. *For any $f \in \mathcal{C}_b^2(\mathbb{R}^m)$ and every $x \in \mathbb{R}^m$,*

$$\lim_{t\to 0} \frac{1}{t}\Big(P_t f(x) - f(x)\Big) = L_m f(x), \tag{3.4}$$

where $L_m = \Delta - x \cdot \nabla = \sum_{i=1}^m \partial^2_{x_i x_i} - \sum_{i=1}^m x_i \partial_{x_i}$.

3. *$(P_t, t \geq 0)$ is a symmetric semigroup on $L^2(\mathbb{R}^m; \mu_m)$.*

Proof. 1) Let X and Y be independent random variables with law μ_m. The law of $\exp(-t)X + \sqrt{1-\exp(-2t)}Y$ is also μ_m. Therefore, $(\mu_m \times \mu_m) \circ T^{-1} = \mu_m$, where $T(x,y) = \exp(-t)x + \sqrt{1-\exp(-2t)}y$. Then, the definition of $P_t f$ and this remark yields

$$\begin{aligned}\int_{\mathbb{R}^m} |P_t f(x)|^p \mu_m(dx) &\leq \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |f(T(x,y))|^p \mu_m(dx)\mu_m(dy) \\ &= \int_{\mathbb{R}^m} |f(x)|^p \mu_m(dx).\end{aligned}$$

2) This follows very easily by applying the Itô formula to the process $f(X_t)$.
3) We must prove that for any $g \in L^2(\mathbb{R}^m; \mu_m)$,

$$\int_{\mathbb{R}^m} P_t f(x) g(x) \mu_m(dx) = \int_{\mathbb{R}^m} f(x) P_t g(x) \mu_m(dx),$$

or equivalently

$$\begin{aligned}&E_m\Big(f(\exp(-t)X + \sqrt{1-\exp(-2t)}Y)g(X)\Big) \\ &= E_m\Big(g(\exp(-t)X + \sqrt{1-\exp(-2t)}Y)f(X)\Big),\end{aligned}$$

where X and Y are two independent standard Gaussian variables. This follows easily from the fact that the vector (Z, X), where

$$Z = \exp(-t)X + \sqrt{1 - \exp(-2t)}Y,$$

has a Gaussian distribution and each component has law μ_m. $\square$

The adjoint of the differential

We are looking for an operator δ_m which is the adjoint of the gradient ∇ in $L^2(\mathbb{R}^m, \mu_m)$. Such an operator must act on functions $\varphi : \mathbb{R}^m \to \mathbb{R}^m$, take values in the space of real-valued functions defined on $\mathbb{R}^m$, and satisfy the duality relation

$$E_m\langle \nabla f, \varphi\rangle = E_m(f \delta_m \varphi), \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^m$. Let $\varphi = (\varphi^1, \dots, \varphi^m)$. Assume first that the functions $f, \varphi^i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$, are continuously differentiable. An usual integration by parts yields

$$\begin{aligned} E_m\langle \nabla f, \varphi\rangle &= \sum_{i=1}^m \int_{\mathbb{R}^m} \partial_i f(x) \varphi^i(x) \mu_m(dx) \\ &= \sum_{i=1}^m \int_{\mathbb{R}^m} f(x) \Big(x_i \varphi^i(x) - \partial_i \varphi^i(x) \Big) \mu_m(dx). \end{aligned}$$

Hence

$$\delta_m \varphi = \sum_{i=1}^m (x_i \varphi^i - \partial_i \varphi^i). \tag{3.6}$$

Notice that on $\mathcal{C}^2(\mathbb{R}^m)$, $\delta_m \circ \nabla = -L_m$.
The definition (3.6) yields the next useful formula

$$\delta_m(f \nabla g) = -\langle \nabla f, \nabla g\rangle - f L_m g, \tag{3.7}$$

for any f, g smooth enough.

Example 3.1 *Let $n \geq 1$; consider the Hermite polynomial of degree n on $\mathbb{R}$, which is defined by*

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right).$$

The operator δ_1 satisfies

$$\begin{aligned}\delta_1 H_n(x) &= xH_n(x) - H_n'(x) = xH_n(x) - H_{n-1}(x)\\ &= (n+1)H_{n+1}(x).\end{aligned}$$

Therefore it increases the order of a Hermite polynomial by one.

An integration by parts formula

Using the operators ∇, δ_m and L_m, and for random vectors $F = (F^1, \ldots, F^n)$ regular enough (meaning that all the differentiations performed throughout this section make sense), we are going to establish an integration by parts formula of the type (2.1).

We start by introducing the finite dimensional *Malliavin matrix*, also termed *covariance matrix*, as follows:

$$A(x) = \Big(\langle \nabla F^i(x), \nabla F^j(x)\rangle\Big)_{1\le i,j\le n}.$$

Notice that by its very definition, $A(x)$ is a symmetric, non-negative definite matrix, for any $x \in \mathbb{R}^m$. Clearly $A(x) = DF(x)DF(x)^T$, where $DF(x)$ is the Jacobian matrix at x and the superscript T means the transpose.

Let us consider a function $\varphi \in \mathcal{C}^1(\mathbb{R}^n)$, and perform some computations showing that $(\partial_i\varphi)(F)$, $i = 1, \ldots, n$, satisfies a linear system of equations. Indeed, by the chain rule,

$$\begin{aligned}\Big\langle \nabla\Big(\varphi(F(x))\Big), \nabla F^l(x)\Big\rangle &= \sum_{j=1}^{m}\sum_{k=1}^{n}(\partial_k\varphi)(F(x))\partial_j F^k(x)\partial_j F^l(x)\\ &= \sum_{k=1}^{n}\langle \nabla F^l(x), \nabla F^k(x)\rangle(\partial_k\varphi)(F(x))\\ &= \Big(A(x)(\nabla^T\varphi)(F(x))\Big)_l, \qquad (3.8)\end{aligned}$$

$l = 1, \ldots, n$. Assume that the matrix $A(x)$ is inversible μ_m-almost everywhere. Then one gets

$$(\partial_i\varphi)(F) = \sum_{l=1}^{n}\Big\langle \nabla\Big(\varphi(F(x))\Big), A^{-1}_{i,l}(x)\nabla F^l(x)\Big\rangle, \qquad (3.9)$$

for every $i = 1, \ldots, n$, μ_m-almost everywhere.

Taking expectations and using (3.7), (3.9) yields

$$E_m\Big((\partial_i\varphi)(F)\Big) = \sum_{l=1}^{n} E_m\langle\nabla\Big(\varphi(F)\Big), A^{-1}_{i,l}\nabla F^l\rangle$$
$$= \sum_{l=1}^{n} E_m\Big(\varphi(F)\delta_m(A^{-1}_{i,l}\nabla F^l)\Big)$$
$$= \sum_{l=1}^{n} E_m\Big(\varphi(F)\Big(-\langle\nabla A^{-1}_{i,l}, \nabla F^l\rangle - A^{-1}_{i,l}L_m F^l\Big). \tag{3.10}$$

Hence we can write

$$E_m\Big(\partial_i\varphi(F)\Big) = E_m\Big(\varphi(F)H_i(F,1)\Big), \tag{3.11}$$

with

$$H_i(F,1) = \sum_{l=1}^{n} \delta_m(A^{-1}_{i.l}\nabla F^l)$$
$$= -\sum_{l=1}^{n}\Big(\langle\nabla A^{-1}_{i,l}, \nabla F^l\rangle + A^{-1}_{i,l}L_m F^l\Big). \tag{3.12}$$

This is an integration by parts formula, as in Definition 2.1, for multi-indices of length one.
For multi-indices of length greater than one, things are a little bit more difficult; essentialy the same ideas would lead to the analogue of formula (2.1) with $\alpha = (1, \cdots, 1)$ and $G = 1$.
The preceding discussion and Proposition 2.2 yield the following result.

Proposition 3.1 *Let F be continuous differentiable up to the second order such that F and its partial derivatives up to order two belong to $L^p(\mathbb{R}^m; \mu_m)$, for any $p \in [1, \infty[$. Assume that:*

(1) The matrix $A(x)$ is invertible for every $x \in \mathbb{R}^m$, μ_m-almost everywhere.

(2) $\det A^{-1} \in L^p(\mathbb{R}^m; \mu_m)$, $\nabla(\det A^{-1}) \in L^r(\mathbb{R}^m; \mu_m)$, for some $p, r \in (1, \infty)$.

Then the law of F is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$.

Proof: The assumptions on F and in (2) show that

$$C_i := \sum_{l=1}^{n} E_m\Big(\Big|\langle \nabla A_{i,l}^{-1}, \nabla F^l\rangle\Big| + \Big|A_{i,l}^{-1} L_m F^l\Big|\Big)$$

is finite. Therefore, one can take expectations on both sides of (3.9). By (3.10), it follows that

$$|E_m(\delta_i\varphi)(F)| \le C_i ||\varphi||_\infty.$$

This finishes the proof of the Proposition. □

Remark 3.1 *The proof of smoothness properties for the density requires an iteration of the procedure presented in the proof of Proposition 3.1.*

3.2 Infinite dimensional framework

This section is devoted to describe an infinite dimensional analogue of the probability space $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \mu_m)$. We start by introducing a family of Gaussian random variables. Let H be a real separable Hilbert space. Denote by $||\cdot||_H$ and $\langle \cdot, \cdot \rangle_H$ the norm and the inner product on H, respectively. There exist a probability space $(\Omega, \mathcal{G}, \mu)$ and a family $\mathcal{M} = \Big(W(h), h \in H\Big)$ of random variables defined on this space, such that the mapping $h \to W(h)$ is linear, each $W(h)$ is Gaussian, $EW(h) = 0$ and $E\Big(W(h_1)W(h_2)\Big) = \langle h_1, h_2\rangle_H$ (see for instance, [63], Chapter 1, Proposition 1.3). Such family is constructed as follows. Let $(e_n, n \ge 1)$ be a complete orthonormal system in H. Consider the canonical probability space $(\Omega, \mathcal{G}, P)$ associated with a sequence $(g_n, n \ge 1)$ of standard independent Gaussian random variables. That is, $\Omega = \mathbb{R}^{\otimes \mathbb{N}}$, $\mathcal{G} = \mathcal{B}^{\otimes \mathbb{N}}$, $\mu = \mu_1^{\otimes \mathbb{N}}$ where, according to the notations of Chapter 1, μ_1 denotes the standard Gaussian measure on $\mathbb{R}$. For each $h \in H$, the series $\sum_{n\ge1}\langle h, e_n\rangle_H g_n$ converges in $L^2(\Omega, \mathcal{G}, \mu)$ to a random variable that we denote by $W(h)$. Notice that the set $\mathcal{M}$ is a closed Gaussian subspace of $L^2(\Omega)$ that is isometric to H. In the sequel, we will replace $\mathcal{G}$ by the σ-field generated by $\mathcal{M}$.

Examples

White Noise

Let $H = L^2(A, \mathcal{A}, m)$, where $(A, \mathcal{A}, m)$ is a separable σ-finite, atomless measure space. For any $F \in \mathcal{A}$ with $m(F) < \infty$, set $W(F) = W(1_F)$. The stochastic Gaussian process $\Big(W(F), F \in \mathcal{A}, m(F) < \infty\Big)$ is such that $W(F)$ and $W(G)$ are independent if F and G are disjoint sets; in this case, $W(F \cup G) = W(F) + W(G)$. Following [78], we call such a process a *white noise based on m*. Then the random variable $W(h)$ coincides with the first order Itô stochastic integral $\int_A h(t)W(dt)$ with respect to W (see [22]).

If $A = \mathbb{R}_+$, $\mathcal{A}$ is the σ-field of Borel sets of $\mathbb{R}_+$ and m is the Lebesgue measure on $\mathbb{R}_+$, then $W(h) = \int_0^\infty h(t)dW_t$ -the Itô integral of a deterministic integrand- where $\Big(W_t, t \geq 0\Big)$ is a standard Brownian motion.

Correlated Noise

Fix $d \geq 1$ and denote by $\mathcal{D}(\mathbb{R}^d)$ the set of Schwartz test functions in $\mathbb{R}^d$, that is, functions of $\mathcal{C}^\infty(\mathbb{R}^d)$ with compact support. Let Γ be a non-negative measure, of non-negative type, and tempered (see [71] for the definitions of these notions).

For φ, ψ in $\mathcal{D}(\mathbb{R}^d)$, define

$$I(\varphi, \psi) = \int_{\mathbb{R}^d} \Gamma(dx) \left(\varphi * \tilde{\psi}\right)(x),$$

where $\tilde{\psi}(x) = \psi(-x)$ and the symbol "$*$" denotes the convolution operator. According to [71], Chap. VII, Théorème XVII, the measure Γ is symmetric. Hence the functional I defines an inner product on $\mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d)$. Moreover, there exists a non-negative tempered measure μ on $\mathbb{R}^d$ whose Fourier transform is Γ (see [71], Chap. VII, Théorème XVIII). Therefore,

$$I(\varphi, \psi) = \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}. \tag{3.13}$$

There is a natural Hilbert space associated with the covariance functional I. Indeed, let $\mathcal{E}$ be the inner-product space consisting of functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$, endowed with the inner-product

$$\langle \varphi, \psi \rangle_{\mathcal{E}} := I(\varphi, \psi) = \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}. \tag{3.14}$$

Let $\mathcal{H}$ denote the completion of $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$. Elements of the Gaussian family $\mathcal{M} = (W(h), h \in \mathcal{H})$ satisfy

$$E\left(W(h_1)W(h_2)\right) = \int_{\mathbb{R}^d} \Gamma(dx) \left(h_1 * \tilde{h}_2\right)(x),$$

$h_1, h_2 \in \mathcal{H}$.

The family $\left(W(1_F), F \in \mathcal{B}_b(\mathbb{R}^d)\right)$ can be rigourously defined by approximating 1_F by a sequence of elements in $\mathcal{H}$. It is called a colored noise with covariance Γ.

We notice that for $\Gamma = \delta_0$,

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \langle \varphi, \psi \rangle_{L^2(\mathbb{R}^d)}.$$

White-Correlated Noise

In the theory of SPDEs, stochastic processes are usually indexed by $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and the role of t and x is different -time and space, respectively. Sometimes the driving noise of the equation is white in time and in space (see the example termed *white noise* before). Another important class of examples are based on noises white in time and correlated in space. We give here the background for this type of noise.

With the same notations and hypotheses as in the preceding example, we consider functions $\varphi, \psi \in \mathcal{D}(\mathbb{R}^{d+1})$ and define

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \left(\varphi * \tilde{\psi}\right)(x). \tag{3.15}$$

By the above quoted result in [71], J defines an inner product. Set $\mathcal{H}_T = L^2([0, T]; \mathcal{H})$. Elements of the Gaussian family $\mathcal{M} = (W(h), h \in \mathcal{H}_T)$ satisfy

$$E\left(W(h_1)W(h_2)\right) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \left(h_1(s) * \tilde{h}_2(s)\right)(x), \tag{3.16}$$

$h_1, h_2 \in \mathcal{H}_T$. We can then consider $\left(W(t, A), t \in [0, \infty[, A \in \mathcal{B}_b(\mathbb{R}^d)\right)$, where $W(t, A) := W(1_{[0,t]} \times 1_A)$ is defined by an approximation procedure. This family is called a Gaussian noise, white in time and stationary correlated (or coloured) in space.

3.3 The derivative and divergence operators

Throughout this section, we consider the probability space $(\Omega, \mathcal{G}, \mu)$, defined in section 3.2 and a Gaussian family $\mathcal{M} = (W(h), h \in H)$, as has been described before.
There are several possibilities to define the Malliavin derivative for random vectors $F : \Omega \to \mathbb{R}^n$. Here we shall follow the *analytic approach* which roughly speaking consists of an extension by a limiting procedure of differentiation in $\mathbb{R}^m$.
To start with, we consider *finite-dimensional* objects, termed *smooth functionals.* They are random variables of the type

$$F = f(W(h_1), \ldots, W(h_n)), \tag{3.17}$$

with $h_1, \ldots, h_n \in H$ and $f : \mathbb{R}^n \to \mathbb{R}$ regular enough.
Different choices of regularity of f lead to different classes of *smooth* functionals. For example, if $f \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, the set of infinitely differentiable functions such that f and its partial derivatives of any order have polynomial growth, we denote the corresponding class of *smooth* functionals by $\mathcal{S}$; if $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, the set of infinitely differentiable functions such that f and its partial derivatives of any order are bounded, we denote by $\mathcal{S}_b$ the corresponding class. If f is a polynomial, then *smooth* functionals are denoted by $\mathcal{P}$. Clearly $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}_b \subset \mathcal{S}$.
We define the operator D on $\mathcal{S}$ (on $\mathcal{P}$, on $\mathcal{S}_b$) with values on the set of H-valued random variables, by

$$DF = \sum_{i=1}^{n} \partial_i f\Big(W(h_1), \ldots, W(h_n)\Big) h_i. \tag{3.18}$$

Fix $h \in H$ and set

$$F^{\epsilon h} = f\Big(W(h_1) + \epsilon\langle h, h_1\rangle_H, \ldots, W(h_n) + \epsilon\langle h, h_n\rangle_H\Big),$$

$\epsilon > 0$. Then it is immediate to check that $\langle DF, h\rangle_H = \frac{d}{d\epsilon} F^{\epsilon h}\Big|_{\epsilon=0}$. Therefore, for smooth functionals, D is a directional derivative. It is also routine to prove that if F, G are smooth functionals then, $D(FG) = FDG + GDF$.
Our next aim is to prove that D is closable as an operator from $L^p(\Omega)$ to $L^p(\Omega; H)$, for any $p \geq 1$. That is, if $\{F_n, n \geq 1\} \subset \mathcal{S}$ is a sequence converging to zero in $L^p(\Omega)$ and the sequence $\{DF_n, n \geq 1\}$ converges to G in $L^p(\Omega; H)$, then $G = 0$. The tool for arguing this is a simple version of an *integration by parts formula* proved in the next lemma.

Lemma 3.2 *For any $F \in \mathcal{S}$, $h \in H$, we have*

$$E\Big(\langle DF, h\rangle_H\Big) = E\Big(FW(h)\Big). \tag{3.19}$$

Proof: Without loss of generality, we shall assume that

$$F = f\Big(W(h_1), \ldots, W(h_n)\Big),$$

where $h_1, \ldots, h_n$ are orthonormal elements of H and $h_1 = h$. Then

$$E\Big(\langle DF, h\rangle_H\Big) = \int_{\mathbb{R}^n} \partial_1 f(x)\mu_n(dx)$$
$$= \int_{\mathbb{R}^n} f(x)x_1\mu_n(dx) = E\Big(FW(h_1)\Big).$$

The proof is complete. □

Formula (3.19) is a statement about duality between the operator D and *a integral* with respect to W.
Let $F, G \in \mathcal{S}$. Applying formula (3.19) to the smooth functional FG yields

$$E\Big(G\langle DF, h\rangle_H\Big) = -E\Big(F\langle DG, h\rangle_H\Big) + E\Big(FGW(h)\Big). \tag{3.20}$$

With this result, we can now prove that D is closable. Indeed, consider a sequence $\{F_n, n \geq 1\} \subset \mathcal{S}$ satisfying the properties stated above. Let $h \in H$ and $F \in \mathcal{S}_b$ be such that $FW(h)$ is bounded. Using (3.20), we obtain

$$\begin{aligned} E\Big(F\langle G, h\rangle_H\Big) &= \lim_{n\to\infty} E\Big(F\langle DF_n, h\rangle_H\Big) \\ &= \lim_{n\to\infty} E\Big(- F_n\langle DF, h\rangle_H + F_n FW(h)\Big) = 0. \end{aligned}$$

Indeed, the sequence $(F_n, n \geq 1)$ converges to zero in L^p and $\langle DF, h\rangle_H$, $FW(h)$ are bounded. This yields $G = 0$. □

Let $\mathbb{D}^{1,p}$ be the closure of the set $\mathcal{S}$ with respect to the seminorm

$$||F||_{1,p} = \Big(E(|F|^p) + E(||DF||_H^p)\Big)^{\frac{1}{p}}. \tag{3.21}$$

The set $\mathbb{D}^{1,p}$ is the domain of the operator D in $L^p(\Omega)$. Notice that $\mathbb{D}^{1,p}$ is dense in $L^p(\Omega)$. The above procedure can be iterated as follows. Clearly, one can recursively define the operator D^k, $k \in \mathbb{N}$, on the set $\mathcal{S}$. This yields an

$H^{\otimes k}$-valued random vector. As for D, one proves that D^k is closable. Then we can introduce the seminorms

$$||F||_{k,p} = \Big(E(|F|^p) + \sum_{j=1}^{k} E(||D^j F||^p_{H^{\otimes j}}\Big)^{\frac{1}{p}}, \tag{3.22}$$

$p \in [1, \infty)$, and define the sets $\mathbb{D}^{k,p}$ to be the closure of $\mathcal{S}$ with respect to the seminorm (3.22). Notice that by definition, $\mathbb{D}^{j,q} \subset \mathbb{D}^{k,p}$ for $k \le j$ and $p \le q$. By convention $\mathbb{D}^{0,p} = L^p(\Omega)$ and $\| \cdot \|_{0,p} = \| \cdot \|_p$, the usual norm in $L^p(\Omega)$.

We now introduce the *divergence operator*, which corresponds to the infinite dimensional analogue of the operator δ_m defined in (3.6).
For this, we notice that the Malliavin derivative D is an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$. Moreover, the domain of D in $L^2(\Omega)$, denoted by $\mathbb{D}^{1,2}$, is dense in $L^2(\Omega)$. Then, by an standard procedure (see for instance [80]) one can define the *adjoint* of D, that we shall denote by δ.
Indeed, the domain of the adjoint, denoted by $\operatorname{Dom}\delta$, is the set of random vectors $u \in L^2(\Omega; H)$ such that for any $F \in \mathbb{D}^{1,2}$,

$$\Big|E\Big(\langle DF, u\rangle_H\Big)\Big| \le c||F||_2,$$

where c is a constant depending on u. If $u \in \operatorname{Dom}\delta$, then δu is the element of $L^2(\Omega)$ characterized by the identity

$$E\Big(F\delta(u)\Big) = E\Big(\langle DF, u\rangle_H\Big), \tag{3.23}$$

for all $F \in \mathbb{D}^{1,2}$.
Equation (3.23) expresses the duality between D and δ. It is called the *integration by parts formula* (compare with (3.19)). The analogy between δ and δ_m defined in (3.6) can be easily established on *finite dimensional* random vectors of $L^2(\Omega; H)$, as follows.
Let $\mathcal{S}_{\mathcal{H}}$ be the set of random vectors of the type

$$u = \sum_{j=1}^{n} F_j h_j,$$

where $F_j \in \mathcal{S}$, $h_j \in H$, $j = 1, \ldots, n$. Let us prove that $u \in \operatorname{Dom}\delta$.

Indeed, owing to formula (3.20), for any $F \in \mathcal{S}$,

$$\Big|E\Big(\langle DF, u\rangle_H\Big)\Big| = \Big|\sum_{j=1}^{n} E\Big(F_j\langle DF, h_j\rangle_H\Big)\Big|$$
$$\leq \sum_{j=1}^{n}\Big(\Big|E\Big(F\langle DF_j, h_j\rangle_H\Big| + \Big|E\Big(FF_jW(h_j)\Big)\Big|\Big)$$
$$\leq C||F||_2.$$

Hence $u \in \operatorname{Dom}\delta$. Moreover, by the same computations,

$$\delta(u) = \sum_{j=1}^{n} F_jW(h_j) - \sum_{j=1}^{n}\langle DF_j, h_j\rangle_H. \tag{3.24}$$

Hence, the gradient operator in the finite dimensional case is replaced by the Malliavin *directional* derivative, and the coordinate variables x_j by the *random coordinates* $W(h_j)$.

Remark 3.2 *The divergence operator coincides with a stochastic integral introduced by Skorohod in [72]. This integral allows for non adapted integrands. It is actually an extension of Itô's integral. Readers interested in this topic are suggested to consult the monographs [46] and [47].*

3.4 Some calculus

In this section we prove several basic rules of calculus for the two operators defined so far. The first result is a chain rule.

Proposition 3.2 *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Let $F = (F^1, \ldots, F^m)$ be a random vector whose components belong to $\mathbb{D}^{1,p}$ for some $p \geq 1$. Then $\varphi(F) \in \mathbb{D}^{1,p}$ and*

$$D(\varphi(F)) = \sum_{i=1}^{m} \partial_i\varphi(F)DF^i. \tag{3.25}$$

The proof of this result is straightforward. First, we assume that $F \in \mathcal{S}$; in this case, formula (3.25) follows by the classical rules of differential calculus. The proof for $F \in \mathbb{D}^{1,p}$ is done by an approximation procedure.
The preceding chain rule can be extended to Lipschitz functions φ. The tool for this improvement is given in the next Proposition. For its proof, we use the Wiener chaos decomposition of $L^2(\Omega, \mathcal{G})$ (see [22]).

Proposition 3.3 *Let* $(F_n, n \geq 1)$ *be a sequence of random variables in* $\mathbb{D}^{1,2}$ *converging to* F *in* $L^2(\Omega)$ *and such that*

$$\sup_n E\Big(||DF_n||_H^2\Big) < \infty. \tag{3.26}$$

Then F *belongs to* $\mathbb{D}^{1,2}$ *and the sequence of derivatives* $(DF_n, n \geq 1)$ *converges to* DF *in the weak topology of* $L^2(\Omega; H)$.

Proof: The assumption (3.26) yields the existence of a subsequence $(F_{n_k}, k \geq 1)$ such that the corresponding sequence of derivatives $(DF_{n_k}, k \geq 1)$ converges in the weak topology of $L^2(\Omega; H)$ to some element $\eta \in L^2(\Omega; H)$. In particular, for any $G \in L^2(\Omega; H)$, $\lim_{k\to\infty} E(\langle DF_{n_k}, J_l G\rangle_H) = E(\langle \eta, J_l G\rangle_H)$, where J_l denotes the projection on the l-th Wiener chaos $\mathcal{H}_l \otimes H$, $l \geq 0$.
The integration by parts formula and the convergence of the sequence $(F_n, n \geq 1)$ yield

$$\begin{aligned}
\lim_{k\to\infty} E(\langle DF_{n_k}, J_l G\rangle_H) &= \lim_{k\to\infty} E(F_{n_k}\delta(J_l G)) \\
&= E(F\delta(J_l G)) = E(\langle DF, J_l G\rangle_H).
\end{aligned}$$

Hence, every weakly convergent subsequence of $DF_n, n \geq 1$, must converge to the same limit and the whole sequence converges. Moreover, the random vectors η and DF have the same projection on each Wiener chaos; consequently, $\eta = DF$ as elements of $L^2(\Omega; H)$. □

Proposition 3.4 *Let* $\varphi : \mathbb{R}^m \to \mathbb{R}$ *be a globally Lipschitz function and* $F = (F^1, \ldots, F^m)$ *be a random vector with components in* $\mathbb{D}^{1,2}$. *Then* $\varphi(F) \in \mathbb{D}^{1,2}$. *Moreover, there exists a bounded random vector* $G = (G_1, \cdots, G_m)$ *such that*

$$D(\varphi(F)) = \sum_{i=1}^{m} G_i DF^i. \tag{3.27}$$

Proof: The idea of the proof is as follows. First we regularize the function φ by convolution with an approximation of the identity. We apply Proposition 3.2 to the sequence obtained in this way. Then we conclude by means of Proposition 3.3.
More explicitely, let $\alpha \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ be nonnegative, with compact support and $\int_{\mathbb{R}^m} \alpha(x)dx = 1$. Define $\alpha_n(x) = n^m \alpha(nx)$ and $\varphi_n = \varphi * \alpha_n$. It is well known

that $\varphi_n \in \mathcal{C}^\infty$ and that the sequence $(\varphi_n, n \geq 1)$ converges to φ uniformly. In addition $\nabla\varphi_n$ is bounded by the Lipschitz constant of φ.
Proposition 3.2 yields,

$$D(\varphi_n(F)) = \sum_{i=1}^{m} \partial_i \varphi_n(F) DF^i. \tag{3.28}$$

Now we apply Proposition 3.3 to the sequence $F_n = \varphi_n(F)$. It is clear that $\lim_{n\to\infty} \varphi_n(F) = \varphi(F)$ in $L^2(\Omega)$. Moreover, by the boundedness property on $\nabla\varphi_n$, the sequence $D(\varphi_n(F)), n \geq 1$, is bounded in $L^2(\Omega; H)$. Hence $\varphi(F) \in \mathbb{D}^{1,2}$ and $D(\varphi_n(F)), n \geq 1$ converges in the weak topology of $L^2(\Omega; H)$ to $D(\varphi(F))$. Since the sequence $(\nabla\varphi_n(F), n \geq 1)$, is bounded, a.s., there exists a subsequence that converges to some random bounded vector G in the weak topology of $L^2(\Omega; H)$. By passing to the limit as $n \to \infty$ the equality (3.28), we finish the proof of the Proposition.

$\square$

Remark 3.3 *Let $\varphi \in \mathcal{C}_p^\infty(\mathbb{R}^m)$ and $F = (F^1, \ldots, F^m)$ be a random vector whose components belong to $\cap_{p\in[1,\infty)}\mathbb{D}^{1,p}$. Then the conclusion of Proposition 3.2 also holds. Moreover, $\varphi(F) \in \cap_{p\in[1,\infty)}\mathbb{D}^{1,p}$.*
The chain rule (3.25) can be iterated; we obtain Leibniz's rule for Malliavin's derivatives. For example, if F is one-dimensional ($m = 1$) then

$$D^k(\varphi(F)) = \sum_{l=1}^{k} \sum_{\mathcal{P}_l} c_l \varphi^{(l)}(F) \Pi_{i=1}^{l} D^{|p_i|} F, \tag{3.29}$$

where $\mathcal{P}_l$ denotes the set of partitions of $\{1, \cdots, k\}$ consisting of l disjoint sets $p_1, \cdots, p_l$, $l = 1, \cdots, k$, $|p_i|$ denotes the cardinal of the set p_i and c_l are positive coefficients.

For any $F \in \mathrm{Dom}D$, $h \in H$ we set $D_h F = \langle DF, h\rangle_H$. The next propositions provide important calculus rules.

Proposition 3.5 *Let $u \in \mathcal{S}_H$. Then*

$$D_h(\delta(u)) = \langle u, h\rangle_H + \delta(D_h u). \tag{3.30}$$

Proof: Fix $u = \sum_{j=1}^{n} F_j h_j$, $F_j \in \mathcal{S}$, $h_j \in H$, $j = 1, \ldots, n$. By virtue of (3.24), we have

$$D_h(\delta(u)) = \sum_{j=1}^{n} \Big((D_h F_j) W(h_j) + F_j \langle h_j, h \rangle - \langle D(D_h F_j), h_j \rangle_H \Big).$$

Notice that by (3.24),

$$\delta(D_h u) = \sum_{j=1}^{n} \Big((D_h F_j) W(h_j) - \langle D(D_h F_j), h_j \rangle_H \Big). \tag{3.31}$$

Hence (3.30) holds.

□

The next result is an *isometry property* for the integral defined by the operator δ.

Proposition 3.6 *Let $u, v \in \mathbb{D}^{1,2}(H)$. Then*

$$E\Big(\delta(u)\delta(v)\Big) = E(\langle u, v \rangle_H) + E(\operatorname{tr}(Du \circ Dv)), \tag{3.32}$$

where $\operatorname{tr}(Du \circ Dv) = \sum_{i,j=1}^{\infty} D_{e_j} \langle u, e_i \rangle_H D_{e_i} \langle v, e_j \rangle_H$, with $(e_i, i \geq 1)$ a complete orthonormal system in H.
Consequently, if $u \in \mathbb{D}^{1,2}(H)$ then $u \in \operatorname{Dom} \delta$ and

$$E\Big(\delta(u)\Big)^2 \leq E(||u||_H^2) + E(||Du||_{H \otimes H}^2). \tag{3.33}$$

Proof: Assume first that $u, v \in \mathcal{S}_H$. The duality relation between D and δ yields

$$E(\delta(u)\delta(v)) = E\Big(\langle v, D(\delta(u)) \rangle_H\Big) = E\Big(\sum_{i=1}^{\infty} \langle v, e_i \rangle_H D_{e_i}(\delta(u))\Big).$$

By virtue of (3.30), this last expression is equal to

$$E\Big(\sum_{i=1}^{\infty} \langle v, e_i \rangle_H (\langle u, e_i \rangle_H + \delta(D_{e_i} u)\Big).$$

The duality relation between D and δ implies

$$\begin{aligned}
&E\Big(\langle v, e_i\rangle_H \delta(D_{e_i} u)\Big) = E\Big(\langle D_{e_i} u, D\langle v, e_i\rangle_H\rangle_H\Big) \\
&= \sum_{j=1}^{\infty} E\Big(\langle D_{e_i}\langle u, e_j\rangle_H e_j, D\langle v, e_i\rangle_H\Big) \\
&= \sum_{j=1}^{\infty} E\Big(D_{e_i}\langle u, e_j\rangle_H D_{e_j}\langle v, e_i\rangle_H\Big).
\end{aligned}$$

This establishes (3.32). Taking $u = v$ and applying Schwarz' inequality yield (3.33).
The extension to $u, v \in \mathbb{D}^{1,2}(H)$ is done by a limit procedure.

$\square$

Remark 3.4 *Proposition 3.6 can be used to extend the validity of (3.30) to $u \in \mathbb{D}^{2,2}(H)$. Indeed, let $u_n \in \mathcal{S}_H$ be a sequence of processes converging to u in $\mathbb{D}^{2,2}(H)$. Formula (3.30) holds true for u_n. We can take limits in $L^2(\Omega; H)$ as n tends to infinity and conclude, because the operators D and δ are closed.*

Proposition 3.7 *Let $F \in \mathbb{D}^{1,2}$, $u \in \operatorname{Dom}\delta$, $Fu \in L^2(\Omega; H)$. If $F\delta(u) - \langle DF, u\rangle_H \in L^2(\Omega)$, then*

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H. \tag{3.34}$$

Proof: Assume first that $F \in \mathcal{S}$ and $u \in \mathcal{S}_H$. Let $G \in \mathcal{S}$. Then by the duality relation between D and δ and the calculus rules on the derivatives, we have

$$\begin{aligned}
E(G\delta(Fu)) &= E(\langle DG, Fu\rangle_H) \\
&= E(\langle u, (D(FG) - GDF)\rangle_H) \\
&= E(G(F\delta(u) - \langle u, DF\rangle_H)).
\end{aligned}$$

By the definition of the operator δ, (3.34) holds under the assumptions of the proposition.

$\square$

4 Criteria for Existence and Regularity of Densities

In lecture 1, we have shown how an integration by parts formula (see Definition 2.1) leads to results on densities of probability laws. The question we tackle in this lecture is how to derive such a formula. In particular we will give an expression for the random variable $H_\alpha(F,G)$. For this, we shall apply the calculus developed in Section 3.
We consider here the probability space associated with a Gaussian family $(W(h), h \in H)$, as has been described in Section 3.2.

4.1 Existence of density

Let us start with a very simple example.

Proposition 4.1 *Let F be a random variable belonging to $\mathbb{D}^{1,2}$. Assume that the random variable $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of δ in $L^2(\Omega; H)$. Then the law of F is absolutely continuous. Moreover, its density is given by*

$$p(x) = E\Big(1_{(F>x)}\delta\Big(\frac{DF}{\|DF\|_H^2}\Big)\Big) \tag{4.1}$$

and therefore it is continuous and bounded.

Proof: We will check that for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R})$,

$$E(\varphi^{'}(F)) = E\Big(\varphi(F)\delta\Big(\frac{DF}{\|DF\|_H^2}\Big)\Big). \tag{4.2}$$

Thus (2.1) holds for $G=1$ with $H_1(F,1) = \delta\big(\frac{DF}{\|DF\|_H^2}\big)$. Then the results follow from part 1 of Proposition 2.1.
The chain rule of Malliavin calculus yields $D(\varphi(F)) = \varphi^{'}(F)DF$. Thus,

$$\varphi^{'}(F) = \Big\langle D(\varphi(F)), \frac{DF}{\|DF\|_H^2}\Big\rangle_H .$$

Therefore, the integration by parts formula implies

$$\begin{aligned} E\Big(\varphi^{'}(F)\Big) &= E\left(\Big\langle D(\varphi(F)), \frac{DF}{\|DF\|_H^2}\Big\rangle_H\right) \\ &= E\left(\varphi(F)\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right), \end{aligned}$$

proving (4.2).

$\square$

Remark 4.1 *Notice the analogy between (4.2) and the finite dimensional formula (3.11).*

Remark 4.2 *Using the explicit formula (4.1) to particular examples and* $L^p(\Omega)$ *estimates of the Skorohod integral leads to interesting estimates for the density (see for instance [47]).*

Remark 4.3 *In Proposition 4.1 we have established the formula*

$$H_1(F,1) = \delta\left(\frac{DF}{||DF||_H^2}\right), \tag{4.3}$$

where $F:\Omega \to \mathbb{R}$.

For random vectors F, $(n>1)$, we can obtain similar results by using matrix calculus, as it is illustrated in the next statement. In the computations, instead of $\|DF\|_H$, we have to deal with the Malliavin matrix, a notion given in the next definition.

Definition 4.1 *Let* $F:\Omega \to \mathbb{R}^n$ *be a random vector with components* $F^j \in \mathbb{D}^{1,2}$, $j=1,\dots,n$. *The Malliavin matrix of* F *is the* $n \times n$ *matrix, denoted by* γ, *whose entries are the random variables* $\gamma_{i,j} = \langle DF^i, DF^j\rangle_H$, $i,j = 1,\dots,n$.

Proposition 4.2 *Let* $F:\Omega \to \mathbb{R}^n$ *be a random vector with components* $F^j \in \mathbb{D}^{1,2}$, $j=1,\dots,n$. *Assume that*

(1) the Malliavin matrix γ *is inversible, a.s.*

(2) For every $i,j=1,\dots,n$, *the random variables* $(\gamma^{-1})_{i,j}DF^j$ *belong to* $\mathrm{Dom}\,\delta$.

Then for any function $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,

$$E(\partial_i\varphi(F)) = E(\varphi(F)H_i(F,1)), \tag{4.4}$$

with

$$H_i(F,1) = \sum_{l=1}^n \delta((\gamma^{-1})_{i,l}DF^l). \tag{4.5}$$

Consequently the law of F *is absolutely continuous.*

Proof: Fix $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$. By virtue of the chain rule, we have $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$\begin{aligned}\langle D(\varphi(F)), DF^l\rangle_H &= \sum_{k=1}^n \partial_k \varphi(F) \langle DF^k, DF^l\rangle_H \\ &= \sum_{k=1}^n \partial_k \varphi(F) \gamma_{k,l},\end{aligned}$$

$l = 1, \ldots, n$. Since γ is inversible a.s., this system of linear equations in $\partial_k \varphi(F)$, $k = 1, \ldots, n$, can be solved, and

$$\partial_i \varphi(F) = \sum_{l=1}^n \langle D(\varphi(F)), (\gamma^{-1})_{i,l} DF^l\rangle_H, \tag{4.6}$$

$i = 1, \ldots, n$, a.s.
The assumption (2), the duality formula along with (4.6) yield

$$\begin{aligned}&\sum_{l=1}^n E\Big(\varphi(F)\delta\Big((\gamma^{-1})_{i,l} DF^l\Big)\Big) \\ &= \sum_{l=1}^n E\Big(\langle D(\varphi(F)), (\gamma^{-1})_{i,l} DF^l\rangle_H\Big) \\ &= E\Big(\partial_i \varphi(F)\Big).\end{aligned}$$

Hence (4.4), (4.5) is proved.
Notice that by assumption $H_i(F, 1) \in L^2(\Omega)$. Thus Proposition 2.2 part 1) yields the existence of the density.

□

Remark 4.4 *The equalities (4.4), (4.5) give the integration by parts formula (in the sense of Definition 2.1) for n–dimensional random vectors, for multi-indices α of length one.*

The assumption of part 2 of Proposition 4.2 may not be easy to check. In the next Proposition we give a statement which is more suitable for applications.

Theorem 4.1 *Let $F : \Omega \longrightarrow \mathbb{R}^n$ be a random vector satisfying the following conditions:*

(a) $F^j \in \mathbb{D}^{2,4}$, for any $j = 1, \ldots, n$,

(b) the Malliavin matrix is inversible, a.s.

Then the law of F has a density with respect to Lebesgue measure on $\mathbb{R}^n$.

Proof: As in the proof of Proposition 4.2, we obtain the system of equations (4.6) for any function $\varphi \in \mathcal{C}_b^\infty$. That is,

$$\partial_i \varphi(F) = \sum_{l=1}^n \langle D(\varphi(F)), (\gamma^{-1})_{i,l} DF^l \rangle_H,$$

$i = 1, \dots, n$, a.s.
We would like to take expectations on both sides of this expression. However, assumption (a) does not ensure the integrability of γ^{-1}. We overcome this problem by localising (4.6), as follows.
For any natural number $N \geq 1$, we define the set

$$C_N = \left\{ \sigma \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) : ||\sigma|| \leq N, |\det \sigma| \geq \frac{1}{N} \right\}.$$

Then we consider a nonnegative function $\psi_N \in \mathcal{C}_0^\infty(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$ satisfying

(i) $\psi_N(\sigma) = 1$, if $\sigma \in C_N$,

(ii) $\psi_N(\sigma) = 0$, if $\sigma \notin C_{N+1}$.

From (4.6), it follows that

$$E\Big(\psi_N(\gamma)\partial_i \varphi(F)\Big) = \sum_{l=1}^n E\Big(\langle D(\varphi(F)), \psi_N(\gamma) DF^l (\gamma^{-1})_{i,l} \rangle_H\Big) \tag{4.7}$$

The random variable $\psi_N(\gamma) DF^l (\gamma^{-1})_{i,l}$ belongs to $\mathbb{D}^{1,2}(H)$, by assumption (a). Consequently $\psi_N(\gamma) DF^l (\gamma^{-1})_{i,l} \in \operatorname{Dom} \delta$ (see Proposition 3.6). Hence, by the duality identity,

$$\begin{aligned} \Big|E\Big(\psi_N(\gamma)\partial_i \varphi(F)\Big)\Big| &= \Big|\sum_{l=1}^n E\Big(\langle D(\varphi(F)), \psi_N(\gamma) DF^l (\gamma^{-1})_{i,l} \rangle_H\Big)\Big| \\ &\leq E\Big(\Big|\sum_{l=1}^n \delta\Big(\psi_N(\gamma) DF^l (\gamma^{-1})_{i,l}\Big)\Big|\Big) ||\varphi||_\infty. \end{aligned}$$

Let P_N be the finite measure on $(\Omega, \mathcal{G})$ absolutely continuous with respect to P with density given by $\psi_N(\gamma)$. Then, by Proposition 2.2, $P_N \circ F^{-1}$ is absolutely continuous with respect to Lebesgue measure. Therefore, for any $B \in \mathcal{B}(\mathbb{R}^n)$ with Lebesgue measure equal to zero, we have

$$\int_{F^{-1}(B)} \psi_N(\gamma) dP = 0.$$

Let $N \to \infty$. Assumption (b) implies that $\lim_{N\to\infty} \psi_N(\gamma) = 1$. Hence, by bounded convergence, we obtain $P(F^{-1}(B)) = 0$. This finishes the proof of the Proposition.

□

Remark 4.5 *The existence of density for the probability law of a random vector F can be obtained under weaker assumptions than in Theorem 4.1 (or Proposition 4.2). Indeed, Bouleau and Hirsch proved a better result using other techniques in the more general setting of Dirichlet forms. For the sake of completenes we give one of their statements, the most similar to Theorem 4.1, and refer the reader to [8] for complete information.*

Proposition 4.3 *Let $F : \Omega \longrightarrow \mathbb{R}^n$ be a random vector satisfying the following conditions:*

(a) $F^j \in \mathbb{D}^{1,2}$, for any $j = 1, \ldots, n$,

(b) the Malliavin matrix is inversible, a.s.

Then the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^n$.

4.2 Smoothness of the density

As we have seen in the first lecture, in order to obtain regularity properties of the density, we need an integration by parts formula for multi-indices of order greater than one. In practice, this can be obtained recursively. In the next proposition we give the details of such a procedure.

An integration by parts formula

Proposition 4.4 *Let $F : \Omega \longrightarrow \mathbb{R}^n$ be a random vector such that $F^j \in \mathbb{D}^\infty$ for any $j = 1, \ldots, n$. Assume that*

$$\det \gamma^{-1} \in \cap_{p\in[1,\infty)} L^p(\Omega). \tag{4.8}$$

Then:

(1) $\det \gamma^{-1} \in \mathbb{D}^\infty$ *and* $\gamma^{-1} \in \mathbb{D}^\infty(\mathbb{R}^m \times \mathbb{R}^m)$.

(2) Let $G \in \mathbb{D}^\infty$. *For any multi-index* $\alpha \in \{1, \ldots, n\}^r$, $r \geq 1$, *there exists a random variable* $H_\alpha(F,G) \in \mathbb{D}^\infty$ *such that for any function* $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,

$$E\Big((\delta_\alpha \varphi)(F) G\Big) = E\Big(\varphi(F) H_\alpha(F,G)\Big). \tag{4.9}$$

The random variables $H_\alpha(F,G)$ *can be defined recursively as follows:*

If $|\alpha| = 1$, $\alpha = i$, *then*

$$H_i(F,G) = \sum_{l=1}^n \delta(G(\gamma^{-1})_{i,l} DF^l), \tag{4.10}$$

and in general, for $\alpha = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r)$,

$$H_\alpha(F,G) = H_{\alpha_r}(F, H_{(\alpha_1,\ldots,\alpha_{r-1})}(F,G)). \tag{4.11}$$

Proof: Consider the sequence of random variables $\Big(Y_N = (\det \gamma + \frac{1}{N})^{-1}, N \geq 1\Big)$. Fix an arbitrary $p \in [1, \infty[$. Assumption (4.8) clearly yields

$$\lim_{N\to\infty} Y_N = \det \gamma^{-1}$$

in $L^p(\Omega)$.
We now prove the following facts:

(a) $Y_N \in \mathbb{D}^\infty$, for any $N \geq 1$,

(b) $(D^k Y_N, N \geq 1)$ is a Cauchy sequence in $L^p(\Omega; H^{\otimes k})$, for any natural number k.

Since the operator D^k is closed, the claim (1) will follow.
Consider the function $\varphi_N(x) = (x + \frac{1}{N})^{-1}$, $x \geq 0$. Notice that $\varphi_N \in \mathcal{C}_b^\infty$. Then Remark 3.3 yields recursively (a). Indeed, $\det \gamma \in \mathbb{D}^\infty$.
Let us now prove (b). The sequence of derivatives $\Big(\varphi_N^{(n)}(\det \gamma), N \geq 1\Big)$ is Cauchy in $L^p(\Omega)$, for any $p \in [1, \infty)$. This can be proved using (4.8) and bounded convergence. The result now follows by expressing the difference $D^k Y_N - D^k Y_M$, $N, M \geq 1$, by means of Leibniz's rule (see (3.29)) and using that $\det \gamma \in \mathbb{D}^\infty$.

Once we have proved that $\det \gamma^{-1} \in \mathbb{D}^\infty$, we trivially obtain $\gamma^{-1} \in \mathbb{D}^\infty(\mathbb{R}^m \times \mathbb{R}^m)$, by a direct computation of the inverse of a matrix and using that $F^j \in \mathbb{D}^\infty$.
The proof of (4.9)–(4.11) is done by induction on the order r of the multi-index α. Let $r = 1$. Consider the identity (4.6), multiply both sides by G and take expectations. We obtain (4.9) and (4.10).
Assume that (4.9) holds for multi-indices of order $r-1$. Fix $\alpha = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r)$. Then,

$$\begin{aligned}
E\Big((\partial_\alpha \varphi)(F)G\Big) &= E\Big(\partial_{(\alpha_1,\ldots,\alpha_{r-1})}((\partial_{\alpha_r}\varphi)(F))G\Big) \\
&= E\Big((\partial_{\alpha_r}\varphi)(F)H_{(\alpha_1,\ldots,\alpha_{r-1})}(F,G)\Big) \\
&= E\Big(\varphi(F)H_{\alpha_r}(F, H_{(\alpha_1,\ldots,\alpha_{r-1})}(F,G)\Big).
\end{aligned}$$

The proof is complete.

□

A criterion for smooth densities

As a consequence of the preceding proposition and part 2 of Proposition 2.1 we have a criterion on smoothness of density, as follows.

Theorem 4.2 *Let $F : \Omega \longrightarrow \mathbb{R}^n$ be a random vector satisfying the assumptions*

(a) $F^j \in \mathbb{D}^\infty$, for any $j = 1, \ldots, n$,

(b) the Malliavin matrix γ is invertible a.s. and

$$\det \gamma^{-1} \in \cap_{p \in [1,\infty)} L^p(\Omega).$$

Then the law of F has an infinitely differentiable density with respect to Lebesgue measure on $\mathbb{R}^n$.

5 Watanabe-Sobolev Differentiability of SPDEs

5.1 A class of linear homogeneous SPDEs

Let L be a second order differential operator acting on real functions defined on $[0,\infty[\times\mathbb{R}^d$. Examples of L where the results of this lecture can be applied gather the heat operator and the wave operator. With some minor modifications, the damped wave operator and some class of parabolic operators with time and space dependent coefficients could also be covered. We are interested in SPDEs of the following type

$$Lu(t,x) = \sigma\left(u(t,x)\right)\dot{W}(t,x) + b\left(u(t,x)\right), \tag{5.1}$$

$t\in]0,T]$, $x\in\mathbb{R}^d$, with suitable initial conditions. This is a Cauchy problem, with finite time horizon $T>0$, driven by the differential operator L, and with a stochastic input given by $W(t,x)$. For the sake of simplicity we shall assume that the initial conditions vanish.

Hypotheses on W

We assume that $(W(\varphi),\varphi\in\mathcal{D}(\mathbb{R}^{d+1}))$ is a Gaussian process, zero mean, and non-degenerate covariance function given by $E\left(W(\varphi_1)W(\varphi_2)\right) = J(\varphi_1,\varphi_2)$, where the functional J is defined in (3.15). By setting $\mu=\mathcal{F}^{-1}\Gamma$, the covariance can be written as

$$E\left(W(\varphi_1)W(\varphi_2)\right) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d}\mu(d\xi)\mathcal{F}\varphi_1(s)(\xi)\overline{\mathcal{F}\varphi_2(s)(\xi)},$$

(see (3.16)).
From this process, we obtain the Gaussian family $(W(h),h\in\mathcal{H}_T)$ (see Section 3.2).

A cylindrical Wiener process derived from $(W(h),h\in\mathcal{H}_T)$

The process $(W_t,t\in[0,T])$ defined by

$$W_t = \sum_{j=1}^{\infty} e_j\beta_j(t),$$

where $(e_j,j\geq 1)$ is a CONS of $\mathcal{H}$ and $\beta_j, j\geq 1$, a sequence of independent standard Wiener processes, defines a cylindrical Wiener process on $\mathcal{H}$ (see

[18], Proposition 4.11, page 96, for a definition of this object). In particular, $W_t(g) := \langle W_t, g\rangle_{\mathcal{H}}$ satisfies

$$E\left(W_t(g_1)W_s(g_2)\right) = (s\wedge t)\langle g_1, g_2\rangle_{\mathcal{H}}.$$

The relationship between $(W_t, t\in[0,T])$ and $(W(h), h\in\mathcal{H}_T)$ can be established as follows. Consider $h\in\mathcal{H}_T$ of the particular type $h = 1_{[0,t]}g$, $g\in\mathcal{H}$. Then, the respective laws of the stochastic processes $W(1_{[0,t]}g), t\in[0,T]$ and $\langle W_t, g\rangle_{\mathcal{H}}, t\in[0,T]$ are the same.
Indeed, by linearity,

$$W(1_{[0,t]}g) = \sum_{j=1}^{\infty}\langle g, e_j\rangle_{\mathcal{H}} W(1_{[0,t]}e_j),$$

By the definition of $(W(h), h\in\mathcal{H}_T)$, the family $(W(1_{[0,t]}e_j), t\in[0,T], j\geq 1)$ is a sequence of independent standard Brownian motions.
On the other hand,

$$\langle W_t, g\rangle_{\mathcal{H}} = \sum_{j=1}^{\infty}\langle g, e_j\rangle_{\mathcal{H}}\beta_j(t).$$

This finishes the proof of the statement.
In connection with the process $(W_t, t\in[0,T])$, we consider the filtration $(\mathcal{F}_t, t\in[0,T])$, where $\mathcal{F}_t$ is the σ-field generated by the random variables $W_s(g), 0\leq s\leq t, g\in\mathcal{H}$. It will be termed the *natural filtration* associated with W.

Hypotheses on L

We shall denote by Λ the fundamental solution of $Lu = 0$, and we shall assume
(H_L) Λ is a deterministic function of t taking values in the space of non-negative measures with rapid decrease (as a distribution), satisfying

$$\int_0^T dt\int_{\mathbb{R}^d}\mu(d\xi)\left|\mathcal{F}\Lambda(t)(\xi)\right|^2 < \infty, \tag{5.2}$$

and

$$\sup_{t\in[0,T]}\Lambda(t)(\mathbb{R}^d) < \infty. \tag{5.3}$$

Examples

1 Heat operator: $L = \partial_t - \Delta_d$, $d \geq 1$.

The fundamental solution of this operator possesses the following property: for any $t \geq 0$, $\xi \in \mathbb{R}^d$,

$$C_1 \frac{t}{1+|\xi|^2} \leq \int_0^t ds |\mathcal{F}\Lambda(s)(\xi)|^2 \leq C_2 \frac{t+1}{1+|\xi|^2}, \tag{5.4}$$

for some positive constants C_i, $i = 1, 2$.

Consequently (5.2) holds if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{5.5}$$

Let us give the proof of (5.4). $\Lambda(t)$ is a function given by

$$\Lambda(t,x) = (2\pi t)^{-\frac{d}{2}} \exp\Big(-\frac{|x|^2}{2t}\Big).$$

Its Fourier transform is

$$\mathcal{F}\Lambda(t)(\xi) = \exp(-2\pi^2 t|\xi|^2).$$

Hence,

$$\int_0^t dt |\mathcal{F}\Lambda(t)(\xi)|^2 = \frac{1-\exp(-4\pi^2 t|\xi|^2}{4\pi^2|\xi|^2}.$$

On the set $(|\xi| > 1)$, we have

$$\frac{1-\exp(-4\pi^2 t|\xi|^2)}{4\pi^2|\xi|^2} \leq \frac{1}{2\pi^2|\xi|^2} \leq \frac{C}{1+|\xi|^2}.$$

On the other hand, on $(|\xi| \leq 1)$, we use the property $1 - e^{-x} \leq x$, $x \geq 0$, and we obtain

$$\frac{1-\exp(-4\pi^2 t|\xi|^2)}{4\pi^2|\xi|^2} \leq \frac{Ct}{1+|\xi|^2}.$$

This yields the upper bound in (5.4).

Moreover, the inequality $1 - e^{-x} \geq \frac{x}{1+x}$, valid for any $x \geq 0$, implies

$$\int_0^t ds |\mathcal{F}\Lambda(t)(\xi)|^2 \geq C \frac{t}{1+4\pi^2 t|\xi|^2}.$$

Assume that $4\pi^2 t|\xi|^2 \geq 1$. Then $1 + 4\pi^2 t|\xi|^2 \leq 8\pi^2 t|\xi|^2$; if $4\pi^2 t|\xi|^2 \leq 1$ then $1 + 4\pi^2 t|\xi|^2 < 2$ and therefore, $\frac{1}{1+4\pi^2 t|\xi|^2} \geq \frac{1}{2(1+|\xi|^2)}$. Hence, we obtain the lower bound in (5.4) and now the equivalence between (5.2) and (5.5) is obvious.
Condition (5.3) is clearly satisfied.

2 Wave operator: $L = \partial^2_{tt} - \Delta_d$, $d \geq 1$.
For any $t \geq 0$, $\xi \in \mathbb{R}^d$, it holds that

$$c_1(t \wedge t^3)\frac{1}{1+|\xi|^2} \leq \int_0^t ds |\mathcal{F}\Lambda(s)(\xi)|^2 \leq c_2(t+t^3)\frac{1}{1+|\xi|^2}, \tag{5.6}$$

for some positive constants c_i, $i = 1, 2$. Thus, (5.2) is equivalent to (5.5).
Let us prove (5.6). It is well known (see for instance [75]) that

$$\mathcal{F}\Lambda(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

Therefore

$$\begin{aligned}
|\mathcal{F}\Lambda(t)(\xi)|^2 &\leq \frac{1}{2\pi^2(1+|\xi|^2)} 1_{(|\xi|\geq 1)} + t^2 1_{(|\xi|\leq 1)} \\
&\leq C\frac{1+t^2}{1+|\xi|^2}.
\end{aligned}$$

This yields the upper bound in (5.6).
Assume that $2\pi t|\xi| \geq 1$. Then $\frac{\sin(4\pi t|\xi|)}{2t|\xi|} \leq \pi$ and consequently,

$$\begin{aligned}
\int_0^t ds \frac{\sin^2(2\pi s|\xi|)}{(2\pi|\xi|)^2} &\geq C\frac{t}{1+|\xi|^2}\int_0^{2\pi t} \sin^2(u|\xi|)du \\
&= C\frac{t}{1+|\xi|^2}\left(2\pi - \frac{\sin(4\pi t|\xi|)}{2t|\xi|}\right) \\
&\geq C\frac{t}{1+|\xi|^2}.
\end{aligned}$$

Next we assume that $2\pi t|\xi| \leq 1$ and we notice that for $r \in [0,1]$, $\frac{\sin^2 r}{r^2} \geq \sin^2 1$. Thus,

$$\begin{aligned}
\int_0^t ds \frac{\sin^2(2\pi s|\xi|)}{(2\pi|\xi|)^2} &\geq C \sin^2 1 \int_0^{2\pi t} s^2 ds \\
&\geq C\frac{t^3}{1+|\xi|^2}.
\end{aligned}$$

This finishes the proof of the lower bound in (5.6).

For $d \le 3$, condition (5.3) holds true. In fact,

$$\Lambda(t,dx) = \begin{cases} \frac{1}{2} 1_{\{|x|<t\}} dx, & d=1, \\ \frac{1}{2\pi}(t^2 - |x|^2)^{-\frac{1}{2}} 1_{\{|x|<t\}} dx, & d=2, \\ \frac{\sigma_t(dx)}{4\pi t}, & d=3, \end{cases}$$

where σ_t stands for the uniform surface measure on the sphere centered at zero and with radius t. Easy computations show that $\Lambda(t)(\mathbb{R}^d) = 1$ in each case.

Dimension $d = 3$ is a threshold value. Indeed, for higher dimensions Λ is not in the class of non-negative measures and therefore the results of this lecture do not apply.

Mild formulation of the SPDE

By a solution of (5.1) we mean a *real-valued* stochastic process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, predictable with respect to the filtration $(\mathcal{F}_t, t \in [0,T])$, such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\left(|u(t,x)|^2\right) < \infty$$

and

$$\begin{aligned} u(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma(u(s,y))W(ds,dy) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} b(u(t-s,x-y))\Lambda(s,dy). \end{aligned} \tag{5.7}$$

Notice that the *pathwise integral* is the integral of a convolution with the measure $\Lambda(s)$:

$$\int_0^t \int_{\mathbb{R}^d} b(u(t-s,x-y))\Lambda(s,dy) = \int_0^t \left[b(u(s,\cdot)) * \Lambda(t-s)\right] ds.$$

As for the *stochastic integral*, it is a *stochastic convolution*. For the construction of this object, we refer the reader to [18]. From this reference, we see that in order to give a meaning to the stochastic convolution $\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma(u(s,y))W(ds,dy)$, the process

$$z(s,y) := \Lambda(t-s,x-y)\sigma(u(s,y)),\ s \in [0,t],\ y \in \mathbb{R}^d,$$

$t \in]0,T]$, $x \in \mathbb{R}^d$, is required to be predictable and to belong to the space $L^2(\Omega \times [0,T]; \mathcal{H})$.
We address this question following the approach of [53] with a few changes, in particular we allow more general covariances Γ (see Lemma 3.2 and Proposition 3.3 in [53]).

Lemma 5.1 *Assume that Λ satisfies (H_L), then $\Lambda \in \mathcal{H}_T$ and*

$$\|\Lambda\|^2_{\mathcal{H}_T} = \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda(t)(\xi)|^2.$$

Proof: Let ψ be a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^d)$ with support contained in the unit ball and such that $\int_{\mathbb{R}^d} \psi(x)dx = 1$. Set $\psi_n(x) = n^d \psi(nx)$, $n \geq 1$. Define $\Lambda_n(t) = \psi_n * \Lambda(t)$. It is well known that $\psi_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ and $\Lambda_n(t) \in \mathcal{S}(\mathbb{R}^d)$. Moreover, for any $\xi \in \mathbb{R}^d$, $|\mathcal{F}\Lambda_n(t)(\xi)| \leq |\mathcal{F}\Lambda(t)(\xi)|$.
By virtue of (5.2), $(\Lambda_n, n \geq 1) \subset \mathcal{H}_T$, and it is Cauchy sequence. Indeed,

$$\|\Lambda_n - \Lambda_m\|_{\mathcal{H}_T} = \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda(t)(\xi)|^2 |\mathcal{F}(\psi_n(\xi) - \psi_m(t))|^2$$

and since $\mathcal{F}(\psi_n(\xi) - \psi_m(\xi))$ converges pointwise to zero as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \|\Lambda_n - \Lambda_m\|_{\mathcal{H}_T} = 0,$$

by bounded convergence. Consequently, $(\Lambda_n, n \geq 1)$ converges in $\mathcal{H}_T$ and the limit is Λ. Finally, by using again bounded convergence,

$$\|\Lambda\|^2_{\mathcal{H}_T} = \lim_{n\to\infty} \|\Lambda_n\|^2_{\mathcal{H}_T} = \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda(t)(\xi)|^2.$$

$\square$

The next Proposition gives a large class of examples for which the stochastic convolution against W can be defined.

Proposition 5.1 *Assume that Λ satisfies (H_L). Let $Z = \{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ be a predictable process, bounded in L^2. Set*

$$C_Z := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\left(|Z(t,x)|^2\right).$$

Then, $z(t,dx) := Z(t,x)\Lambda(t,dx)$ *is a predictable process belonging to* $L^2(\Omega \times [0,T];\mathcal{H})$ *and*

$$E\left(\|z\|^2_{\mathcal{H}_T}\right) \leq C_Z \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\Lambda(t)(\xi)|^2.$$

Hence, the stochastic integral $\int_0^T \int_{\mathbb{R}^d} zdW$ *is well-defined as an* $L^2(\Omega)$*-valued random variable and*

$$\begin{aligned}\left\|\int_0^T \int_{\mathbb{R}^d} zdW\right\|^2_{L^2(\Omega)} &= E\left(\|z\|^2_{\mathcal{H}_T}\right)\\ &\leq C_Z \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\Lambda(t)(\xi)|^2. \qquad (5.8)\end{aligned}$$

Proof: By decomposing the process Z into its positive and negative part, it suffices to consider non-negative processes Z. Since $\Lambda(t)$ is a tempered measure, so is $z(t)$. Hence we can consider the sequence of $\mathcal{S}(\mathbb{R}^d)$-valued functions $z_n(t) = \psi_n * z(t)$, $n \geq 1$, where ψ_n is the approximation of the identity defined in the proof of the preceding lemma.
Using Fubini's theorem and the boundedness property of Z we obtain

$$\begin{aligned}E(\|z_n\|^2_{\mathcal{H}_T}) &= E\left(\int_0^T dt \int_{\mathbb{R}^d} \Gamma(dx)\left[z_n(t) * \tilde{z}_n(t)\right](x)\right)\\ &\leq C_Z \int_0^T dt \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dz \Lambda_n(t,d(-z))\Lambda_n(t,d(x-z))\\ &= C_Z \int_0^T dt \int_{\mathbb{R}^d} \Gamma(dx)\left[\Lambda_n(t) * \tilde{\Lambda}_n(t)\right](x)\\ &= C_Z \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda_n(t)(\xi)|^2\\ &\leq C_Z \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(t)(\xi)|^2 < \infty.\end{aligned}$$

Hence $\sup_{n\geq 1} E(\|z_n\|^2_{\mathcal{H}_T}) < \infty$.
Moreover, assume that for any $n, m \geq 1$ we can prove the following identity:

$$E(\|z_n - z_m\|^2_{\mathcal{H}_T}) = E\left(\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}(Z(t)\Lambda(t))(\xi)|^2|\mathcal{F}(\psi_n(\xi) - \psi_m(\xi))|^2\right). \qquad (5.9)$$

then, using similar arguments as in the preceding lemma, we can prove that $(z_n, n \geq 1)$ converges in $L^2(\Omega \times [0,T]; \mathcal{H})$ to z, finishing the proof of the proposition.
For the proof of (5.9), we proceed as follows. Firstly, to simplify the expresions, we write $\tilde{z}_{n,m}$ instead of $z_n - z_m$, and $\tilde{\psi}_{n,m}$ for $\psi_n - \psi_m$.

$$
\begin{aligned}
E(\|\tilde{z}_{n,m}\|^2_{\mathcal{H}_T}) &= E\left(\int_0^T dt \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy \tilde{z}_{n,m}(t, x-y)\tilde{z}_{n,m}(t,-y)\right) \\
&= E\Big(\int_0^T dt \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy \left(\int_{\mathbb{R}^d} dz' \tilde{\psi}_{n,m}(x-y-z') z(t,z')\right) \\
&\quad \times \left(\int_{\mathbb{R}^d} dz'' \tilde{\psi}_{n,m}(-y-z'') z(t,z'')\right)\Big) \\
&= E\Big(\int_0^T dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Z(t,z')Z(t,z'')\Lambda(t,dz')\Lambda(t,dz'') \\
&\quad \times \int_{\mathbb{R}^d} \Gamma(dx)[\tilde{\psi}_{n,m}(.-z') * \tilde{\psi}_{n,m}(.+z'')]\Big) \\
&= E\Big(\int_0^T dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Z(t,z')Z(t,z'')\Lambda(t,dz')\Lambda(t,dz'') \\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi) e^{-2\pi i \xi \cdot (z'-z'')} |\mathcal{F}\tilde{\psi}_{n,m}(\xi)|^2\Big).
\end{aligned} \tag{5.10}
$$

Then, since the Fourier transform of a convolution is the product of the Fourier transform of the corresponding factors, using Fubini's theorem this last expression is equal to

$$
E\left(\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}(Z(t)\Lambda(t))(\xi)|^2 |\mathcal{F}\tilde{\psi}_{n,m}(\xi)|^2\right).
$$

Hence (5.9) is established.
Finally, (5.8) is obtained by the isometry property of the stochastic convolution, combined with the estimate of the integrand proved before.

□

Remark 5.1 *Assume that the process Z is bounded away from zero, that it $\inf_{(t,x)\in[0,T]\times\mathbb{R}^d} |Z(t,x)| \geq c_0 > 0$. Then*

$$
E(\|z_n\|^2_{\mathcal{H}_T}) \geq c_0^2 \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda(t)(\xi)|^2 |\mathcal{F}\psi_n(\xi)|^2. \tag{5.11}
$$

Indeed, arguing as in (5.10), we see that

$$\begin{aligned}
E(\|z_n\|^2_{\mathcal{H}_T}) &= E\Big(\int_0^T dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Z(t,z')Z(t,z'')\Lambda(t,dz')\Lambda(t,dz'') \\
&\quad \times \int_{\mathbb{R}^d}\Gamma(dx)[\psi_n(.-z')*\psi_n(.+z'']\Big) \\
&\ge c_0^2\int_0^T dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Lambda(t,dz')\Lambda(t,dz'') \\
&\quad \times \int_{\mathbb{R}^d}\Gamma(dx)[\psi_n(.-z')*\psi_n(.+z''] \\
&= c_0^2\int_0^T dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Lambda(t,dz')\Lambda(t,dz'') \\
&\quad \times \int_{\mathbb{R}^d}\mu(d\xi)e^{-2\pi i\xi\cdot(z'-z'')}|\mathcal{F}\psi_n(\xi)|^2 \\
&= c_0^2\int_0^T dt \int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Lambda(t)(\xi)|^2|\mathcal{F}\psi_n(\xi)|^2.
\end{aligned}$$

Remark 5.2 *Assume that the coefficient σ in (5.7) has linear growth and that the process u satisfies the conditions given at the beginning of the section. Then $Z(s,y) := \sigma(u(s,y))$ satisfies the assumption of Proposition 5.1 and the stochastic integral (stochastic convolution) in (5.7) is well-defined.*

A result on existence and uniqueness of solution

Theorem 5.1 *Assume that $\sigma, b : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions and that Λ satisfies (H_L). Then there exists a unique mild solution to Equation (5.1). Such a solution is a random field indexed by $(t,x) \in [0,T] \times \mathbb{R}^d$, continuous in $L^2(\Omega)$, and for any $p \in [1,\infty[$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\left(|u(t,x)|^p\right) < \infty. \tag{5.12}$$

The proof of this theorem can be done using Picard's iteration scheme defined as follows:

$$\begin{aligned}
u^0(t,x) &= 0, \\
u^n(t,x) &= \int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-y)\sigma(u^{n-1}(s,y))W(ds,dy) \\
&\quad + \int_0^t\int_{\mathbb{R}^d} b(u^{n-1}(t-s,x-y))\Lambda(s,dy),
\end{aligned} \tag{5.13}$$

for $n \geq 1$. We refer the reader to Theorem 13 of [16] for the details of the proof of the convergence of the Picard sequence and the extensions of Gronwall's lemma suitable thereof (see also Theorem 6.2 and Lemma 6.2 in [69]).

5.2 The Malliavin derivative of a SPDE

Consider a SPDE in its mild formulation (see 5.7). We would like to study its differentiability in the Watanabe-Sobolev sense. There are two aspects of the problem:

(A) to prove differentiability,

(B) to give an equation satisfied by the Malliavin derivative.

A useful tool to prove differentiability of Wiener functionals is provided by the next result, which is an immediate consequence from the fact that the N–th order Malliavin derivative is a closed operator defined on $L^p(\Omega)$ with values in $L^p(\Omega; H^{\otimes N})$, for any $p \in [1, \infty[$. In our context $H := \mathcal{H}_T$. A result of the same vein has been presented in Proposition 3.3.

Lemma 5.2 *Let $(F_n, n \geq 1)$ be a sequence of random variables belonging to $\mathbb{D}^{N,p}$. Assume that:*

(a) there exists a random variable F such that F_n converges to F in $L^p(\Omega)$, as n tends to ∞,

(b) the sequence $(D^N F_n, n \geq 1)$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$, as n tends to ∞,

Then F belongs to $\mathbb{D}^{N,p}$ and $D^N F = L^p(\Omega; \mathcal{H}_T^{\otimes N}) - \lim_{n\to\infty} D^N F_n$.

We shall apply this lemma to $F := u(t, x)$, the solution of Equation (5.7). Therefore, we have to find out an approximating sequence of the SPDE satisfying the assumptions (a) and (b) above. A possible candidate is provided in the proof of the existence and uniqueness of solution: the Picard approximations defined in (5.13). The verification of condition (a) for the sequence $(u^n(t, x), n \geq 0)$, for fixed $(t, x) \in [0, T] \times \mathbb{R}^d$ is part of the proof of Theorem 5.1

As regards condition (b), we will avoid too many technicalities by focussing on the first order derivative and taking $p = 2$. For this we need the functions σ and b to be of class $\mathcal{C}^1$.

A possible strategy might consists in proving recursively that $u^n(t,x)$ belongs to $\mathbb{D}^{1,2}$, then applying rules of Malliavin calculus (for instance, an extension of (3.30)) we will obtain

$$\begin{aligned}
Du^0(t,x) &= 0\\
Du^n(t,x) &= \Lambda(t-\cdot, x-*)\sigma(u^{n-1}(\cdot,*))\\
&\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma'(u^{n-1}(s,y))Du^{n-1}(s,y)W(ds,dy)\\
&\quad + \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)b'(u^{n-1}(t-s,x-y))Du^{n-1}(t-s,x-y),
\end{aligned} \tag{5.14}$$

$n \geq 1$.
A natural candidate for the limit of this sequence is the process satisfying (5.16).
At this point some comments are pertinent:

1. The Malliavin derivative is a random vector with values in $\mathcal{H}_T$. Therefore, Equations (5.14) and (5.16) correspond to the mild formulation of a Hilbert-valued SPDE.

2. The notation $\Lambda(t-\cdot, x-*)\sigma(u^{n-1}(\cdot,*))$ aims to show up the dependence on the time variable (written with a dot) and on the space variable (written with a star). By Proposition 5.1 and Remark 5.2 such a term is in $L^2(\Omega\times[0,T];\mathcal{H})$.

3. The stochastic convolution term in (5.14) is not covered by the previous discussion, since the process $\sigma'(u^{n-1}(s,y))Du^{n-1}(s,y)$ takes values on $\mathcal{H}_T$. A sketch of the required extension is given in the next paragraphs.

Stochastic convolution with Hilbert-valued integrands

Let $\mathcal{K}$ be a separable real Hilbert space with inner-product and norm denoted by $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$, respectively. Let $K=\{K(s,z),(s,z)\in[0,T]\times\mathbb{R}^d\}$ be a $\mathcal{K}$-valued predictable process satisfying

$$C_K := \sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E\left(||K(s,z)||^2_{\mathcal{K}}\right) < \infty.$$

Consider a complete orthonormal system of $\mathcal{K}$, that we denote by $\{e_j, j\geq 0\}$. Set $K^j(s,z)=\langle K(s,z),e_j\rangle_{\mathcal{K}}$, $(s,z)\in[0,T]\times\mathbb{R}^d$. By Proposition

5.1, $z_j(t,dx) = K^j(t,x)\Lambda(t,dx)$ is a predictable process and belongs to $L^2(\Omega\times[0,T];\mathcal{H})$, and then $K(t,x)\Lambda(t,dx)$ is also a predictable process and belongs to $L^2(\Omega\times[0,T];\mathcal{H}\otimes\mathcal{K})$. The $\mathcal{K}$-valued stochastic convolution $\int_0^T\int_{\mathbb{R}^d}\Lambda(t,x)K(t,x)W(dt,dx)$ is defined as

$$\left(\int_0^T\int_{\mathbb{R}^d}\Lambda(t,x)K^j(t,x)W(dt,dx), j\geq 0\right)$$

and satisfies

$$\begin{aligned} E\left(\left\|\int_0^T\int_{\mathbb{R}^d}\Lambda(t,x)K(t,x)W(dt,dx)\right\|_{\mathcal{K}}^2\right) &= E\left(\|\Lambda K\|_{\mathcal{H}\otimes\mathcal{K}}^2\right)\\ &\leq C_K\int_0^T dt\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Lambda(t)(\xi)|^2. \end{aligned} \tag{5.15}$$

Going back to the application of Lemma 5.2, we might guess as limit of the sequence (5.14) a $\mathcal{H}_T$-valued process $(Du(t,x),(t,x)\in[0,T]\times\mathbb{R}^d)$ satisfying the equation

$$\begin{aligned} Du(t,x) &= \Lambda(t-\cdot,x-*)\sigma(u(\cdot,*))\\ &\quad+\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-y)\sigma'(u(s,y))Du(s,y)W(ds,dy)\\ &\quad+\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dy)b'(u(t-s,x-y))Du(t-s,x-y). \end{aligned} \tag{5.16}$$

Yet another result on existence and uniqueness of solution

Theorem 5.1 is not general enough to cover SPDEs like (5.16). In this section we set up a suitable framework for this (actually to deal with Malliavin derivatives of any order). For more details we refer the reader to [69], Chapter 6.

Let $\mathcal{K}_1$, $\mathcal{K}$ be two separable Hilbert spaces. If there is no reason for misunderstanding we will use the same notation, $||\cdot||$, $\langle\cdot,\cdot\rangle$, for the norms and inner products in these two spaces, respectively.

Consider two mappings

$$\sigma, b:\mathcal{K}_1\times\mathcal{K}\longrightarrow\mathcal{K}$$

satisfying the next two conditions for some positive constant C:

(c1)

$$\sup_{x\in\mathcal{K}_1}\Big(||\sigma(x,y)-\sigma(x,y')||+||b(x,y)-b(x,y')||\Big)\leq C||y-y'||,$$

(c2) there exists $q \in [1, \infty)$ such that

$$||\sigma(x,0)|| + ||b(x,0)|| \leq C(1 + ||x||^q),$$

$x \in \mathcal{K}_1$, $y, y' \in \mathcal{K}$.

Notice that (c1) and (c2) clearly imply

(c3) $||\sigma(x,y)|| + ||b(x,y)|| \leq C(1 + ||x||^q + ||y||)$.

Let $V = \Big(V(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\Big)$ be a predictable $\mathcal{K}_1$-valued process such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(||V(t,x)||^p\Big) < \infty, \tag{5.17}$$

for any $p \in [1, \infty)$.
Consider also a predictable $\mathcal{K}$-valued process $U_0 = \Big(U_0(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\Big)$ satisfying the analogue of (5.17).
Set

$$\begin{aligned} U(t,x) = U_0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma\Big(V(s,y), U(s,y)\Big) W(ds,dy) \\ + \int_0^t ds \int_{\mathbb{R}^d} b\Big(V(t-s,x-y), U(t-s,x-y)\Big)\Lambda(s,dy). \end{aligned} \tag{5.18}$$

A solution to Equation (5.18) is a $\mathcal{K}$-valued predictable stochastic process $\Big(U(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\Big)$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(||U(t,x)||^2\Big) < \infty$$

and satisfies the relation (5.18).

Theorem 5.2 *We assume that the coefficients σ and b satisfy the conditions (c1) and (c2) above. Then, Equation (5.18) has a unique solution.*
In addition the solution satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(||U(t,x)||^p\Big) < \infty, \tag{5.19}$$

for any $p \in [1, \infty)$.

Main result

We will now apply Lemma 5.2 to prove that for any fixed $(t,x) \in [0,T] \times \mathbb{R}^d$, $u(t,x) \in \mathbb{D}^{1,2}$. The next results provide a verification of conditions (a) and (b) of the Lemma. We shall assume that the functions σ and b are differentiable with bounded derivatives.

Lemma 5.3 *The sequence of random variables* $\Big(u^n(t,x), n \geq 0\Big)$ *defined recursively in (5.13) is a subset of* $\mathbb{D}^{1,2}$.
In addition,

$$\sup_{n\geq 0} \sup_{(t,x)\in[0,T]\mathbb{R}^d} E\left(\|Du^n(t,x)\|^2_{\mathcal{H}_T}\right) < \infty. \tag{5.20}$$

Proof: It is done by a recursive argument on n. Clearly the property is true for $n = 0$. Assume it holds up to the $(n-1)$-th iteration. By the rules of Malliavin calculus (in particular, Proposition 3.5 and Remark 3.4), the right hand-side of (5.13) belongs to $\mathbb{D}^{1,2}$. Hence $u^n(t,x) \in \mathbb{D}^{1,2}$ and moreover (5.14) holds.
We now prove (5.20). Denote by $B_{i,n}$, $i = 1,2,3$, each one of the terms on the right hand-side of (5.14), respectively. By applying Proposition 5.1 to $Z(t,x) := \sigma(u(t,x))$ along with the linear growth of the function σ, we obtain

$$\begin{aligned} &E\Big(||B_{1,n}||^2_{\mathcal{H}_T}\Big) \\ &\quad \leq C\Big(1 + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u^{n-1}(t,x)|^2)\Big) \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2, \end{aligned}$$

which is uniformly bounded with respect to n (see (ii) in the proof of Theorem 6.2 in [69]).
Set

$$J(t) = \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Lambda(t)(\xi)|^2\,,\ t \geq 0.$$

Consider now the second term $B_{2,n}(t,x)$. By the construction of the stochastic convolution and the properties of σ, we have

$$\begin{aligned} &E(\|B_{2,n}(t,x)\|^2_{\mathcal{H}_T}) \leq C \int_0^t ds \sup_{z\in\mathbb{R}^d} E\Big(||\sigma'(u^{n-1}(s,z))Du^{n-1}(s,z)||^2_{\mathcal{H}_T}\Big) J(t-s) \\ &\leq C \int_0^t ds \sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d} E\Big(||Du^{n-1}(\tau,z)||^2_{\mathcal{H}_T}\Big) J(t-s). \end{aligned}$$

Finally, for the third term $B_{3,n}(t,x)$ we use Schwarz's inequality with respect to the finite measure $\Lambda(s,dz)ds$. Then, the assumptions on b and Λ yield

$$E(\|B_{3,n}(t,x)\|^2_{\mathcal{H}_T} \leq C \int_0^t ds \sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d} E\Big(||Du^{n-1}(\tau,z)||^2_{\mathcal{H}_T}\Big).$$

Therefore,

$$\begin{aligned}
&\sup_{(s,z)\in[0,t]\times\mathbb{R}^d} E\Big(||Du^n(s,z)||^2_{\mathcal{H}_T}\Big) \\
&\leq C\Big(1+\int_0^t ds \sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d} E\Big(||Du^{n-1}(\tau,z)||^2_{\mathcal{H}_T}\Big)(J(t-s)+1)\Big).
\end{aligned}$$

Then, by Gronwall's Lemma (see Lemma 6.2 in [69]), we finish the proof.

□

Lemma 5.4 *Under the standing hypotheses, the sequence $Du^n(t,x)$, $n \geq 0$, converges in $L^2(\Omega;\mathcal{H}_T)$, uniformly in $(t,x)\in[0,T]\times\mathbb{R}^d$, to the $\mathcal{H}_T$-valued stochastic processes $\Big(U(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\Big)$ solution of the equation*

$$\begin{aligned}
&U(t,x) = H(t,x) \\
&+\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-z)U(s,z)\sigma'(u(s,z))W(ds,dz) \\
&+\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dz)U(t-s,x-z)b'(u(t-s,x-z)), \qquad (5.21)
\end{aligned}$$

with $H(t,x)=\sigma(u(\cdot,*))\Lambda(t-\cdot,x-*)$.

Proof : We must prove

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\Big\|Du^n(t,x)-U(t,x)\Big\|^2_{\mathcal{H}_T}\Big)\to 0, \qquad (5.22)$$

as n tends to infinity.

Set

$$I_Z^{n,N}(t,x) = \Lambda(t-\cdot, x-*)\Big(\sigma(u^{n-1}(\cdot,*)) - \sigma(u(\cdot,*))\Big),$$

$$I_\sigma^n(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u^{n-1}(s,z))Du^{n-1}(s,z)W(ds,dz)$$
$$- \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u(s,z))U(s,z)W(ds,dz),$$

$$I_b^n(t,x) = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)\Big(b'(u^{n-1}(t-s,x-z))Du^{n-1}(t-s,x-z)$$
$$- b'(u(t-s,x-z))U(t-s,x-z)\Big).$$

The Lipschitz property of σ yields

$$E(||I_Z^{n,N}(t,x)||^2_{\mathcal{H}_T}) \le C \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u^{n-1}(t,x)-u(t,x)|^2) \int_0^t ds$$
$$\times \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2$$
$$\le C \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u^{n-1}(t,x)-u(t,x)|^2).$$

Hence,

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(||I_Z^{n,N}(t,x)||^2_{\mathcal{H}_T}) = 0. \tag{5.23}$$

Consider the decomposition

$$E(\|I_\sigma^n(t,x)\|^2_{\mathcal{H}_T}) \le C(D_{1,n}(t,x) + D_{2,n}(t,x),$$

where

$$D_{1,n}(t,x) = E\Big(\| \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)[\sigma'(u^{n-1}(s,z))$$
$$- \sigma'(u(s,z))]Du^{n-1}(s,z)W(ds,dz)\|^2_{\mathcal{H}_T}\Big),$$

$$D_{2,n}(t,x) = E\Big(\| \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u(s,z))[Du^{n-1}(s,z)$$
$$- U(s,z)]W(ds,dz)\|^2_{\mathcal{H}_T}\Big).$$

The isometry property of the stochastic integral, Cauchy-Schwarz's inequality and the properties of σ yield

$$D_{1,n}(t,x) \le C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \Big(E(|u^{n-1}(s,y)-u(s,y)|^4)E(\|Du^{n-1}(s,y)\|^4_{\mathcal{H}_T})\Big)^{\frac{1}{2}}$$
$$\times \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2.$$

Owing to and Lemma 5.3 we conclude that

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} D_{1,n}(t,x) = 0.$$

Similarly,

$$D_{2,n}(t,x) \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(\|Du^{n-1}(\tau,y) - U(\tau,y)\|^2_{\mathcal{H}_T}) J(t-s). \tag{5.24}$$

For the pathwise integral term, we have

$$E(\|I_b^n(t,x)\|^2_{\mathcal{H}_T}) \le C(b_{1,n}(t,x) + b_{2,n}(t,x)),$$

with

$$\begin{aligned}
&b_{1,n}(t,x) = E\Big(|| \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)[b'(u^{n-1}(t-s,x-z)) - b'(u(t-s,x-z))] \\
&\quad \times Du^{n-1}(t-s,x-z)||^2_{\mathcal{H}_T}\Big), \\
&b_{2,n}(t,x) = E\Big(\| \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) b'(u(t-s,x-z)) \\
&\quad \times [Du^{n-1}(t-s,x-z) - U(t-s,x-z)]\|^2_{\mathcal{H}_T}\Big).
\end{aligned}$$

By the properties of the deterministic integral of Hilbert-valued processes, the assumptions on b and Cauchy-Schwarz's inequality we obtain

$$\begin{aligned}
&b_{1,n}(t,x) \le \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) E\Big(|b'(u^{n-1}(t-s,x-z)) - b'(u(t-s,x-z))|^2 \\
&\quad \times \|Du^{n-1}(t-s,x-z)\|^2_{\mathcal{H}_T}\Big) \\
&\le \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \Big(E|u^{n-1}(s,y) - u(s,y)|^4 \, E\|Du^{n-1}(s,y)\|^4_{\mathcal{H}_T}\Big)^{1/2} \int_0^t ds \Lambda(s,dz).
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} b_{1,n}(t,x) = 0.$$

Similar arguments yield

$$b_{2,n}(t,x) \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(\|Du^{n-1}(\tau,y) - U(\tau,y)\|^2_{\mathcal{H}_T}).$$

Therefore we have obtained that

$$\begin{aligned}
&\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E(||Du^n(s,x)-U(s,x)||^2_{\mathcal{H}_T})\\
&\leq C_n + C\int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E(||Du^{n-1}(\tau,x)-U(\tau,x)||^2_{\mathcal{H}_T})(J(t-s)+1),
\end{aligned}$$

with $\lim_{n\to\infty} C_n = 0$. Thus applying a version of Gronwall's lemma (see Lemma 6.2 in [69]) we complete the proof of (5.22).

□

6 Analysis of Non-Degeneracy

In comparison with SDEs, the application of the criteria for existence and smoothness of density for Gaussian functionals (see for instance Proposition 4.3) and Theorem 4.2) to SPDEs is not a well developed topic. Most of the results for SPDEs are proved under *ellipticity* conditions. In this lecture, we shall discuss the non-degeneracy of the Malliavin matrix for the class of SPDEs studied in the preceding lecture, in a very simple situation: in dimension one and assuming ellipticity.

6.1 Existence of moments of the Malliavin covariance

Throughout this section, we fix $(t,x) \in]0,T] \times \mathbb{R}^d$ and consider the random variable $u(t,x)$ obtained as a solution of (5.7). Hence we are in the framework of Section 5 and therefore, we are assuming in particular that Λ satisfies hypotheses (H_L).

Following Definition 4.1, the *Malliavin matrix* is the random variable $\|Du(t,x)\|_{\mathcal{H}_T}$. In this section, we want to study the property

$$E\left(\|Du(t,x)\|_{\mathcal{H}_T}^{-p}\right) < \infty, \tag{6.1}$$

for some $p \in]0,\infty[$.

A reason for this is to apply Proposition 4.3 and to deduce the existence of density for the law of $u(t,x)$. We have already proved in the preceding lecture that $u(t,x) \in \mathbb{D}^{1,2}$. Hence, it remains to check that $\|Du(t,x)\|_{\mathcal{H}_T} > 0$, a.s. Clearly, having (6.1) for some $p > 0$ is a sufficient condition for this property to hold.

The classical connection between moments and distribution functions

Lemma 6.1 *Fix $p \in]0,\infty]$. The property (6.1) holds if and only if there exists $\epsilon_0 > 0$, depending on p, such that*

$$\int_0^{\epsilon_0} \epsilon^{-(1+p)} P(||Du(t,x)||_{\mathcal{H}_T}^2 < \epsilon) d\epsilon < \infty. \tag{6.2}$$

Proof: It is well known that for any positive random variable Y,

$$E(Y) = \int_0^\infty P(Y > \eta) d\eta.$$

In fact, this follows easily from Fubini's theorem.
Apply this formula to $Y := ||Du(t,x)||_{\mathcal{H}_T}^{-2p}$. We obtain

$$E(||Du(t,x)||_{\mathcal{H}_T}^{-2p}) = m_1 + m_2,$$

with

$$\begin{aligned}
m_1 &= \int_0^{\eta_0} P(||Du(t,x)||_{\mathcal{H}_T}^{-2p} > \eta)d\eta, \\
m_2 &= \int_{\eta_0}^{\infty} P(||Du(t,x)||_{\mathcal{H}_T}^{-2p} > \eta)d\eta.
\end{aligned}$$

Clearly, $m_1 \leq \eta_0$. The change of variable $\eta = \epsilon^{-p}$ implies

$$\begin{aligned}
m_2 &= \int_{\eta_0}^{\infty} P(||Du(t,x)||_{\mathcal{H}_T}^{-2p} > \eta)d\eta \\
&= \int_{\eta_0}^{\infty} P(||Du(t,x)||_{\mathcal{H}_T}^{2} < \eta^{-\frac{1}{p}})d\eta \\
&= p\int_0^{\eta_0^{-\frac{1}{p}}} \epsilon^{-(1+p)} P(||Du(t,x)||_{\mathcal{H}_T}^{2} < \epsilon)d\epsilon.
\end{aligned}$$

This finishes the proof.

□

Moments of low order

Knowing the size in ϵ of the term $P(||Du(t,x)||_{\mathcal{H}_T}^{2} < \epsilon)$ will help us to verify the integrability of $\epsilon^{-(1+p)}P(||Du(t,x)||_{\mathcal{H}_T}^{2} < \epsilon)$ at zero, and *a posteriori* to establish the validity of (6.1). The next proposition gives a result in this direction.

Proposition 6.1 *We assume that*

(1) there exists $\sigma_0 > 0$ *such that* $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$,

(2) there exist θ *such that for any* $t \in (0,1)$,

$$C_1 t^{\theta} \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2, \tag{6.3}$$

Then for any $\epsilon \in]0,1[$,

$$P\left(\|Du(t,x)\|^2_{\mathcal{H}_T} < \epsilon\right) \le C\epsilon^{1\wedge\frac{1}{\theta}}. \tag{6.4}$$

Consequently, (6.1) holds for any $p < 1 \wedge \frac{1}{\theta}$.

Proof: Fix $\delta > 0$ such that $t-\delta \ge 0$. From (5.16), the definition of $\mathcal{H}_T$, and the triangular inequality, we clearly have

$$\begin{aligned}\|Du(t,x)\|^2_{\mathcal{H}_T} &\ge \int_{t-\delta}^t ds \|D_{s,*}u(t,x)\|^2_{\mathcal{H}}\\ &\ge \frac{1}{2}\int_{t-\delta}^t ds \|\Lambda(t-s,x-*)\sigma(u(s,*))\|^2_{\mathcal{H}} - I(t,x;\delta),\end{aligned}$$

where

$$\begin{aligned}I(t,x;\delta) = \int_{t-\delta}^t ds \| &\int_s^t \int_{\mathbb{R}^d} \Lambda(t-r,x-z)\sigma'(u(r,z))D_{s,*}u(r,z)W(dr,dz)\\ &+ \int_s^t dr \int_{\mathbb{R}^d} \Lambda(t-r,dz)b'(u(r,x-z))D_{s,*}u(r,x-z)\|^2_{\mathcal{H}}.\end{aligned}$$

Set

$$\begin{aligned}M_1(\delta) &= \int_0^\delta ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2,\\ M_2(\delta) &= \int_0^\delta ds \int_{\mathbb{R}^d} \Lambda(s,y)dy.\end{aligned}$$

Notice that by (5.3), $M_2(\delta) \le C\delta$.
Our aim is to prove

$$\int_{t-\delta}^t ds \|\Lambda(t-s,x-*)\sigma(u(s,*))\|^2_{\mathcal{H}} \ge \sigma_0^2 M_1(\delta), \tag{6.5}$$

$$E\left(I(t,x;\delta)\right) \le CM_1(\delta)\left(M_1(\delta)+M_2(\delta)\right). \tag{6.6}$$

For any $\epsilon \in]0,1[$, we can choose $\delta := \delta(\epsilon) > 0$ such that $M_1(\delta) = \frac{4\epsilon}{\sigma_0^2}$. Notice that by (6.3) this is possible. Then, $\delta < \left(\frac{4}{\sigma_0^2}\right)^{\frac{1}{\theta}} \epsilon^{\frac{1}{\theta}}$, and assuming that (6.5), (6.6) hold true, we have

$$\begin{aligned}
P\left(\|Du(t,x)\|^2_{\mathcal{H}_T} < \epsilon\right) &\le P\left(\int_{t-\delta}^t ds \|D_{s,*}u(t,x)\|^2_{\mathcal{H}} < \epsilon\right) \\
&\le P\left(I(t,x;\delta) \ge \frac{\sigma_0^2}{2} M_1(\delta) - \epsilon\right) \\
&\le C\epsilon^{-1} E\left(I(t,x;\delta(\epsilon))\right) \\
&\le C\epsilon^{-1}\left(\epsilon^2 + \epsilon^{1+\frac{1}{\theta}}\right) \\
&\le C\epsilon^{1\wedge\frac{1}{\theta}}.
\end{aligned}$$

This is (6.4).

Proof of (6.5):

By a change of variables

$$\int_{t-\delta}^t ds \|\Lambda(t-s, x-*)\sigma(u(s,*))\|^2_{\mathcal{H}} = \int_0^\delta ds \|\Lambda(s, x-*)\sigma(u(t-s,*))\|^2_{\mathcal{H}}.$$

Then, the inequality (5.11) applied to $Z(s,y) = |\sigma(u(t-s,y))|$ and $T := \delta$ yields (6.5). Indeed, for this choice of Z and T,

$$\int_0^\delta ds \|\Lambda(s, x-*)\sigma(u(t-s,*))\|^2_{\mathcal{H}} = \lim_{n\to\infty} E(\|z_n\|^2_{\mathcal{H}_\delta}) \ge \sigma_0 M_1(\delta).$$

Proof of (6.6):

We shall give a bound for the mathematical expectation of each one of the terms

$$I_1(t,x;\delta) = \int_0^\delta ds \left\|\int_{t-s}^t \int_{\mathbb{R}^d} \Lambda(t-r, x-z)\sigma'(u(r,z)) D_{t-s,*}u(r,z) W(dr,dz)\right\|^2_{\mathcal{H}},$$

$$I_2(t,x;\delta) = \int_0^\delta ds \left\|\int_{t-s}^t dr \int_{\mathbb{R}^d} \Lambda(t-r, dz) b'(u(r,x-z)) D_{t-s,*}u(r,x-z)\right\|^2_{\mathcal{H}}.$$

Since σ' is bounded, the inequality (5.15) yields

$$E\left(I_1(t,x;\delta)\right) \le C \sup_{(s,y)\in[0,\delta]\times\mathbb{R}^d} E\left(\|D_{t-\cdot,*}u(t-s,y)\|^2_{\mathcal{H}_\delta}\right) M_1(\delta).$$

For the pathwise integral, it is easy to prove that

$$E\left(I_2(t,x;\delta)\right) \le C \sup_{(s,y)\in[0,\delta]\times\mathbb{R}^d} E\left(\|D_{t-\cdot,*}u(t-s,y)\|^2_{\mathcal{H}_\delta}\right) M_2(\delta).$$

Since

$$\sup_{(s,y)\in[0,\delta]\times\mathbb{R}^d} E\left(\|D_{t-\cdot,*}u(t-s,y)\|^2_{\mathcal{H}_\delta}\right) \le CM_1(\delta),$$

(see for instance [61] or Lemma 8.2 in [69]), we get

$$\begin{aligned} E\left(I_1(t,x;\delta)\right) &\le CM_1(\delta)^2,\\ E\left(I_2(t,x;\delta)\right) &\le CM_1(\delta)M_2(\delta). \end{aligned}$$

This finishes the proof of (6.7) and therefore of (6.4).
The statement about the validity of (6.1) for the given range of p is a consequence of Lemma 6.1.

□

An example: the stochastic wave equation in dimension $d \le 3$

Proposition 6.1 can be applied for instance to the stochastic wave equation. Indeed, let Λ be the fundamental solution of $L = 0$ with $L = \partial^2_{tt} - \Delta_d$, $d = 1, 2, 3$. Assume that the measure μ satisfies

$$0 < \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty.$$

Then from (5.6) it follows that

$$C_1(t\wedge t^3) \le \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2 \le C_2(t+t^3),$$

where C_i, $i = 1, 2$ are positive constants independent of t.
In particular, for $t \in [0,1)$,

$$C_1 t^3 \le \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2 \le C_2 t.$$

Thus, condition (6.3) of Proposition 6.1 is satisfied with $\theta = 3$, and consequently $E\left(\|Du(t,x)\|^{-p}\right) < \infty$, for any $p < \frac{1}{3}$.

Remark 6.1 *Assume that for some $\delta_0 \in]0,t[$,*

$$\int_0^{\delta_0} dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi) > 0 \tag{6.7}$$

and consider the same assumptions on the coefficients as in Proposition 6.1. Then $\|Du(t,x)\|_{\mathcal{H}_T} > 0$, a.s. (see Theorem 5.2 in [53]). This conclusion is weaker that (6.1), but it suffices to give the existence of density for $u(t,x)$.

Remark 6.2 *Proposition 8.1 [69] is a weaker version of Proposition 6.1. The proof of of Proposition 6.1 follows [53], Theorem 5.2.*

Existence of density

To end this section, and as a summary, we give a result on existence of density for the solution of a class of SPDEs, as follows.

Theorem 6.1 *Consider the stochastic process $\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ solution of (5.7). We assume:*

(1) the functions σ, b belong to $\mathcal{C}^1$ and have bounded derivatives,

(2) there exists $\sigma_0>0$ such that $\inf\{|\sigma(z)|, z\in\mathbb{R}\}\geq\sigma_0$,

(3) Λ satisfies the assumptions (H_L),

(4) there exist $\theta>0$, such that for any $t\in(0,1)$,

$$C_1 t^\theta \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2$$

Then, for any fixed $(t,x)\in]0,T]\times\mathbb{R}^d$, the random variable $u(t,x)$ belongs to $\mathbb{D}^{1,2}$, and for any $p<1\wedge\frac{1}{\theta}$, $E(\|Du(t,x)\|_{\mathcal{H}_T}^{-p})<\infty$. Consequently, the probability law of $u(t,x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proof: The results of Section 5.2 imply $u(t,x)\in\mathbb{D}^{1,2}$. While the property about existence of moments has been established in Proposition 6.1. Eventually, the conclusion about the law of $u(t,x)$ follows from Proposition 4.3. □

Moments of any order

It is easy to improve the conclusions of Proposition 6.1 so that we can obtain (6.1) for any $p\geq 0$.
Indeed, moving back to the proof of this Proposition, we recall that we have obtained

$$P\left(\|Du(t,x)\|_{\mathcal{H}_T}^2<\epsilon\right)\leq P\left(I(t,x;\delta)\geq\frac{\sigma_0^2}{2}M_1(\delta)-\epsilon\right).$$

At this point, we can apply Chebychev's inequality to obtain

$$P\left(I(t,x;\delta) \geq \frac{\sigma_0^2}{2} M_1(\delta) - \epsilon\right) \leq C\epsilon^{-q} E\left(I(t,x;\delta(\epsilon))\right)^q,$$

for any $q > 1$.
Using $L^q(\Omega)$- estimates for the Hilbert-valued stochastic convolutions (see for instance [69], Theorem 6.1), and for pathwise integrals as well, yield

$$E\left(I(t,x;\delta(\epsilon))\right)^q \leq C\delta(\epsilon)^{q-1}\left(M_1(\delta(\epsilon))^{2q} + M_1(\delta(\epsilon))^q(\delta(\epsilon))^q\right).$$

By the choice of $\delta(\epsilon)$, this implies

$$P\left(\|Du(t,x)\|^2_{\mathcal{H}_T} < \epsilon\right) \leq C\epsilon^{\frac{q-1}{\theta}+[q\wedge\frac{q}{\theta}]}.$$

Since q can be chosen arbitrarily large, we obtain (6.1) for any $p \geq 0$.

Regularity of the density

Proceeding recursively, it is possible to extend the results in Section 5.2 and prove that under suitable assumptions, the solution of (5.7) is infinitely differentiable in the Watanabe-Sobolev sense (see [69], Chapter 7). Then, owing to the results discussed in the preceding paragraphs, applying Theorem 4.2 yields the following:

Theorem 6.2 *Consider the stochastic process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ solution of (5.7). We assume:*

(1) the functions σ, b belong to $\mathcal{C}^\infty$ and have bounded derivatives of any order,

(2) there exists $\sigma_0 > 0$ such that $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$,

(3) Λ satisfies the assumptions (H_L),

(4) there exist $\theta > 0$, such that for any $t \in (0,1)$,

$$C_1 t^\theta \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2$$

Then, for any fixed $(t,x) \in]0,T] \times \mathbb{R}^d$, the random variable $u(t,x)$ belongs to $\mathbb{D}^\infty$, and for any $p > 0$, $E(\|Du(t,x)\|^{-p}_{\mathcal{H}_T}) < \infty$. Consequently, the probability law of $u(t,x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and has a $\mathcal{C}^\infty$ density.

Notice that this result applies to the stochastic wave equation in dimension $d = 1, 2, 3$.

6.2 Some references

To end this lecture, we mention some references on existence and smoothness of density of probability laws, as a guide for the reader to have a further insight into the subject.

The first application of Malliavin calculus to SPDEs may be found in [51]; it concerns the hyperbolic equation on $\mathbb{R}^n$

$$\frac{\partial^2}{\partial s \partial t} X(s,t) = A(X(s,t))\dot{W}_{s,t} + A_0(X(s,t)), \tag{6.8}$$

with $s,t \in]0,1]$ and initial condition $X(s,t) = 0$ if $s \times t = 0$. Here

$$A : \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n, \ A_0 : \mathbb{R}^n \to \mathbb{R}^n,$$

are smooth functions, and W a d-dimensional Brownian sheet on $[0,1]^2$, that is, $W = (W_{s,t} = (W^1_{s,t}, \dots W^d_{s,t}), (s,t) \in [0,1]^2)$, with independent Gaussian components, zero mean and covariance function given by

$$E(W^i_{s_1,t_1} W^i_{s_2,t_2}) = (s_1 \wedge s_2)(t_1 \wedge t_2),$$

$i = 1, \dots, d$.

In dimension $n = 1$, this equation is transformed into the standard wave equation after a rotation of forty-five degrees. Otherwise, (6.8) is an extension to a two-parameter space of the Itô equation. The existence and smoothness of density for the probability law of the solution to (6.8) at a fixed time parameter (s,t), with $s \times t = 0$ has been proved under a specific type of Hörmander's condition on the vector fields A_i, $i = 1, \dots, d$, which does not coincide with Hörmander's condition for diffusions. An extension to a non restricted Hörmander's condition, that is, including the vector field A_0, has been done in [52].

The one-dimensional wave equation perturbed by space-time white noise, as an initial value problem but also as a boundary value problem, has been studied in [13]. The degeneracy conditions on the free terms of the equation are different from those in Theorem 6.2.

Existence for the density of equation (4.1) in dimension one, with $L = \Delta$ and space-time white noise, has been first studied in [58]. The authors consider a Dirichlet boundary value problem on $[0,1]$, with initial condition $u(0,x) = u_0(x)$. The required non-degeneracy condition reads as follows:

$$\sigma(u_0(y)) \neq 0, \text{for some } y \in]0,1[. \tag{6.9}$$

The same equation has been analyzed in [3]. In this reference, the authors consider the random vector $(u(t,x_1),\dots,u(t,x_m))$ obtained by looking at the solution of the equation at time $t \neq 0$ and different points $x_1,\dots,x_m \in]0,1[$. Under assumption (2) of Theorem 6.2, they obtain the smoothness of the density.

Recently in [45], this result has been improved. The authors prove that for $m = 1$, the assumption 6.9 yields the smoothness of the density as well.

The first application of Malliavin calculus to SPDEs with correlated noise appears in [43], and a first attempt for an unified approach of stochastic heat and wave equations is done in [37]. Hörmander's type conditions in a general context of Volterra equations have been given in [64].

7 Small perturbations of the density

Consider the SPDE (5.1) that we write in its mild form, as in (5.7). We replace $\dot{W}(t,x)$ by $\epsilon\dot{W}(t,x)$, with $\epsilon \in]0,1[$ and we are interested in the behaviour, as $\epsilon \to 0$ of the solution of the modified equation, that we will denote by u^ϵ. At the moment, this is a very vague plan; roughly speaking, one would like to know the effect of small noise on a deterministic evolution equation. Several questions may be addressed. For instance, denoting by μ^ϵ the probability law of the solution u^ϵ, we may want to prove a *large deviation principle* on spaces where the solution lives.

We recall that a family $(\mu^\epsilon, \epsilon \in]0,1[)$ of probability measures on a Polish space $\mathbb{E}$ is said to satisfy a *large deviation principle* with *rate functional* I if $I : \mathbb{E} \to [0,\infty]$ is a lower semicontinous function such that the level sets $\{I(x) \le a\}$ are compact, and for any Borel set $B \subset \mathbb{E}$,

$$-\inf_{x\in\mathring{B}} I(x) \le \liminf_{\epsilon\to 0} \epsilon^2 \log\left(\mu^\epsilon(B)\right) \le \limsup_{\epsilon\to 0} \epsilon^2 \log\left(\mu^\epsilon(B)\right) \le -\inf_{x\in\bar{B}} I(x).$$

In many of the applications that have motivated the theory of large deviations, as $\epsilon \to 0$, μ^ϵ degenerates to a delta Dirac measure at zero. Hence, typically a large deviation principle provides the rate of convergence and an accurate description of the degeneracy.

Suppose that the measures μ^ϵ live in $\mathbb{R}^d$ and have a density with respect to the Lebesgue measure. A natural question is whether from a large deviation principle one could obtain a precise lower and upper bound for the density. This question has been addressed by several authors in the context of diffusion processes (Azencott, Ben Arous and Léandre, Varadhan, to mention some of them), and the result is known as the *logarithmic estimates* and also as the *Varadhan estimates*. We recall this result since it is the inspiration for extensions to SPDEs.

Consider the family of stochastic differential equations on $\mathbb{R}^n$ (in the Stratonovich formulation)

$$X_t^\epsilon = x + \epsilon \int_0^t A(X_s^\epsilon) \circ dB_s + \int_0^t A_0(X_s^\epsilon) ds,$$

$t \in [0,1]$, $\epsilon > 0$, where $A : \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$, $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ and B is a d-dimensional Brownian motion. For each h in the Cameron-Martin space H associated with B, we consider the ordinary (deterministic) equation

$$S_t^h = x + \int_0^t A(S_s^h)\dot{h}_s ds + \int_0^t A_0(S_s^h) ds,$$

$t \in [0,1]$, termed the *skeleton* of X. For $y \in \mathbb{R}^n$, set

$$d^2(y) = \inf\{\|h\|_H^2; S_1^h = y\}$$
$$d_R^2(y) = \inf\{\|h\|_H^2; S_1^h = y, \det \gamma_{S_1^h} > 0\},$$

where $\gamma_{S_1^h}$ denotes the n-dimensional matrix whose entries are $\langle D(S_1^h)^i, D(S_1^h)^j \rangle$, $i, j = 1, \ldots, n$. Here D stands for the Fréchet differential operator on Banach spaces and we assume that the random fields $A_1, \ldots, A_d$ (the components of A) and A_0 are smooth enough. Notice that we use the same notation for the deterministic matrix $(\langle D(S_1^h)^i, D(S_1^h)^j \rangle)_{i,j}$ than for the Malliavin matrix. In this context, the former is often termed the *deterministic Malliavin matrix*. The quantities $d^2(y)$, $d_R^2(y)$ are related with the energy needed by a system described by the skeleton to leave the initial condition $x \in \mathbb{R}^n$ and reach $y \in \mathbb{R}^n$ at time $t = 1$.

This is the result for SDEs.

Theorem 7.1 *Let $A : \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$, $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be infinite differentiable functions with bounded derivatives of any order. Assume:*

(HM) *There exists $k_0 \geq 1$ such that the vector space spanned by the vector fields*

$$[A_{j_1}, [A_{j_{k_2}}, [\ldots [A_{j_k}, A_{j_0}]] \cdots], \quad 0 \leq k \leq k_0,$$

where $j_0 \in \{1, 2, \ldots, d\}$ and $j_i \in \{0, 1, 2, \ldots, d\}$ if $1 \leq i \leq k$ at the point $x \in \mathbb{R}^n$ has dimension n.

Then, the random variable X_1^ϵ has a smooth density p_ϵ, and

$$-d_R^2(y) \leq \liminf_{\epsilon \to 0} 2\epsilon^2 \log p_\epsilon(y) \leq \limsup_{\epsilon \to 0} 2\epsilon^2 \log p_\epsilon(y) \leq -d^2(y). \tag{7.1}$$

In addition, if $\inf\{\det \gamma_{S_1^h};$ for h such that $S_1^h = y\} > 0$, then $d^2(y) = d_R^2(y)$ and consequently,

$$\lim_{\epsilon \to 0} 2\epsilon^2 \log p_\epsilon(y) = d^2(y). \tag{7.2}$$

The assumption **(HM)** in this theorem is termed *Hörmander's unrestricted assumption*; the notation $[\cdot, \cdot]$ refers to the Lie brackets.

The proof of this theorem given in [11] admits an extension to the general framework of an abstract Wiener space. This fact has been noticed and applied in [33], and then written in [47]; since then, it has been applied to several examples of SPDEs. In this lecture we shall give this general result

and then some hints on its application to an example of stochastic heat equation.
Throughout this section, $\{W(h), h \in H\}$ is a Gaussian family, as has been defined in Section 3.2. We will consider *non-degenerate* random vectors F, which in this context means that $F \in \mathbb{D}^\infty$ and

$$\det \gamma_F^{-1} \in \cap_{p\in[1,\infty[} L^p(\Omega),$$

where γ_F denotes the Malliavin matrix of F. Notice that by Theorem 4.2, non-degenerate random vectors F have an infinitely differentiable density.

7.1 General results

Lower bound

Proposition 7.1 *Let $\{F^\epsilon, \epsilon \in]0,1]\}$ be a family of non-degenerate n-dimensional random vectors and let $\Phi \in \mathcal{C}_p^\infty(H;\mathbb{R}^n)$ (the space of $\mathcal{C}^\infty$ functions with polynomial growth) be such that for each $h \in H$, the limit*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(F^\epsilon\left(\omega+\frac{h}{\epsilon}\right)-\Phi(h)\right)=Z(h) \tag{7.3}$$

exists in the topology of $\mathbb{D}^\infty$ and defines a n-dimensional random vector with absolute continuous distribution.
Then, setting

$$d_R^2(y)=\inf\{\|h\|_H^2 : \Phi(h)=y, \det\gamma_{\gamma_{\Phi(h)}}>0\},$$

$y \in \mathbb{R}^n$, we have

$$-d_R^2(y) \le \liminf_{\epsilon\to 0}\epsilon^2\log p_\epsilon(y). \tag{7.4}$$

Proof: Let $y \in \mathbb{R}^n$ be such that $d_R^2(y) < \infty$. For any $\eta > 0$ there exists $h \in H$ such that $\Phi(h) = y$, $\det\gamma_{\gamma_{\Phi(h)}} > 0$ and $\|h\|_H^2 \le d_R^2(y) + \eta$. For any function $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we can write

$$E\left(f(F^\epsilon)\right)=\exp\left(-\frac{\|h\|_H^2}{2\epsilon^2}\right)E\left(f\left(F^\epsilon\left(\omega+\frac{h}{\epsilon}\right)\right)\exp\left(-\frac{W(h)}{\epsilon}\right)\right), \tag{7.5}$$

by Girsanov's theorem.

Consider a smooth approximation of $1_{[-\eta,\eta]}$ given by a function $\rho \in \mathcal{C}^\infty$, $0 \le \rho \le 1$, such that $\rho(t) = 0$, if $t \notin [-2\eta, 2\eta]$, $\rho(t) = 1$ if $t \in [-\eta, \eta]$. Then using (7.5), and assuming that f is a positive function, we have

$$E\left(f(F^\epsilon)\right) \ge \exp\left(-\frac{\|h\|_H^2 + 4\eta}{2\epsilon^2}\right) E\left(f\left(F^\epsilon\left(\omega + \frac{h}{\epsilon}\right)\right)\rho(\epsilon W(h))\right).$$

We now apply this inequality to a sequence $f_n, n \ge 1$, of smooth approximations of the delta Dirac function at y. Passing to the limit and taking logarithms, we obtain

$$2\epsilon^2 \log p_\epsilon(y) \ge -(\|h\|_H^2 + 4\eta) + 2\epsilon^2 \log E\left(\delta_y\left(F^\epsilon\left(\omega + \frac{h}{\epsilon}\right)\right)\rho(\epsilon W(h))\right).$$

Hence, to complete the proof we need checking that

$$\lim_{\epsilon \to 0} \epsilon^2 \log E\left(\delta_y\left(F^\epsilon\left(\omega + \frac{h}{\epsilon}\right)\right)\rho(\epsilon W(h))\right) = 0. \tag{7.6}$$

Since $y = \Phi(h)$, we clearly have

$$E\left(\delta_y\left(F^\epsilon\left(\omega + \frac{h}{\epsilon}\right)\right)\rho(\epsilon W(h))\right) = \epsilon^{-m} E\left(\delta_0\left(\frac{F^\epsilon\left(\omega + \frac{h}{\epsilon}\right) - \Phi(h)}{\epsilon}\right)\rho(\epsilon W(h))\right).$$

The expression

$$E\left(\delta_0\left(\frac{F^\epsilon\left(\omega + \frac{h}{\epsilon}\right) - \Phi(h)}{\epsilon}\right)\rho(\epsilon W(h))\right)$$

tends to the density of $Z(h)$ at zero, as $\varepsilon \to 0$, as can be proved using the integration by parts formula (4.9)–(4.11). Hence (7.6) holds true and this ends the proof of the Proposition.

□

Upper bound

Stating the upper bound for the logarithm of the density needs more demanding assumptions. Among others, the family $\{F^\epsilon, \epsilon \in]0,1]\}$ must satisfy a large deviation principle, and there should be a control of the norm of the inverse of the Malliavin matrix in terms of powers of ϵ.

Proposition 7.2 *Let $\{F^\epsilon, \epsilon \in]0,1]\}$ be a family of non-degenerate n-dimensional random vectors and let $\Phi \in \mathcal{C}_p^\infty(H;\mathbb{R}^n)$ be such that:*

1. $\sup_{\epsilon\in]0,1]} \|F^\epsilon\|_{k,p} < \infty$, *for each integer $k \geq 1$ and any real number $p \in]1,\infty[$,*

2. *For any $p \in]1,\infty[$, there exist $\epsilon_p > 0$ and $N(p) \in]1,\infty[$ such that $\|(\gamma_{F^\epsilon})^{-1}\|_p \leq \epsilon^{-N(p)}$, for each integer $\epsilon \leq \epsilon_p$,*

3. *$\{F^\epsilon, \epsilon \in]0,1]\}$ satisfies a large deviation principle on $\mathbb{R}^n$ with rate function $I(y)$, $y \in \mathbb{R}^n$.*

Then,

$$\limsup_{\epsilon\to 0} 2\epsilon^2 \log p_\epsilon(y) \leq -I(y). \tag{7.7}$$

Proof: It is an application of the integration by parts formula (4.9)–(4.11). Indeed, fix $y \in \mathbb{R}^n$ and a smooth function $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $0 \leq \rho \leq 1$ such that ρ is equal to one in a nighbourhood of y. Then we can write

$$p_\epsilon(y) = E\left(\rho(F^\epsilon)\delta_y(F^\epsilon)\right).$$

Applying Hölder's inequality with $p,q \in]1,\infty[$ with $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned} E\left(\rho(F^\epsilon)\delta_y(F^\epsilon)\right) &= E\left(1_{\{F^\epsilon \geq y\}} H_{1,\dots,1}(F^\epsilon, \rho(F^\epsilon))\right) \\ &\leq E\left(|H_{1,\dots,1}(F^\epsilon, \rho(F^\epsilon))|\right) \\ &= E\left(|H_{1,\dots,1}(F^\epsilon, \rho(F^\epsilon))| 1_{\{F^\epsilon \in supp\ \rho\}}\right)) \\ &\leq \left(P\{F^\epsilon \in \text{supp } \rho\}\right)^{\frac{1}{q}} \|H_{1,\dots,1}(F^\epsilon, \rho(F^\epsilon))\|_p. \end{aligned}$$

By the L^p estimates of the Skorohod integral (see for instance [79], or Proposition 3.2.2 in [47]), there exist real numbers greater than one, $\tilde{p}$, a,b, $\tilde{a},\tilde{b}$ and

$$\|H_{1,\dots,1}(F^\epsilon, \rho(F^\epsilon))\|_p \leq C\|(\gamma_{F^\epsilon})^{-1}\|_{\tilde{p}} \|F^\epsilon\|_{a,b} \|\rho(F^\epsilon)\|_{\tilde{a},\tilde{b}}.$$

The assumptions (1) (2) below ensure

$$\limsup_{\epsilon\to 0} 2\epsilon^2 \log \|H_{1,\dots,1}(F^\epsilon, \rho(F^\epsilon))\|_p = 0,$$

while (3) implies

$$\limsup_{\epsilon\to 0} 2\epsilon^2 \log P\{F^\epsilon \in \text{supp}\rho\} \leq -\inf(I(y), y \in supp\rho)$$

and consequently,

$$\limsup_{\epsilon\to 0} 2\epsilon^2 \log p_\epsilon(y) \leq -\frac{1}{q}\inf(I(y), y \in supp\rho).$$

Set $I(supp\rho) = \inf(I(y), y \in supp\rho)$. For any $\delta > 0$, there exists $q > 1$ such that $\frac{1}{q}I(supp\rho) \geq I(supp\rho) - \delta$. Then, by taking a sequence of smooth functions ρ_n (with the same properties as ρ) such that $supp\rho_n$ decreases to $\{y\}$, we see that

$$\limsup_{\epsilon\to 0} 2\epsilon^2 \log p_\epsilon(y) \leq -I(y) + \delta,$$

and since δ is arbitrary, we obtain (7.7).
This ends the proof.

$\square$

7.2 An example: the stochastic heat equation

In this section, we consider the SPDE

$$\begin{aligned} u(t,x) = &\int_0^t \int_{\mathbb{R}} G(t-s, x-y)\sigma(u(s,y))W(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s, x-y) b(u(s,y)) ds dy, \end{aligned} \tag{7.8}$$

$t \in [0,T]$, where $G(t,x) = (2\pi t)^{-\frac{1}{2}} \exp\left(-\frac{|x|^2}{2t}\right)$ and W is space-time white noise. In the framework of Section 5 this corresponds to a stochastic heat equation in dimension $d = 1$ and to a spatial covariation measure given by $\Gamma(dy) = \delta_0(y)$.
The companion perturbed family we would like to study is

$$\begin{aligned} u^\epsilon(t,x) = &\,\epsilon\int_0^t \int_{\mathbb{R}} G(t-s, x-y)\sigma(u^\epsilon(s,y))W(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s, x-y) b(u^\epsilon(s,y)) ds dy, \end{aligned} \tag{7.9}$$

$\epsilon \in]0,1[$.
Our purpose is to apply Propositions 7.1 and 7.2 and therefore to obtain logarithm estimates for the density.
Throughout this section we will assume the following condition, although some of the results hold under weaker assumptions.

(C) The functions σ, b are $\mathcal{C}^\infty$ with bounded derivatives.

The Hilbert space H we should take into account here is the Cameron-Martin space associated with W. It consists of the set of functions $h : [0,T]\times\mathbb{R} \to \mathbb{R}$ absolutely continuous with respect to the product Lebesgue measure $dtdx$ and such that $\|h\|_H^2 := \left(\int_0^T \int_{\mathbb{R}} |\dot{h}_{t,x}|^2 dtdx\right)^{\frac{1}{2}} < \infty$, where $\dot{h}_{t,x}$ stands for the second order derivative $\frac{\partial^2 h(t,x)}{\partial t \partial x}$, which exist almost everywhere.
For each $h \in H$ we consider the deterministic evolution equation

$$\begin{aligned}\Psi^h(t,x) &= \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma(\Psi^h(s,y))\dot{h}(s,y)dsdy \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s,x-y)b(\Psi^h(s,y))dsdy, \qquad (7.10)\end{aligned}$$

Existence and uniqueness of solution for (7.10) is proved in an analogue way than for (7.8), using a Picard iteration scheme.

Some known results

We quote some known results on (7.8) that shall allow to follow the procedure explained in the previous section.

1 Non degeneracy and existence of density

Fix $(t,x) \in]0,T] \times \mathbb{R}$. Assume **(C)** and also

(ND) $\inf\{|\sigma(y)|, y \in \mathbb{R}\} \geq c > 0$.

Then the family $(u^\epsilon, \epsilon \in]0,1[)$ is *non-degenerate* and therefore the random variable $u^\epsilon(t,x)$ possesses a $\mathcal{C}^\infty$ density p_ϵ.
This result has been proved in [3] (see also sections 5 and 6).

2 Large deviation principle

Large deviation principles for the family $(u^\epsilon, \epsilon \in]0,1[)$ in the topology of Hölder continuous functions have been established under different type of assumptions by Sowers ([66]) and Chenal and Millet. Here, for the sake of simplicity we shall consider the topology of uniform convergence on compact sets and denote by $\mathcal{C}([0,T]\times\mathbb{R})$ the set of continuous functions defined on $[0,T]\times\mathbb{R}$ with respect to this topology. It is known that $(u^\epsilon, \epsilon \in]0,1[)$ satisfies a large deviation principle on $\mathcal{C}([0,T]\times\mathbb{R})$ with rate function

$$I(f) = \inf\left\{\frac{\|h\|_H^2}{2}; h \in H, \Psi^h = f\right\}.$$

This is a *functional* large deviation principle; the *contraction principle* (a transfer principle of large deviations through continuous functionals) gives rise to the following statement:
Fix $(t,x) \in]0,T] \times \mathbb{R}$. Assuming **(C)**, *$u^\epsilon(t,x)$ satisfies a large deviation principle on $\mathbb{R}$ with rate function*

$$I(y) = \inf \left\{ \frac{\|h\|_H^2}{2}; h \in H, \Psi^h(t,x) = y \right\}, \ y \in \mathbb{R}. \tag{7.11}$$

A result for the stochastic heat equation

For the family defined in (7.9), we have the following theorem ([40], Theorem 2.1)

Theorem 7.2 *Assume that the functions σ, b satisfy* **(C)** *and also* **(ND)**. *Fix $(t,x) \in]0,T] \times \mathbb{R}$ and let $I : \mathbb{R} \to \mathbb{R}$ be defined in (7.11). Then the densities $(p^\epsilon_{t,x}, \epsilon \in]0,1[)$ of $(u^\epsilon(t,x,\epsilon \in]0,1[)$, satisfy*

$$\lim_{\epsilon \to 0} \epsilon^2 \log p^\epsilon_{t,x}(y) = -I(y). \tag{7.12}$$

.

Proof: We shall consider the abstract Wiener space associated with the space-time white noise W and check that $F^\epsilon := u^\epsilon(t,x)$, $\epsilon \in]0,1[$, satisfy the assumptions of Propositions 7.1, 7.2. We give some hints for this in the sequel.

Proposition 7.2, Assumption 1.

Following the results of Section 5, we already know that $u(t,x) \in \mathbb{D}^\infty$, that is, $\|u(t,x)\|_{k,p} < \infty$, for any $p \in [1,\infty[$, $k \in \mathbb{N}$. With the same proof, it is easy to check that

$$\sup_{\epsilon \in]0,1[} \|u^\epsilon(t,x)\|_{k,p} < \infty.$$

Actually, in the different estimates to be checked, ϵ appears as a factor that can be bounded by one.

Proposition 7.2, Assumption 2.

Set $\gamma_\epsilon = \gamma_{u^\epsilon(t,x)}$ and $Q_\epsilon = \epsilon^{-2}\gamma_\epsilon$. Since $u^\epsilon(t,x)$ is a random variable, γ_ϵ and Q_ϵ are random variables as well. Assume we can prove

$$\sup_{\epsilon \in]0,1[} E\left(|Q_\epsilon|^{-p} \right) < \infty, \tag{7.13}$$

for any $p \in [1, \infty[$. Then we will have

$$E\left(|\gamma_\epsilon|^{-p}\right) \le \epsilon^{-2p} \sup_{\epsilon \in]0,1[} E\left(|Q_\epsilon|^{-p}\right) \le C\epsilon^{-2p},$$

and we will obtain the desired conclusion with $N(p) = 2$.

Let us give some hints for the proof of (7.13). Remember that $\gamma_\epsilon = \|Du^\epsilon(t,x)\|_H^2$. The H–valued stochastic process $Du^\epsilon(t,x)$ satisfies an equation similar to (5.16), where σ is replaced by $\epsilon\sigma$ (and consequently σ' replaced by $\epsilon\sigma'$). By uniqueness of solution, it holds that $Du^\epsilon(t,x) = \epsilon\sigma(u^\epsilon(.,*))Y^\epsilon(t,x)$, where $Y^\epsilon(t,x)$ is a H–valued stochastic process solution to the equation

$$\begin{aligned} Y^\epsilon(t,x) = G(t-\cdot, x-*) + \epsilon \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma'(u^\epsilon(s,y))Y^\epsilon(s,y)W(ds,dy) \\ + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)b'(u^\epsilon(s,y))Y^\epsilon(s,y)dsdy. \end{aligned}$$

Then Q_ϵ corresponds to $\|\sigma(u^\epsilon(.,*))Y^\epsilon(t,x)\|_H^2$, which essentially behaves like $\|Du(t,x)\|_H^2$.

Proposition 7.2, Assumption 3.

See the result under the heading *large deviation principle*.

Assumptions of Proposition 7.1.

The non-degeneracy of the family $u^\epsilon(t,x), \epsilon \in]0,1[$ has already been discussed. Now we will discuss the existence of limit (7.3), which is a hypothesis on existence of directional derivative with respect to ω. Let us proceed formally.

For any $h \in H$, set $Z^{\epsilon,h}(t,x)(\omega) = u^\epsilon(t,x)(\omega + \frac{h}{\epsilon})$. By uniqueness of solution $Z^{\epsilon,h}(t,x)$ is given by

$$\begin{aligned} Z^{\epsilon,h}(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\Big[\epsilon\sigma(Z^{\epsilon,h}(s,y)W(ds,dy) \\ + \left\{\sigma(Z^{\epsilon,h}(s,y)\dot{h}_{s,y} + b(Z^{\epsilon,h}(s,y))\right\} dsdy\Big]. \end{aligned}$$

It is now clear that the candidate for $\Phi(h)$ in Proposition 7.1 should be $Z^{0,h}(t,x)$, and by uniqueness of solution $Z^{0,h}(t,x) = \Psi^h(t,x)$. Hence, we have to check that the mapping $\epsilon \in]0,1[\mapsto Z^{\epsilon,h}(t,x)$ is differentiable at $\epsilon = 0$ in the topology of $\mathbb{D}^\infty$. We refer the reader to [33] for a proof of a similar result.

Going on with formal arguments, we see that $Z^h(t,x) := \frac{\partial}{\partial\epsilon} Z^{\epsilon,h}(t,x)|_{\epsilon=0}$ must be the solution of

$$Z^h(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\Big[\sigma(\Psi^h(s,y)W(ds,dy) \\ + \Big\{\sigma'(\Psi^h(s,y))\dot{h}_{s,y} + b'(\Psi^h(s,y))\Big\} Z^h(s,y)dsdy\Big].$$

This equation does not differ very much from (5.7). As for the latter equation, one can prove that $Z \in \mathbb{D}^\infty$. Notice that the stochastic process $Z^h(t,x)$ does not appear in the integrand of the stochastic integral.
The Malliavin derivative of $Z^h(t,x)$ is a deterministic random variable obtained by solving the equation

$$D_{r,z}Z^h(t,x) = 1_{\{r<t\}}\Big\{G(t-r, x-z)\sigma(\Psi^h(r,z)) \\ + \int_r^t \int_{\mathbb{R}} \Big[G(t-s, x-y))\sigma'(\Psi^h(s,y))\dot{h}_{s,y} + b'(\Psi^h(s,y))\Big] D_{r,z}Z^h(s,y)dsdy\Big\}.$$

Thus $Z^h(t,x)$ is a Gaussian random variable. We notice that, by uniqueness of solution $DZ^h(t,x) = D\Psi^h(t,x)$.
With this, we finish the checking of the assumptions.
In this example $d_R^2(y) = I(y)$. In fact, for any $h \in H$, $\det \gamma_{\Psi^h} > 0$. This property can be proved following the same ideas as for the analysis of the Malliavin variance $\|Du(t,x)\|_H^2$ (see Lemma 2.5 in [40]).

□

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