Advanced Classical Electromagnetic Theory

Hong Guo
LP01 Group, School of Electronics Engineering
& Computer Science, Peking University,
Beijing 100871, P.R. China
LP01 Group, Laboratory of Light Transmission Optics,
South China Normal University,
Guangzhou 510631, P.R. China

August 21, 2003

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Preface to Edition 2003

This is an advanced text of classical electrodynamics for graduate students of optical physics, optics, and radio physics. It is assumed that all the graduate students have had some basis of classical electrodynamics in undergraduate level and have some experiences of using Maxwell's equations and advanced mathematics to treat the electromagnetic field related problems. Hence, in this text, a new system of electrodynamics is adopted and the emphasis is focused on some new properties of electromagnetic field which may be ignored in undergraduate grades, such as the conservation properties of field-particle system, the transverse and longitudinal properties of the electromagnetic field, the gauge properties, and the covariant forms of electromagnetic field. Further, some preliminaries of free classical electromagnetic field theory are also given.

The units adopted in this text is Gaussian (cgs) since in many theoretical problems this unit is a preferable one, which can make the analysis clearer and physics therein easier to be understood.

In the references, those very original ones are listed and emphasized, such as the original work of the experimental laws discovered by C. A. de Coulomb, A. M. Ampère, and M. Faraday., who, respectively, gives the Coulomb's law, Ampère's law and Faraday's law of induction. Also, J. C. Maxwell's original work are also listed cause it is him who introduced the so-called electric displacement and derived the foundations of the electromagnetic theory, i.e., Maxwell's equations, which are a set of complete, unique and self-consistent equations for exploring the interaction between charged particles and electromagnetic field in classical regime. The work of H. von Helmholtz, G. Kirchhoff, P. de Fermat, A. Fresnel and A. J. W. Sommerfeld are listed since they also contribute to the development of electromagnetic theory or applications in optics, radio physics, etc. The famous work of M. Born and E. Wolf, Principle of Optics, is the first one which treated the problems of optics, systematically, using completely the electromagnetic theory and also gives the relations among electromagnetic theory with geometrical optics and diffraction integral theories. The work of John David Jackson, Ewan Wright and Renchuan Wang are listed since we adopted some of their materials in constituting the viewgraph of our text.

The author hopes that after learning this text, the graduate students can have a much better understanding of Maxwell's equations and an improvement in handling the electromagnetic field related problems.

The author acknowledges the contribution work of his several students, Mr. Shixiong Liu, Miss Dongmei Deng and Miss Hua Tang, for large amount of typing this viewgraph and for their critical problems encountered during the period of preparing this viewgraph.

Chapter 1

Experimental laws

- Coulomb's law:
$$\vec{E} = \int_{V'} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|^3} (\vec{x} - \vec{x}') d^3\vec{x}'. \tag{1.1}$$

- Biot-Savart's law:
$$\vec{B} = \frac{1}{c} \int_{V'} \frac{\vec{j}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d^3\vec{x}'. \tag{1.2}$$

- Faraday's law of induction:
$$\oint_L \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}. \tag{1.3}$$

- Ampère's law:
$$\oint_L \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int_S \vec{j} \cdot d\vec{S}. \tag{1.4}$$

- Conservation of charge:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{1.5}$$

- Lorentz equation:

$$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}), \qquad \text{(Lorentz force)} \tag{1.6}$$

$$\vec{f} = \rho(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}). \qquad \text{(Lorentz force density)} \tag{1.7}$$

Chapter 2

Derivation of Maxwell's equations

2.1 Differential equations for electric field

To evaluate a point charge in a closed surface S, since

$$\vec{E}\cdot d\vec{S}=\frac{q}{r^2}\cos\theta dS=qd\Omega,\qquad (\cos\theta dS=r^2d\Omega)$$

$$\oint_S \vec{E}\cdot d\vec{S}=\oint_S qd\Omega=q\oint_S d\Omega=\begin{cases} 4\pi q, & \text{(the charge is inside the surfaces);}\\ 0, & \text{(outside).}\end{cases}$$

Hence

$$\oint_S \vec{E}\cdot d\vec{S}=4\pi q.$$

Generally, for a discrete charge system with many charges, q_i $(i=1,2,3,\cdots)$, one has

$$\oint_S \vec{E}\cdot d\vec{S}=4\pi\sum_i q_i=4\pi Q.$$

To be more general, for a continuous charge system, one has

$$\oint_S \vec{E}\cdot d\vec{S}=4\pi\int_V \rho(\vec{x})d^3\vec{x}.$$

This is Gauss's law. Then according to Gauss's theorem (mathematical)

$$\oint_S \vec{E}\cdot d\vec{S}=\int_V \nabla\cdot\vec{E}d^3\vec{x},$$

one yields

$$\int_V(\nabla\cdot\vec{E}-4\pi\rho)d^3\vec{x}=0$$

for an arbitrary volume V. So, we can, in the usual way, simply put the integrand equal to zero, i.e.,

$$\nabla\cdot\vec{E}=4\pi\rho, \tag{2.1}$$

which is the differential form of Gauss's law of electrostatics. Eq. (2.1) can also be derived by directly using Coulomb's law, i.e., Eq. (1.1):

$$\begin{aligned}
\nabla_{\vec{x}} \cdot \vec{E} &= \nabla_{\vec{x}} \cdot \int_{V'} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')d^3\vec{x}' \\
&= \int_{V'} \left(\nabla_{\vec{x}} \cdot \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right) \rho(\vec{x}')d^3\vec{x}' \\
&= \int_{V'} 4\pi\delta(\vec{x}-\vec{x}')\rho(\vec{x}')d^3\vec{x}' \\
&= 4\pi\rho(\vec{x}),
\end{aligned}$$

where we used $\nabla \cdot \frac{\vec{x}}{r^3} = 4\pi\delta(\vec{x})$ (Verify, Problem 3.1).
Further, since

$$\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\nabla \frac{1}{|\vec{x}-\vec{x}'|}, \qquad \text{(verify Problem 3.2)} \tag{2.2}$$

one has

$$\vec{E} = -\nabla \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' = -\nabla\phi. \tag{2.3}$$

From Faraday's law, one easily derives, using Stokes's theorem, that

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}. \tag{2.4}$$

2.2 Differential equations for magnetic field

Based on Biot-Sarart's law, one has

$$\begin{aligned}
\vec{B}(\vec{x}) &= \frac{1}{c}\int_{V'} \vec{j}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} d^3\vec{x}' \qquad (2.5) \\
&= -\frac{1}{c}\int_{V'} \vec{j}(\vec{x}') \times \nabla \frac{1}{|\vec{x}-\vec{x}'|} d^3\vec{x}' \\
&= \frac{1}{c}\int_{V'} \nabla \times \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' \\
&= \nabla \times \vec{A},
\end{aligned}$$

where

$$\vec{A} = \frac{1}{c}\int_{V'} \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'.$$

Hence

$$\nabla \cdot \vec{B} = \nabla \cdot \nabla \times \vec{A} = 0. \qquad \text{(verify, Problem 3.3)} \tag{2.6}$$

Also, since

$$\nabla \times \vec{B} = \frac{1}{c}\nabla \times \nabla \times \int_{V'} \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'.$$

and

$$\nabla \times \nabla \times = \nabla(\nabla \cdot) - \nabla^2, \qquad \text{(verify, Problem 3.4)} \tag{2.7}$$

one has

$$\begin{aligned} & \frac{1}{c}\nabla \times \nabla \times \int_{V'} \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' \\ = & \frac{1}{c}\nabla \int_{V'} \vec{j}(\vec{x}') \cdot \nabla \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) d^3\vec{x}' - \frac{1}{c} \int_{V'} \vec{j}(\vec{x}') \nabla^2 \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) d^3\vec{x}'. \end{aligned}$$

Using the relations

$$\nabla \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) = -\nabla' \left(\frac{1}{|\vec{x}-\vec{x}'|} \right), \qquad \text{(verify, Problem 3.5)} \tag{2.8}$$

$$\nabla^2 \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) = -4\pi\delta(\vec{x}-\vec{x}'), \qquad \text{(verify, Problem 3.6)} \tag{2.9}$$

yields,

$$\begin{aligned} \nabla \times \vec{B} &= -\frac{1}{c}\nabla \int_{V'} \vec{j}(\vec{x}') \cdot \nabla' \frac{1}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \frac{4\pi}{c}\vec{j}(\vec{x}) \\ &= \frac{1}{c}\nabla \int_{V'} d^3\vec{x}' \frac{\nabla' \cdot \vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{4\pi}{c}\vec{j}(\vec{x}). \qquad \text{(integration by parts, verify, Problem 3.7)} \end{aligned}$$

Since, in magneto-statics, $\nabla \cdot \vec{j} = 0 \;\; (\partial_t \rho = 0)$, hence

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}(\vec{x}). \tag{2.10}$$

This can also be verified by directly using Ampère's law, i.e.,

$$\oint_L \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int_S \vec{j} \cdot d\vec{S},$$

together with Stokes's theorem

$$\oint_L \vec{B} \cdot d\vec{l} = \int_S \nabla \times \vec{B} \cdot d\vec{S},$$

therefore

$$\int_S (\nabla \times \vec{B} - \frac{4\pi}{c}\vec{j}) \cdot d\vec{S} = 0,$$

then

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}.$$

2.3 Maxwell's equations

In vacuum:

$$\nabla \cdot \vec{E} = 4\pi\rho, \tag{2.11}$$

$$\nabla \cdot \vec{B} = 0, \tag{2.12}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}, \tag{2.13}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}. \tag{2.14}$$

To fulfill the charge conservation, one must consider that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0. \tag{2.15}$$

However, Eq. (2.14) and (2.11) will lead to conflicts:

$$\frac{\partial}{\partial t}\rho = \frac{1}{4\pi}\frac{\partial}{\partial t}\nabla \cdot \vec{E} = \frac{1}{4\pi}\nabla \cdot \frac{\partial}{\partial t}\vec{E} \neq 0, \tag{2.16}$$

$$\nabla \cdot \vec{j} = \frac{c}{4\pi}\nabla \cdot \nabla \times \vec{B} = 0. \tag{2.17}$$

Hence, Maxwell introduced the so-called displacement current, i.e.,

$$\vec{j}_D = \frac{1}{4\pi}\frac{\partial}{\partial t}\vec{E} \tag{2.18}$$

into Eq. (2.14), then

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j}. \tag{2.19}$$

in fact, one can also get from following

$$\begin{aligned} \frac{1}{c}\nabla \int_{V'} d^3\vec{x}' \frac{\nabla' \cdot \vec{j}(\vec{x}')}{|\vec{x} - \vec{x}'|} &= -\frac{1}{c}\nabla \frac{\partial}{\partial t}\int_{V'} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \\ &= -\frac{1}{c}\nabla \frac{\partial}{\partial t}\phi \\ &= -\frac{1}{c}\frac{\partial}{\partial t}\nabla\phi \\ &= \frac{1}{c}\frac{\partial}{\partial t}\vec{E}, \end{aligned} \tag{2.20}$$

and thus

$$\begin{aligned} \nabla \cdot (\nabla \times \vec{B}) &= \nabla \cdot \left(\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j}\right) \\ &= \frac{1}{c}\left(\frac{\partial}{\partial t}4\pi\rho + 4\pi\nabla \cdot \vec{j}\right) \\ &= \frac{4\pi}{c}\left(\frac{\partial}{\partial t}\rho + \nabla \cdot \vec{j}\right) \\ &= 0. \qquad \text{(consistent)} \end{aligned} \tag{2.21}$$

After introducing the electric polarization and magnetization, i.e.,

$$\vec{D} = \vec{E} + 4\pi\vec{P}, \tag{2.22}$$

$$\vec{B} = \vec{H} + 4\pi\vec{M}, \tag{2.23}$$

Maxwell's equations hold:

$$\begin{cases} \nabla \cdot \vec{D} = 4\pi\rho, & (\mathrm{Coulomb's\ \ law}) \\ \nabla \cdot \vec{B} = 0, & (\mathrm{Biot-Savart's\ \ law}) \\ \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}, & (\mathrm{Faraday's\ \ law}) \\ \nabla \times \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}. & (\mathrm{Amp\grave{e}re's\ \ law + displacement\ \ current}) \end{cases} \tag{2.24}$$

Chapter 3

Properties of Maxwell's equations

3.1 Conservation Properties

3.1.1 Maxwell-Lorentz equations

In vacuum, Maxwell-Lorentz equation hold:

$$\nabla \cdot \vec{E}(\vec{x}, t) = 4\pi\rho(\vec{x}, t), \tag{3.1}$$

$$\nabla \cdot \vec{B}(\vec{x}, t) = 0, \tag{3.2}$$

$$\nabla \times \vec{E}(\vec{x}, t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}(\vec{x}, t), \tag{3.3}$$

$$\nabla \times \vec{B}(\vec{x}, t) = \frac{4\pi}{c}\vec{j}(\vec{x}, t) + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}(\vec{x}, t), \tag{3.4}$$

$$\rho(\vec{x}, t) = \sum_{\alpha} q_\alpha \delta(\vec{x} - \vec{x}_\alpha(t)), \tag{3.5}$$

$$\vec{j}(\vec{x}, t) = \sum_{\alpha} q_\alpha \vec{v}_\alpha(t)\delta(\vec{x} - \vec{x}_\alpha(t)), \tag{3.6}$$

$$m_\alpha \ddot{\vec{x}}_\alpha(t) = q_\alpha \left[\vec{E}(\vec{x}_\alpha(t), t) + \frac{\vec{v}_\alpha(t) \times \vec{B}(\vec{x}_\alpha(t), t)}{c}\right]. \tag{3.7}$$

3.1.2 Conservation of electric charge and current

1.
$$\frac{\partial}{\partial t}\rho(\vec{x}, t) + \nabla \cdot \vec{j}(\vec{x}, t) = 0, \tag{3.8}$$

2. Define $Q = \int d^3\vec{x}\rho(\vec{x}, t)$, then
$$\frac{dQ}{dt} = 0. \tag{3.9}$$

[Proof-1]

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{1}{4\pi}\frac{\partial}{\partial t}\nabla\cdot\vec{E}\\ &= \frac{1}{4\pi}\nabla\cdot\frac{\partial\vec{E}}{\partial t}\\ &= \frac{1}{4\pi}\left(c\nabla\times\vec{B}-4\pi\vec{j}\right)\\ &= \frac{c}{4\pi}\nabla\cdot\nabla\times\vec{B}-\nabla\cdot\vec{j}\\ &= -\nabla\cdot\vec{j},\end{aligned}$$

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\vec{j}=0.$$

[Proof-2]

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \sum_\alpha q_\alpha\delta'(\vec{x}-\vec{x}_\alpha(t))\cdot[-\frac{d\vec{x}_\alpha(t)}{dt}]\\ &= -\sum_\alpha q_\alpha\delta'(\vec{x}-\vec{x}_\alpha(t))\vec{v}_\alpha(t),\end{aligned}$$

$$\nabla\cdot\vec{j}=\sum_\alpha q_\alpha\delta'(\vec{x}-\vec{x}_\alpha(t))\vec{v}_\alpha(t),$$

likewise,

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\vec{j}=0.$$

$$\frac{dQ}{dt}=\int_V d^3\vec{x}\frac{\partial\rho}{\partial t}=-\int_V d^3\vec{x}\nabla\cdot\vec{j}=-\oint_S\vec{j}\cdot d\vec{S}=0.$$

[EOP]

3.1.3 Total energy of the charged particles and the electromagnetic fields

$$H=\sum_\alpha\frac{1}{2}m_\alpha\vec{v}_\alpha^2(t)+\frac{1}{8\pi}\int d^3\vec{x}[\vec{E}^2(\vec{x},t)+\vec{B}^2(\vec{x},t)], \tag{3.10}$$

$$\frac{dH}{dt}=0. \tag{3.11}$$

[Proof]

$$\begin{aligned}\left(\frac{dH}{dt}\right)_P &= \sum_\alpha m_\alpha\vec{v}_\alpha\cdot\dot{\vec{v}}_\alpha(t) \qquad (3.12)\\ &= \sum_\alpha q_\alpha\vec{v}_\alpha\cdot\left[\vec{E}(\vec{x}_\alpha(t),t)+\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c}\right]\\ &= \sum_\alpha q_\alpha\vec{v}_\alpha\cdot\vec{E}(\vec{x}_\alpha(t),t),\end{aligned}$$

where we have used the identity

$$\vec{v}\cdot\vec{v}\times\vec{B}=0.$$

Actually, the above identity can be easily verified using Levi-Civita tensor (ϵ_{ijk}):

$$\begin{aligned}\vec{v}\cdot\vec{v}\times\vec{B} &= v_i\epsilon_{ijk}v_jB_k\\ &= \epsilon_{ijk}v_iv_jB_k\\ &= \frac{1}{2}\epsilon_{ijk}(v_iv_j-v_jv_i)B_k\\ &= 0.\end{aligned}$$

That is to say, Lorentz force does not contribute to work.

$$\begin{aligned}\left(\frac{dH}{dt}\right)_f &= \frac{1}{8\pi}\int d^3\vec{x}\left[2\vec{E}(\vec{x},t)\cdot\frac{\partial\vec{E}(\vec{x},t)}{\partial t}+2\vec{B}(\vec{x},t)\cdot\frac{\partial\vec{B}(\vec{x},t)}{\partial t}\right] \\ &= \frac{1}{4\pi}\int d^3\vec{x}\left\{\vec{E}(\vec{x},t)\cdot\left[c\nabla\times\vec{B}(\vec{x},t)-4\pi\vec{j}(\vec{x},t)\right]\right.\\ &\quad\left.-\vec{B}(\vec{x},t)\cdot\left[c\nabla\times\vec{E}(\vec{x},t)\right]\right\}.\end{aligned} \tag{3.13}$$

Noting that

$$\nabla\cdot(\vec{E}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{E})-\vec{E}\cdot(\nabla\times\vec{B}),\quad \text{(verifity, using } \epsilon_{\rm ijk}\text{, Problem 4.1)}$$

one has

$$\left(\frac{dH}{dt}\right)_f=\frac{1}{4\pi}\int d^3\vec{x}\left[-c\nabla\cdot(\vec{E}\times\vec{B})-4\pi\vec{j}\cdot\vec{E}\right], \tag{3.14}$$

in which

$$-\frac{c}{4\pi}\int d^3\vec{x}\nabla\cdot(\vec{E}\times\vec{B})=-\frac{c}{4\pi}\oint_S\vec{E}\times\vec{B}\cdot d\vec{S}=0, \tag{3.15}$$

$$\begin{aligned}\frac{1}{4\pi}\int d^3\vec{x}(-4\pi\vec{j}\cdot\vec{E}) &= -\int d^3\vec{x}\vec{j}\cdot\vec{E} \\ &= -\int d^3\vec{x}\sum_\alpha q_\alpha\vec{v}_\alpha(t)\delta(\vec{x}-\vec{x}_\alpha(t))\cdot\vec{E}(\vec{x},t)\\ &= -\sum_\alpha q_\alpha\vec{v}_\alpha(t)\cdot\vec{E}(\vec{x}_\alpha(t),t).\end{aligned} \tag{3.16}$$

So,

$$\left(\frac{dH}{dt}\right)_f=-\sum_\alpha q_\alpha\vec{v}_\alpha(t)\cdot\vec{E}_\alpha(\vec{x}_\alpha(t),t). \tag{3.17}$$

Compacting with $(dH/dt)_P$, one has

$$\frac{dH}{dt}=0.$$

[EOP]

3.1.4 Total linear momentum of the charge particles and the electromagnetic fields

$$\vec{P} = \sum_\alpha m_\alpha \vec{v}_\alpha(t) + \frac{1}{4\pi c}\int d^3\vec{x}[\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)], \tag{3.18}$$

$$\frac{d\vec{P}}{dt} = 0. \tag{3.19}$$

[Proof]

$$\left(\frac{d\vec{P}}{dt}\right)_P = \sum_\alpha m_\alpha \dot{\vec{v}}_\alpha = \sum_\alpha q_\alpha\left[\vec{E}(\vec{x}_\alpha(t),t) + \frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c}\right]. \tag{3.20}$$

$$\begin{aligned}\left(\frac{d\vec{P}}{dt}\right)_f &= \frac{1}{4\pi c}\int d^3\vec{x}\left[\frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}\right] \\ &= \frac{1}{4\pi c}\int d^3\vec{x}\left[(c\nabla\times\vec{B} - 4\pi\vec{j})\times\vec{B} + \vec{E}\times(-c\nabla\times\vec{E})\right].\end{aligned} \tag{3.21}$$

Noting that

$$\begin{aligned}(\nabla\times\vec{B})\times\vec{B} &= (\vec{B}\cdot\nabla)\vec{B} - (\nabla\vec{B})\cdot\vec{B} \\ &= (\vec{B}\cdot\nabla)\vec{B} - \frac{1}{2}\nabla B^2,\end{aligned} \tag{3.22}$$

$$\begin{aligned}-\vec{E}\times(\nabla\times\vec{E}) &= (\vec{E}\cdot\nabla)\vec{E} - (\nabla\vec{E})\cdot\vec{E} \\ &= (\vec{E}\cdot\nabla)\vec{E} - \frac{1}{2}\nabla E^2,\end{aligned} \tag{3.23}$$

(verify Eqs. (3.22) and (3.23), Problem 4.2)
Also, since

$$\begin{aligned}\nabla\cdot(\vec{E}\vec{E}) &= (\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E} \\ &= 4\pi\rho\vec{E} + (\vec{E}\cdot\nabla)\vec{E},\end{aligned} \tag{3.24}$$

$$\begin{aligned}\nabla\cdot(\vec{B}\vec{B}) &= (\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B} \\ &= (\vec{B}\cdot\nabla)\vec{B}.\end{aligned} \tag{3.25}$$

(verify Eqs. (3.24) and (3.25), Problem 4.3)
then

$$\begin{aligned}(\nabla\cdot\vec{E})\vec{E} - \vec{E}\times(\nabla\times\vec{E}) &= (\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E} - \frac{1}{2}\nabla E^2 \\ &= \nabla\cdot(\vec{E}\vec{E}) - \frac{1}{2}\nabla\cdot(IE^2) \\ &= \nabla\cdot(\vec{E}\vec{E} - \frac{1}{2}IE^2),\end{aligned} \tag{3.26}$$

$$\begin{aligned}(\nabla\cdot\vec{B})\vec{B}+(\nabla\times\vec{B})\times\vec{B} &= (\nabla\cdot\vec{B})\vec{B}+(\vec{B}\cdot\nabla)\vec{B}-\frac{1}{2}\nabla B^2 \\ &= \nabla\cdot(\vec{B}\vec{B})-\frac{1}{2}\nabla\cdot(IB^2) \\ &= \nabla\cdot(\vec{B}\vec{B}-\frac{1}{2}IB^2).\end{aligned} \tag{3.27}$$

so

$$\begin{aligned}(\nabla\times\vec{B})\times\vec{B} &= \nabla\cdot(\vec{B}\vec{B}-\frac{1}{2}IB^2)-(\nabla\cdot\vec{B})\vec{B} \\ &= \nabla\cdot(\vec{B}\vec{B}-\frac{1}{2}IB^2),\end{aligned} \tag{3.28}$$

$$\begin{aligned}(\nabla\times\vec{E})\times\vec{E} &= \nabla\cdot(\vec{E}\vec{E}-\frac{1}{2}IE^2)-(\nabla\cdot\vec{E})\vec{E} \\ &= \nabla\cdot(\vec{E}\vec{E}-\frac{1}{2}IE^2)-4\pi\rho\vec{E}.\end{aligned} \tag{3.29}$$

So

$$\begin{aligned}&(\nabla\times\vec{B})\times\vec{B}-\vec{E}\times(\nabla\times\vec{E}) \\ =\ &\nabla\cdot(\vec{E}\vec{E}+\vec{B}\vec{B}-\frac{1}{2}IE^2-\frac{1}{2}IB^2)-4\pi\rho\vec{E}.\end{aligned} \tag{3.30}$$

Therefore,

$$\begin{aligned}&\frac{1}{4\pi c}\int c\nabla\cdot(\vec{E}\vec{E}+\vec{B}\vec{B}-\frac{1}{2}IE^2-\frac{1}{2}IB^2)d^3\vec{x} \\ =\ &\frac{1}{4\pi}\int\nabla\cdot(\vec{E}\vec{E}+\vec{B}\vec{B}-\frac{1}{2}IE^2-\frac{1}{2}IB^2)d^3\vec{x} \\ =\ &\frac{1}{4\pi}\oint_S(\vec{E}\vec{E}+\vec{B}\vec{B}-\frac{1}{2}IE^2-\frac{1}{2}IB^2)\cdot d\vec{S} \\ =\ &0,\end{aligned} \tag{3.31}$$

$$\begin{aligned}\frac{1}{4\pi c}\int c(-4\pi\rho\vec{E})d^3\vec{x} &= -\int d^3\vec{x}\sum_\alpha q_\alpha\delta(\vec{x}-\vec{x}_\alpha(t))\vec{E}(\vec{x}) \\ &= -\sum_\alpha q_\alpha\vec{E}(\vec{x}_\alpha(t),t),\end{aligned} \tag{3.32}$$

$$\begin{aligned}\frac{1}{4\pi c}\int(-4\pi\vec{j}\times\vec{B})d^3\vec{x} &= -\frac{1}{c}\int d^3\vec{x}\vec{j}\times\vec{B} \\ &= -\frac{1}{c}\int d^3\vec{x}\sum_\alpha q_\alpha\vec{v}_\alpha(t)\delta(\vec{x}-\vec{x}_\alpha(t))\times\vec{B}(\vec{x},t) \\ &= -\sum_\alpha q_\alpha\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c},\end{aligned} \tag{3.33}$$

$$\begin{aligned}\frac{d\vec{P}}{dt} &= \sum_\alpha q_\alpha\left[\vec{E}(\vec{x}_\alpha(t),t)+\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c}\right] \\ &\quad -\sum_\alpha q_\alpha\vec{E}(\vec{x}_\alpha(t),t)-\sum_\alpha q_\alpha\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c} \\ &= 0.\end{aligned} \tag{3.34}$$

[EOP]

3.1.5 Total angular momentum of the charged particles and electromagnetic fields

$$\vec{J}=\sum_\alpha \vec{x}_\alpha(t)\times m_\alpha\vec{v}_\alpha+\frac{1}{4\pi c}\int d^3\vec{x}\left[\vec{x}\times\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)\right]. \tag{3.35}$$

$$\frac{d\vec{J}}{dt}=0. \tag{3.36}$$

[Proof]

$$\begin{aligned}\left(\frac{d\vec{J}}{dt}\right)_P &= \sum_\alpha[\vec{v}_\alpha(t)\times m_\alpha\vec{v}_\alpha(t)+\vec{x}_\alpha(t)\times m_\alpha\dot{\vec{v}}_\alpha(t)] \\ &= \sum_\alpha \vec{x}_\alpha(t)\times\left\{q_\alpha\left[\vec{E}(\vec{x}_\alpha(t),t)+\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c}\right]\right\} \\ &= \sum_\alpha\left[q_\alpha\vec{x}_\alpha(t)\times\vec{E}(\vec{x}_\alpha(t),t)+q_\alpha\frac{\vec{x}_\alpha(t)\times\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c}\right].\end{aligned} \tag{3.37}$$

$$\begin{aligned}\left(\frac{d\vec{J}}{dt}\right)_f &= \frac{1}{4\pi c}\int d^3\vec{x}\left[\vec{x}\times\dot{\vec{E}}\times\vec{B}+\vec{x}\times\vec{E}\times\dot{\vec{B}}\right] \\ &= \frac{1}{4\pi c}\int d^3\vec{x}\left[\vec{x}\times(c\nabla\times\vec{B}-4\pi\vec{j})\times\vec{B}+\vec{x}\times\vec{E}\times(-c\nabla\times\vec{E})\right] \\ &= \frac{1}{4\pi}\int d^3\vec{x}\left\{\vec{x}\times\left[(\nabla\times\vec{B})\times\vec{B}+(\nabla\times\vec{E})\times\vec{E}\right]\right\}-\frac{1}{c}\int d^3\vec{x}(\vec{x}\times\vec{j}\times\vec{B}).\end{aligned} \tag{3.38}$$

Since, from Eq. (3.30), one has

$$\begin{aligned}&(\nabla\times\vec{B})\times\vec{B}+(\nabla\times\vec{E})\times\vec{E} \\ &= \nabla\cdot(\vec{E}\vec{E}+\vec{B}\vec{B}-\frac{1}{2}IE^2-\frac{1}{2}IB^2)-4\pi\rho\vec{E},\end{aligned} \tag{3.39}$$

where

$$\begin{aligned}
&\frac{1}{4\pi c}\int d^3\vec{x}c\vec{x}\times\left[(\nabla\times\vec{B})\times\vec{B}+(\nabla\times\vec{E})\times\vec{E}\right] \\
=\ &\frac{1}{4\pi}\int d^3\vec{x}[\vec{x}\times(-4\pi\rho\vec{E})] \\
=\ &-\int d^3\vec{x}\vec{x}\times\sum_\alpha q_\alpha\delta(\vec{x}-\vec{x}_\alpha(t))\times\vec{E}(\vec{x},t) \\
=\ &-\sum_\alpha q_\alpha\vec{x}_\alpha(t)\times\vec{E}(\vec{x}_\alpha(t),t),
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
&-\frac{1}{c}\int d^3\vec{x}(\vec{x}\times\vec{j}\times\vec{B}) \\
=\ &-\frac{1}{c}\int d^3\vec{x}\vec{x}\times\sum_\alpha q_\alpha\vec{v}_\alpha(t)\times\vec{B}\delta(\vec{x}-\vec{x}_\alpha(t)) \\
=\ &-\sum_\alpha q_\alpha\frac{\vec{x}_\alpha(t)\times\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c},
\end{aligned} \tag{3.41}$$

So

$$\left(\frac{d\vec{J}}{dt}\right)=\left(\frac{d\vec{J}}{dt}\right)_P+\left(\frac{d\vec{J}}{dt}\right)_f=0 \tag{3.42}$$

[EOP]

3.2 Transverse and longitudinal properties of electromagnetic fields

3.2.1 Fourier Transform

$$F(\vec{x})=\frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\tilde{F}(\vec{k})e^{i\vec{k}\cdot\vec{x}}, \tag{3.43}$$

i.e.,

$$F(\vec{x})\quad\overset{\mathcal{F}}{\longleftrightarrow}\quad\tilde{F}(\vec{k}),$$

$$\tilde{F}(\vec{k})=\frac{1}{(2\pi)^{3/2}}\int d^3\vec{x}F(\vec{x})e^{-i\vec{k}\cdot\vec{x}}, \tag{3.44}$$

i.e.,

$$\tilde{F}(\vec{k})\quad\overset{\mathcal{F}^{-1}}{\longleftrightarrow}\quad F(\vec{x}),$$

$$\int d^3\vec{k}e^{\pm i\vec{k}\cdot\vec{x}}=(2\pi)^3\delta(\vec{x}). \tag{3.45}$$

3.2.2 General relations

Helmholtz's theorem:
$\forall \vec{V}(\vec{x})$, it can be decomposed into two parts, i.e., transverse and longitudinal parts,

$$\vec{V}(\vec{x}) = \vec{V}_{\perp}(\vec{x}) + \vec{V}_{\parallel}(\vec{x}), \tag{3.46}$$

and at the some time, transverse and longitudinal parts, which satisfy, respectively

$$\nabla \cdot \vec{V}_{\perp}(\vec{x}) = 0, \tag{3.47}$$

$$\nabla \times \vec{V}_{\parallel}(\vec{x}) = 0. \tag{3.48}$$

So

$$\vec{V}(\vec{x}) = \vec{V}_{\perp}(\vec{x}) + \vec{V}_{\parallel}(\vec{x}), \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \vec{\tilde{V}}(\vec{k}) = \vec{\tilde{V}}_{\perp}(\vec{k}) + \vec{\tilde{V}}_{\parallel}(\vec{k}), \tag{3.49}$$

$$\nabla \cdot \vec{V}_{\perp}(\vec{x}) = 0, \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \vec{k} \cdot \vec{\tilde{V}}_{\perp}(\vec{k}) = 0, \tag{3.50}$$

$$\nabla \times \vec{V}_{\parallel}(\vec{x}) = 0, \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \vec{k} \times \vec{\tilde{V}}_{\parallel}(\vec{k}) = 0, \tag{3.51}$$

$$\vec{\tilde{V}}_{\parallel}(\vec{k}) = \vec{k}^0(\vec{k}^0 \cdot \vec{\tilde{V}}(\vec{k})) = \vec{k}^0\vec{k}^0 \cdot \vec{\tilde{V}}(\vec{k}), \tag{3.52}$$

$$\vec{\tilde{V}}_{\perp}(\vec{k}) \;=\; \vec{\tilde{V}}(\vec{k}) - \vec{\tilde{V}}_{\parallel}(\vec{k}) = (I - \vec{k}^0\vec{k}^0) \cdot \vec{\tilde{V}}(\vec{k}) = O(\vec{k}^0) \cdot \vec{\tilde{V}}(\vec{k}), \tag{3.53}$$

$$\nabla \times \nabla \times \vec{V}(\vec{x}) \;=\; \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V} \tag{3.54}$$

$$\overset{\mathcal{F}}{\longleftrightarrow} \quad i\vec{k} \times i\vec{k} \times \vec{\tilde{V}}(\vec{k}) = i\vec{k}(i\vec{k} \cdot \vec{\tilde{V}}(\vec{k})) - (-k^2)\vec{\tilde{V}}(\vec{k}) \tag{3.55}$$

i.e.,

$$\begin{aligned} \vec{k} \times \vec{k} \times \vec{\tilde{V}}(\vec{k}) &= k^2 I \cdot \vec{\tilde{V}}(\vec{k}) - k^2 \vec{k}^0\vec{k}^0 \cdot \vec{\tilde{V}}(\vec{k}) \\ &= k^2 O(\vec{k}^0) \cdot \vec{\tilde{V}}(\vec{k}), \end{aligned} \tag{3.56}$$

therefore

$$\vec{k}^0 \times \vec{k}^0 \times \vec{\tilde{V}}(\vec{k}) \;=\; O(\vec{k}^0) \cdot \vec{\tilde{V}}(\vec{k}) = \vec{\tilde{V}}_{\perp}(\vec{k}). \tag{3.57}$$

$$\vec{\tilde{V}}_{\perp}(\vec{k}) = (I - \vec{k}^0\vec{k}^0) \cdot \vec{\tilde{V}}(\vec{k}), \tag{3.58}$$

$$\begin{aligned}\vec{V}_\perp(\vec{x}) &= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\tilde{\vec{V}}_\perp(\vec{k})e^{i\vec{k}\cdot\vec{x}} \\ &= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}(I-\vec{k}^0\vec{k}^0)\cdot\int \vec{V}(\vec{x'})e^{-i\vec{k}\cdot\vec{x'}}e^{i\vec{k}\cdot\vec{x}}d^3\vec{x'} \\ &= \frac{1}{(2\pi)^3}\int d^3\vec{k}d^3\vec{x'}(I-\vec{k}^0\vec{k}^0)\cdot\vec{V}(\vec{x'})e^{-i\vec{k}\cdot(\vec{x'}-\vec{x})},\end{aligned} \tag{3.59}$$

$$\begin{aligned}\vec{V}_{\perp i}(\vec{x}) &= \frac{1}{(2\pi)^3}\int d^3\vec{k}d^3\vec{x'}(\delta_{ij}-k_i^0k_j^0)\vec{V}_j(\vec{x'})e^{i\vec{k}\cdot(\vec{x}-\vec{x'})} \\ &= \sum_j\int d^3\vec{x}'\delta_{ij}^\perp(\vec{x}-\vec{x}')V_j(\vec{x}'),\end{aligned} \tag{3.60}$$

where

$$\frac{1}{(2\pi)^3}\int d^3\vec{k}\delta_{ij}\vec{V}_j(\vec{x'})e^{i\vec{k}\cdot(\vec{x}-\vec{x'})} = \delta_{ij}\delta(\vec{x}-\vec{x'})\vec{V}(\vec{x'}). \tag{3.61}$$

$$\delta_{ij}^\perp(\vec{x}) = \frac{2}{3}\delta_{ij}\delta(\vec{x}) - \frac{1}{4\pi r^3}(\delta_{ij}-\frac{3x_ix_j}{r^2}). \tag{3.62}$$

Next, evaluate

$$-\frac{1}{(2\pi)^3}\int d^3\vec{k}k_i^0k_j^0\vec{V}_j(\vec{x'})e^{i\vec{k}\cdot(\vec{x}-\vec{x'})} \tag{3.63}$$

[Proof: (Not finished) consult Quantum field Theory]

3.2.3 Maxwell-Lorentz equations in configuration and reciprocal space

$$\nabla\cdot\vec{D}(\vec{x},t) = 4\pi\rho(\vec{x},t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad i\vec{k}\cdot\tilde{\vec{D}}(\vec{k},t) = 4\pi\tilde{\rho}(\vec{k},t), \tag{3.64}$$

$$\nabla\cdot\vec{B}(\vec{x},t) = 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \vec{k}\cdot\tilde{\vec{B}}(\vec{k},t) = 0, \tag{3.65}$$

$$\nabla\times\vec{E}(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}(\vec{x},t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad i\vec{k}\times\tilde{\vec{E}}(\vec{k},t) = -\frac{1}{c}\dot{\tilde{\vec{B}}}(\vec{k},t), \tag{3.66}$$

$$\begin{aligned}\nabla\times\vec{H}(\vec{x},t) &= \frac{1}{c}\frac{\partial}{\partial t}\vec{D}(\vec{x},t) + \frac{4\pi}{c}\vec{j}(\vec{x},t), \\ \overset{\mathcal{F}}{\longleftrightarrow} \quad i\vec{k}\times\tilde{\vec{H}}(\vec{k},t) &= \frac{1}{c}\dot{\tilde{\vec{D}}}(\vec{k},t) + \frac{4\pi}{c}\tilde{\vec{j}}(\vec{k},t),\end{aligned} \tag{3.67}$$

$$\rho(\vec{x},t) = \sum_\alpha q_\alpha\delta(\vec{x}-\vec{x}_\alpha(t)), \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \tilde{\rho}(\vec{k},t) = \frac{1}{(2\pi)^{3/2}}\sum_\alpha q_\alpha e^{-i\vec{k}\cdot\vec{x}_\alpha(t)}, \tag{3.68}$$

$$\vec{j}(\vec{x},t) = \sum_\alpha q_\alpha\vec{v}_\alpha(t)\delta(\vec{x}-\vec{x}_\alpha(t), \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \tilde{\vec{j}}(\vec{k},t) = \frac{1}{(2\pi)^{3/2}}\sum_\alpha q_\alpha\vec{v}_\alpha(t)e^{-i\vec{k}\cdot\vec{x}_\alpha(t)}, \tag{3.69}$$

$$m\ddot{\vec{x}}_\alpha = q_\alpha \vec{E}(\vec{x}_\alpha(t), t) + \frac{\vec{v}_\alpha(t) \times \vec{B}(\vec{x}_\alpha(t), t)}{c} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{(No reciprocal correspondence)}, \tag{3.70}$$

$$\frac{\partial \rho(\vec{x}, t)}{\partial t} + \nabla \cdot \vec{j}(\vec{x}, t) = 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \dot{\tilde{\rho}}(\vec{k}, t) + i\vec{k} \cdot \tilde{\vec{j}}(\vec{k}, t) = 0, \tag{3.71}$$

$$\begin{aligned} H_{em} &= \frac{1}{8\pi} \int d^3\vec{x} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \\ &= \frac{1}{8\pi} \int d^3\vec{k} [\tilde{\vec{E}}(\vec{k}, t) \cdot \tilde{\vec{D}}^*(\vec{k}, t) + \tilde{\vec{H}}(\vec{k}, t) \cdot \tilde{\vec{B}}^*(\vec{k}, t)], \end{aligned} \tag{3.72}$$

$$\begin{aligned} \vec{P}_{em} &= \frac{1}{4\pi c} \int d^3\vec{x} [\vec{E}(\vec{x}, t) \times \vec{H}(\vec{x}, t)] \\ &= \frac{1}{4\pi c} \int d^3\vec{k} [\tilde{\vec{E}}(\vec{k}, t) \times \tilde{\vec{H}}^*(\vec{k}, t)], \end{aligned} \tag{3.73}$$

$$\vec{J}_{em} = \frac{1}{4\pi c} \int d^3\vec{x} \vec{x} \times (\vec{E} \times \vec{H}) \tag{3.74}$$

3.2.4 Fourier Transform in space-time

$$F(\vec{x}, t) = \frac{1}{(2\pi)^2} \int d^3\vec{k} d\omega \tilde{F}(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \tag{3.75}$$

i.e.,

$$F(\vec{x}, t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \tilde{F}(\vec{k}, \omega), \tag{3.76}$$

$$\tilde{F}(\vec{k}, \omega) = \frac{1}{(2\pi)^2} \int d^3\vec{x} dt F(\vec{x}, t) e^{-i(\vec{k} \cdot \vec{x} - \omega t)} \tag{3.77}$$

i.e.,

$$\tilde{F}(\vec{k}, \omega) \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad F(\vec{x}, t), \tag{3.78}$$

$$\nabla \cdot \vec{D}(\vec{x}, t) = 4\pi \rho(\vec{x}, t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad i\vec{k} \cdot \tilde{\vec{D}}(\vec{k}, \omega) = 4\pi \tilde{\rho}(\vec{k}, \omega), \tag{3.79}$$

$$\nabla \cdot \vec{B}(\vec{x}, t) = 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \vec{k} \cdot \tilde{\vec{B}}(\vec{k}, \omega) = 0, \tag{3.80}$$

$$\nabla \times \vec{E}(\vec{x}, t) = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B}(\vec{x}, t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \vec{k} \times \tilde{\vec{E}}(\vec{k}, \omega) = \frac{\omega}{c} \tilde{\vec{B}}(\vec{k}, \omega), \tag{3.81}$$

$$\begin{aligned} &\nabla \times \vec{H}(\vec{x}, t) = \frac{1}{c} \frac{\partial}{\partial t} \vec{D}(\vec{x}, t) + \frac{4\pi}{c} \vec{j}(\vec{x}, t) \\ &\overset{\mathcal{F}}{\longleftrightarrow} \quad \vec{k} \times \tilde{\vec{H}}(\vec{k}, \omega) = -\frac{\omega}{c} \tilde{\vec{D}}(\vec{k}, t) - i\frac{4\pi}{c} \tilde{\vec{j}}(\vec{k}, t), \end{aligned} \tag{3.82}$$

$$\rho(\vec{x},t) = \sum_\alpha q_\alpha \delta(\vec{x} - \vec{x}_\alpha(t)), \tag{3.83}$$

$$\vec{j}(\vec{x},t) = \sum_\alpha q_\alpha \vec{v}_\alpha(t) \delta(\vec{x} - \vec{x}_\alpha(t)), \tag{3.84}$$

$$m\ddot{\vec{x}}_\alpha = q_\alpha \left[\vec{E}(\vec{x}_\alpha(t),t) + \frac{\vec{v}_\alpha(t) \times \vec{B}(\vec{x}_\alpha(t),t)}{c} \right]. \tag{3.85}$$

3.2.5 Gauge, gauge transformation and gauge invariance

$$\vec{B}(\vec{x},t) = \nabla \times \vec{A}(\vec{x},t), \tag{3.86}$$

$$\vec{E}(\vec{x},t) = -\nabla U(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x},t), \tag{3.87}$$

$$\vec{E}(\vec{x},t) = \vec{D}(\vec{x},t), \tag{3.88}$$

$$\vec{B}(\vec{x},t) = \vec{H}(\vec{x},t), \tag{3.89}$$

$$\vec{B}(\vec{x},t) = \vec{B}_\perp(\vec{x},t) + \vec{B}_\parallel(\vec{x},t), \tag{3.90}$$

$$\Rightarrow \begin{cases} \vec{B}_\perp(\vec{x},t) = \nabla \times \vec{A}_\perp(\vec{x},t), \\ \vec{B}_\parallel(\vec{x},t) = \nabla \times \vec{A}_\parallel(\vec{x},t) = 0 \end{cases} \tag{3.91}$$

$$\Rightarrow \vec{A}_\parallel(\vec{x},t) = \nabla\chi(\vec{x},t), \tag{3.92}$$

$$\vec{E}(\vec{x},t) = \vec{E}_\perp(\vec{x},t) + \vec{E}_\parallel(\vec{x},t) \tag{3.93}$$

$$\Rightarrow \vec{E}_\perp(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}_\perp(\vec{x},t), \tag{3.94}$$

$$\vec{E}_\parallel(\vec{x},t) = -\nabla U(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}_\parallel(\vec{x},t), \tag{3.95}$$

Gauge transform: under the gauge transform, i.e.,

$$\begin{aligned} \vec{A} \to \vec{A}'(\vec{x},t) &= \vec{A}(\vec{x},t) + \nabla\chi(\vec{x},t), \\ U \to \vec{U}'(\vec{x},t) &= U(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{x},t), \end{aligned} \tag{3.96}$$

we can show that $\vec{E}$ and $\vec{B}$ does not change. Also, since $\vec{A}_\parallel(\vec{x},t)$ play a role of gauge, so we know that

1. $\vec{E}_\perp(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x},t)$ is gange free, or in other words, $\vec{E}_\perp$ has nothing to do with gange.

2. $\vec{B}_{\parallel}(\vec{x},t) = 0$, i.e., the magnetic field is always transverse.

3. Longitudinal components of the electric field:
   Since
$$\nabla \cdot \vec{E}_{\parallel}(\vec{x},t) = 4\pi\rho(\vec{x},t),$$
   so
$$i\vec{k} \cdot \tilde{\vec{E}}_{\parallel}(\vec{k},t) = 4\pi\tilde{\rho}(\vec{k},t),$$
   then
$$\tilde{\vec{E}}_{\parallel}(\vec{k},t) = -i4\pi\tilde{\rho}(\vec{k},t)\frac{\vec{k}}{k^2}.$$
   Hence
$$\begin{aligned}
\tilde{\vec{E}}_{\parallel}(\vec{x},t) &= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}4\pi\tilde{\rho}(\vec{k},t)\left(\frac{-i\vec{k}}{k^2}\right)e^{i\vec{k}\cdot\vec{x}} \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}4\pi\left[\frac{1}{(2\pi)^{3/2}}\int \rho(\vec{x'},t)e^{-i\vec{k}\cdot\vec{x'}}d^3\vec{x'}\right] \\
&\quad \times\left[\frac{(2\pi)^{3/2}}{(2\pi)^{3/2}}\int \frac{\vec{x''}}{4\pi r''^3}e^{-i\vec{k}\cdot\vec{x''}}d^3\vec{x''}\right]e^{i\vec{k}\cdot\vec{x}} \\
&= \frac{1}{(2\pi)^3}\int \rho(\vec{x'},t)\frac{\vec{x''}}{r''^3}\left[\int d^3\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{x'}-\vec{x''})}\right]d^3\vec{x'}d^3\vec{x''} \\
&= \frac{1}{(2\pi)^3}\int \rho(\vec{x'},t)\frac{\vec{x''}}{r''^3}(2\pi)^3\delta(\vec{x''}-(\vec{x}-\vec{x'}))d^3\vec{x'}d^3\vec{x''} \\
&= \int \rho(\vec{x'},t)\frac{\vec{x}-\vec{x'}}{|\vec{x}-\vec{x'}|^3}d^3\vec{x'} \\
&= \int \sum_\alpha q_\alpha\delta(\vec{x'}-\vec{x}_\alpha(t))\frac{\vec{x}-\vec{x'}}{|\vec{x}-\vec{x'}|^3}d^3\vec{x'} \\
&= \sum_\alpha q_\alpha(t)\frac{\vec{x}-\vec{x}_\alpha(t)}{|\vec{x}-\vec{x}_\alpha(t)|^3} \\
&= -\nabla\sum_\alpha q_\alpha(t)\frac{1}{|\vec{x}-\vec{x}_\alpha(t)|}.
\end{aligned}$$

   The above expression indicates that the longitudinal electric field responds instantaneously to changes in the charge density which would seem to violate special relativity. The resolution of this problem lies in the fact that it is only the total electric field, longitudinal pulse transverse, that has a physical meaning, and the total electric field is always retarded.

4. In Coulomb gange, $\nabla\cdot\vec{A}(\vec{x},t)=0$, The vector potential is transverse, i.e., $\vec{A}_{\parallel}(\vec{x},t)=0$. Hence

$$\vec{E}_{\perp}(\vec{x},t)=-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x},t), \tag{3.97}$$

$$\vec{E}_{\parallel}(\vec{x},t)=-\nabla U(\vec{x},t). \tag{3.98}$$

Together with what we have just derived,

$$\vec{E}_{\parallel}(\vec{x},t)=\int\rho(\vec{x'},t)\frac{\vec{x}-\vec{x'}}{|\vec{x}-\vec{x'}|^3}d^3\vec{x'}=\sum_{\alpha}q_{\alpha}\frac{\vec{x}-\vec{x}_{\alpha}(t)}{|\vec{x}-\vec{x}_{\alpha}(t)|^3}, \tag{3.99}$$

one yields

$$U(\vec{x},t)=\int\frac{\rho(\vec{x'},t)}{|\vec{x}-\vec{x'}|}d^3\vec{x'}=\sum_{\alpha}q_{\alpha}\frac{1}{|\vec{x}-\vec{x}_{\alpha}(t)|}. \tag{3.100}$$

5. Coulomb electrostatic energy:

$$H_{long}=\frac{1}{8\pi}\int d^3\vec{x}\vec{E}_{\parallel}^2(\vec{x},t)=\frac{1}{8\pi}\int d^3\vec{k}|\tilde{\vec{E}}_{\parallel}(\vec{k},t)|^2.\ \text{(Parseval identity)} \tag{3.101}$$

Also, since

$$U(\vec{x},t)=\int\frac{\rho(\vec{x'},t)}{|\vec{x}-\vec{x'}|}d^3\vec{x'}=\sum_{\alpha}q_{\alpha}\frac{1}{|\vec{x}-\vec{x}_{\alpha}(t)|},$$

$$\tilde{\vec{E}}_{\parallel}(\vec{k},t)=-i4\pi\tilde{\rho}(\vec{k},t)\frac{\vec{k}}{k^2},$$

$$\vec{E}_{\parallel}(\vec{x},t)=\int\rho(\vec{x'},t)\frac{\vec{x}-\vec{x'}}{|\vec{x}-\vec{x'}|^3}d^3\vec{x'},$$

hence

$$H_{long}=2\pi\int d^3\vec{k}\frac{|\tilde{\rho}(\vec{k},t)|^2}{k^2}=\frac{1}{2}\int d^3\vec{x}\int d^3\vec{x'}\frac{\rho^*(\vec{x},t)\rho(\vec{x'},t)}{|\vec{x}-\vec{x'}|}. \tag{3.102}$$

[Proof]
From Parserval identify, one has

$$2\pi\int d^3\vec{k}|\tilde{\rho}(\vec{k},t)|^2\frac{1}{k^2}=2\pi\int d^3\vec{x}\mathcal{F}^{-1}\{|\tilde{\rho}(\vec{k},t)|^2\}\mathcal{F}^{-1}\left\{\frac{1}{k^2}\right\}.$$

Also, since

$$\frac{1}{(2\pi)^{3/2}}\frac{1}{k^2}=\mathcal{F}\left\{\frac{1}{4\pi r}\right\},\qquad \mathcal{F}^{-1}\left\{\frac{1}{k^2}\right\}=(2\pi)^{3/2}\frac{1}{4\pi r},$$

and

$$\begin{aligned}
\mathcal{F}^{-1}\Big\{|\tilde{\rho}(\vec{k},t)|^2\Big\} &= \frac{1}{(2\pi)^{3/2}}\int |\tilde{\rho}(\vec{k},t)|^2 e^{i\vec{k}\cdot\vec{x}} d^3\vec{k} \qquad (3.103)\\
&= \frac{1}{(2\pi)^{3/2}}\int \left[\frac{1}{(2\pi)^{3/2}}\int \rho(\vec{x'},t)e^{-i\vec{k}\cdot\vec{x'}}d^3\vec{x'}\right]\\
&\quad\times\left[\frac{1}{(2\pi)^{3/2}}\int \rho^*(\vec{x''},t)e^{i\vec{k}\cdot\vec{x''}}d^3\vec{x''}\right]e^{i\vec{k}\cdot\vec{x}}d^3\vec{k}\\
&= \frac{1}{(2\pi)^{3/2}}\frac{1}{(2\pi)^3}\int \rho(\vec{x'},t)\rho^*(\vec{x''},t)\left[\int e^{i\vec{k}\cdot(\vec{x}-\vec{x'}+\vec{x''})}d^3\vec{k}\right]d^3\vec{x'}d^3\vec{x''}\\
&= \frac{1}{(2\pi)^{3/2}}\int \rho(\vec{x'},t)\rho^*(\vec{x''},t)\delta(\vec{x}-\vec{x'}+\vec{x''})d^3\vec{x'}d^3\vec{x''}\\
&= \frac{1}{(2\pi)^{3/2}}\int \rho(\vec{x'},t)\rho^*(\vec{x'}-\vec{x},t)d^3\vec{x'},
\end{aligned}$$

one yields

$$\begin{aligned}
2\pi\int d^3\vec{k}|\tilde{\rho}(\vec{k},t)|^2\frac{1}{k^2} &= 2\pi\int d^3\vec{x}\left[\frac{1}{(2\pi)^{3/2}}\int \rho(\vec{x'},t)\rho^*(\vec{x'}-\vec{x},t)d^3\vec{x'}\right](2\pi)^{3/2}\frac{1}{4\pi r}\\
&= \frac{1}{2}\int d^3\vec{x}\int d^3\vec{x'}\frac{\rho^*(\vec{x},t)\rho(\vec{x'},t)}{|\vec{x}-\vec{x'}|}.
\end{aligned}$$

Further,

$$\begin{aligned}
H_{long} &= \frac{1}{2}\int d^3\vec{x}\int d^3\vec{x'}\frac{\rho^*(\vec{x},t)\rho(\vec{x'},t)}{|\vec{x}-\vec{x'}|} \qquad (3.104)\\
&= \frac{1}{2}\int d^3\vec{x}\int d^3\vec{x'}\frac{1}{|\vec{x}-\vec{x'}|}\sum_\alpha q_\alpha\delta(\vec{x}-\vec{x}_\alpha(t))\times\sum_\beta q_\beta\delta(\vec{x'}-\vec{x}_\beta(t))\\
&= \frac{1}{2}\sum_{\alpha,\beta}\frac{q_\alpha q_\beta}{|\vec{x}_\alpha(t)-\vec{x}_\beta(t)|}\\
&= \sum_\alpha \epsilon^\alpha_{\text{coul}} + \frac{1}{2}\sum_\alpha\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\vec{x}_\alpha(t)-\vec{x}_\beta(t)|},
\end{aligned}$$

in which $\epsilon^\alpha_{\text{coul}}$ is the Coulomb self-energy of the α^{th} charged particle, and the second term is simply the Coulomb potential between pairs of particle $\alpha \neq \beta$. One should just keep in mind that ϵ^α_{coul} = constant (this can only be treated in QED).

$$H_{coul} = V_{coul} = \frac{1}{2}\sum_\alpha\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\vec{x}_\alpha(t)-\vec{x}_\beta(t)|}, \qquad (3.105)$$

$$H_{total} = \frac{1}{2}\sum_\alpha m_\alpha\vec{v}_\alpha^2 + V_{coul} + H_{trans}, \qquad (3.106)$$

$$H_{trans} = \frac{1}{8\pi}\int d^3\vec{x}[\vec{E}^2_\perp(\vec{x},t)+\vec{B}^2(\vec{x},t)]. \qquad (3.107)$$

3.2.6 Transverse field

$$\nabla \times \vec{E}_\perp(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}_\perp(\vec{x},t), \tag{3.108}$$

$$\nabla \times \vec{B}_\perp(\vec{x},t) = \frac{4\pi}{c}\vec{j}_\perp(\vec{x},t) + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{x},t). \tag{3.109}$$

From the scalar and vector potential and their Fourier transforms, one has

$$\vec{E}_\perp(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}_\perp(\vec{x},t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \tilde{\vec{E}}_\perp(\vec{k},t) = -\frac{1}{c}\dot{\tilde{\vec{A}}}_\perp(\vec{k},t), \tag{3.110}$$

$$\vec{B}_\perp(\vec{x},t) = \nabla \times \vec{A}_\perp(\vec{x},t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \tilde{\vec{B}}_\perp(\vec{k},t) = -i\vec{k} \times \dot{\tilde{\vec{A}}}_\perp(\vec{k},t), \tag{3.111}$$

$$i\vec{k} \times i\vec{k} \times \tilde{\vec{A}}_\perp(\vec{k},t) = \frac{4\pi}{c}\tilde{\vec{j}}_\perp(\vec{k},t) - \frac{1}{c^2}\ddot{\tilde{\vec{A}}}_\perp(\vec{k},t),$$

therefore,

$$\frac{1}{c^2}\ddot{\tilde{\vec{A}}}_\perp(\vec{k},t) + k^2\tilde{\vec{A}}_\perp(\vec{k},t) = \frac{4\pi}{c}\tilde{\vec{j}}_\perp(\vec{k},t). \tag{3.112}$$

In real space,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A}_\perp(\vec{x},t) = \frac{4\pi}{c}\vec{j}_\perp(\vec{x},t). \tag{3.113}$$

3.2.7 Longitudinal field

In real space,

$$\left.\begin{aligned} \nabla \cdot \vec{D}_\parallel(\vec{x},t) &= 4\pi\rho, \\ 4\pi\vec{j}_\parallel(\vec{x},t) + \frac{\partial}{\partial t}\vec{D}_\parallel(\vec{x},t) &= 0. \end{aligned}\right\} \tag{3.114}$$

Hence

$$\frac{\partial}{\partial t}\rho(\vec{x},t) + \nabla \cdot \vec{j}_\parallel(\vec{x},t) = 0. \tag{3.115}$$

In vacuum, there is no polarization, hence, $\vec{D}(\vec{x},t) = \vec{E}(\vec{x},t)$, so one has

$$\left.\begin{aligned} i\vec{k} \cdot \tilde{\vec{E}}_\parallel(\vec{k},t) &= 4\pi\tilde{\rho}(\vec{k},t), \\ 4\pi\tilde{\vec{j}}_\parallel(\vec{k},t) + \frac{\partial}{\partial t}\tilde{\vec{E}}_\parallel(\vec{k},t) &= 0. \end{aligned}\right\} \tag{3.116}$$

Hence

$$\frac{\partial}{\partial t}\tilde{\rho}(\vec{k},t) + i\vec{k} \cdot \tilde{\vec{j}}_\parallel(\vec{k},t) = 0. \tag{3.117}$$

Since

$$\vec{E}_{\parallel}(\vec{x},t) = -\nabla U(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}_{\parallel}(\vec{x},t) \tag{3.118}$$

so

$$\tilde{\vec{E}}_{\parallel}(\vec{k},t) = -i\vec{k}\tilde{U}(\vec{k},t) - \frac{1}{c}\dot{\tilde{\vec{A}}}_{\parallel}(\vec{k},t), \tag{3.119}$$

$$i\vec{k}\cdot\tilde{\vec{E}}_{\parallel}(\vec{k},t) = k^2\tilde{U}(\vec{k},t) - i\frac{\vec{k}\cdot\dot{\tilde{\vec{A}}}_{\parallel}(\vec{k},t)}{c}. \tag{3.120}$$

$$k^2\tilde{U}(\vec{k},t) = 4\pi\tilde{\rho}(\vec{k},t) + i\frac{\vec{k}}{c}\cdot\dot{\tilde{\vec{A}}}_{\parallel}(\vec{k},t). \tag{3.121}$$

In Coulomb gange, $\vec{A}_{\parallel}(\vec{x},t) = \tilde{\vec{A}}_{\parallel}(\vec{k},t) = 0$, so we get Poisson equation

$$k^2\tilde{U}(\vec{k},t) = 4\pi\tilde{\rho}(\vec{k},t), \tag{3.122}$$

in real space

$$\nabla^2 U(\vec{x},t) = -4\pi\rho(\vec{x},t). \tag{3.123}$$

3.2.8 Maxwell's wave equations

Generally, the Maxwell's wave equations for electromagnetic field field holds:

$$\begin{aligned}\nabla^2\vec{E}(\vec{x},t) - \nabla[\nabla\cdot\vec{E}(\vec{x},t)] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{x},t) \\ = \frac{4\pi}{c}\frac{\partial}{\partial t}\vec{j}(\vec{x},t) + \frac{4\pi}{c}\frac{\partial^2}{\partial t^2}\vec{P}(\vec{x},t).\end{aligned} \tag{3.124}$$

[Proof]
Maxwell's equations read,

$$\nabla\cdot\vec{D}(\vec{x},t) = 4\pi\rho(\vec{x},t),$$

$$\nabla\cdot\vec{B}(\vec{x},t) = 0,$$

$$\nabla\times\vec{E}(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}(\vec{x},t),$$

$$\nabla\times\vec{H}(\vec{x},t) = \frac{4\pi}{c}\vec{j}(\vec{x},t) + \frac{1}{c}\frac{\partial}{\partial t}\vec{D}(\vec{x},t),$$

$$\vec{B}(\vec{x},t) = \vec{H}(\vec{x},t)$$

$$\vec{D}(\vec{x},t) = \vec{E}(\vec{x},t) + 4\pi\vec{P}(\vec{x},t).$$

$$\begin{aligned}
\nabla\times\nabla\times\vec{E} &= \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} \\
&= -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{B} \\
&= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\vec{D}\right) \\
&= -\frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\vec{E} + 4\pi\frac{\partial^2}{\partial t^2}\vec{P}\right).
\end{aligned}$$

$$\nabla^2\vec{E}(\vec{x},t) - \nabla[\nabla\cdot\vec{E}(\vec{x},t)] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{x},t) = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}(\vec{x},t) + \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P}(\vec{x},t).$$

On the other hand, from

$$\begin{cases}
\nabla\cdot\vec{D}_\parallel = 4\pi\rho, \\
\nabla\times\vec{E}_\perp = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}_\perp, \\
\nabla\times\vec{H}_\perp = \frac{1}{c}\frac{\partial}{\partial t}\vec{D}_\perp + \frac{4\pi}{c}\vec{j}_\perp, \\
4\pi\vec{j}_\parallel + \frac{\partial}{\partial t}\vec{D}_\parallel = 0,
\end{cases}$$

one has

$$\frac{\partial^2}{\partial t^2}\vec{D}_\parallel(\vec{x},t) + 4\pi\frac{\partial}{\partial t}\vec{j}_\parallel(\vec{x},t) = 0.$$

Also,

$$\begin{aligned}
\nabla\times\nabla\times\vec{E}_\perp &= \nabla(\nabla\cdot\vec{E}_\perp) - \nabla^2\vec{E}_\perp \\
&= -\nabla^2\vec{E}_\perp \\
&= -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{B}_\perp \\
&= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial}{\partial t}\vec{D}_\perp + \frac{4\pi}{c}\vec{j}_\perp\right) \\
&= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{D}_\perp - \frac{4\pi}{c}\frac{\partial}{\partial t}\vec{j}_\perp,
\end{aligned}$$

so

$$\nabla^2\vec{E}_\perp - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{D}_\perp - \frac{4\pi}{c}\frac{\partial}{\partial t}\vec{j}_\perp = 0,$$

i.e.,

$$\nabla^2\vec{E}_\perp - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_\perp - \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P}_\perp - \frac{4\pi}{c}\frac{\partial}{\partial t}\vec{j}_\perp = 0.$$

Then

$$\nabla^2\vec{E}_\perp - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{E}_\perp + \vec{E}_\parallel) - \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}(\vec{P}_\perp + \vec{P}_\parallel) - \frac{4\pi}{c}\frac{\partial}{\partial t}(\vec{j}_\perp + \vec{j}_\parallel) = 0, \tag{3.125}$$

$$\nabla^2\vec{E}_\perp - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P} + \frac{4\pi}{c}\frac{\partial}{\partial t}\vec{j} \tag{3.126}$$

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \nabla^2\vec{E}_\parallel = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P} + \frac{4\pi}{c}\frac{\partial}{\partial t}\vec{j}. \tag{3.127}$$

Next, evaluate $\vec{E}_\parallel$ related:
Since

$$\nabla\times\nabla\times\vec{E}_\parallel = \nabla(\nabla\cdot\vec{E}_\parallel) - \nabla^2\vec{E}_\parallel = 0, \tag{3.128}$$

so

$$\begin{aligned} -\nabla^2\vec{E}_\parallel &= -\nabla(\nabla\cdot\vec{E}_\parallel) \\ &= -\nabla[\nabla\cdot(\vec{E}-\vec{E}_\perp)] \\ &= -\nabla(\nabla\cdot\vec{E}). \end{aligned} \tag{3.129}$$

Hence

$$\nabla^2\vec{E} - \nabla(\nabla\cdot\vec{E}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P} + \frac{4\pi}{c}\frac{\partial}{\partial t}\vec{j}. \tag{3.130}$$

3.2.9 Newton-Lorentz equation in Coulomb gauge

Since

$$\vec{E}_\parallel = -\nabla U, \tag{3.131}$$

so

$$m_\alpha\ddot{\vec{x}}_\alpha(t) = -q_\alpha\nabla_{\vec{x}_\alpha}U(\vec{x}_\alpha,t) + q_\alpha\left[\vec{E}_\perp(\vec{x}_\alpha(t),t) + \frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{x}_\alpha(t),t)}{c}\right],$$

$$U(\vec{x}_\alpha,t) = U_{self} + \frac{1}{2}\sum_{\alpha\neq\beta}\frac{q_\beta}{|\vec{x}_\alpha(t)-\vec{x}_\beta(t)|}. \tag{3.132}$$

Since U_{self} = constant, so $\nabla_{\vec{x}_\alpha}U_{self} = 0$ and it does not paly a role in the Newton-Lorentz equation. Also, we can find that longitudinal field provide the Coulomb interaction between charge particles.

3.2.10 Charge and current densities:

Since

$$\rho(\vec{x},t) = \sum_\alpha q_\alpha \delta(\vec{x}-\vec{x}_\alpha(t)), \tag{3.133}$$

$$\vec{j}(\vec{x},t) = \sum_\alpha q_\alpha \vec{v}_\alpha(t)\delta(\vec{x}-\vec{x}_\alpha(t)). \tag{3.134}$$

So

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \sum_\alpha q_\alpha[\nabla_{\vec{x}}\delta(\vec{x}-\vec{x}_\alpha(t))]\cdot\left[-\frac{d\vec{x}_\alpha(t)}{dt}\right] \\ &= -\sum_\alpha q_\alpha \vec{v}_\alpha(t)\cdot\nabla_{\vec{x}}\delta(\vec{x}-\vec{x}_\alpha(t)), \end{aligned} \tag{3.135}$$

$$\begin{aligned} \nabla\cdot\vec{j}(\vec{x},t) &= \sum_\alpha q_\alpha \nabla_{\vec{x}}\cdot[\delta(\vec{x}-\vec{x}_\alpha(t))\vec{v}_\alpha(t)] \\ &= \sum_\alpha q_\alpha \vec{v}_\alpha(t)\cdot\nabla_{\vec{x}}\delta(\vec{x}-\vec{x}_\alpha(t)), \end{aligned} \tag{3.136}$$

then

$$\nabla\cdot\vec{j}(\vec{x},t) + \frac{\partial\rho(\vec{x},t)}{\partial t} = 0. \tag{3.137}$$

3.2.11 Potential

$$\nabla^2 U(\vec{x},t) = -4\pi\rho(\vec{x},t) \tag{3.138}$$

gives

$$-k^2\tilde{U}(\vec{k},t) = -4\pi\tilde{\rho}(\vec{k},t), \tag{3.139}$$

$$\tilde{U}(\vec{k},t) = 4\pi\frac{\tilde{\rho}(\vec{k},t)}{k^2} \tag{3.140}$$

$$\begin{aligned} U(\vec{x},t) &= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\,4\pi\frac{\tilde{\rho}(\vec{k},t)}{k^2}e^{i\vec{k}\cdot\vec{x}} \\ &= \frac{4\pi}{(2\pi)^{3/2}}\int d^3\vec{x''}\mathcal{F}^{-1}\left\{\frac{1}{k^2}\right\}\mathcal{F}^{-1}\{\tilde{\rho}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}\}, \end{aligned} \tag{3.141}$$

where

$$\mathcal{F}^{-1}\left\{\frac{1}{k^2}\right\} = (2\pi)^{3/2}\frac{1}{4\pi r''}, \tag{3.142}$$

$$\mathcal{F}^{-1}\{\tilde{\rho}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}\} = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\tilde{\rho}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}e^{-i\vec{k}\cdot\vec{x''}} = \rho(\vec{x}-\vec{x''},t). \tag{3.143}$$

$$\begin{aligned} U(\vec{x},t) &= \frac{4\pi}{(2\pi)^{3/2}}\int d^3\vec{x''}\frac{1}{4\pi r''}(2\pi)^{3/2}\rho(\vec{x}-\vec{x''},t) \\ &= \int d^3\vec{x'}\frac{\rho(\vec{x'},t)}{|\vec{x}-\vec{x'}|} \\ &= \sum_\alpha \frac{q_\alpha}{|\vec{x}-\vec{x}_\alpha(t)|}. \end{aligned} \tag{3.144}$$

Chapter 4

Special Relativistic Theory

4.1 Preliminavies of Differential geometry

- Coordinate transform:

$$d\xi^a = \frac{\partial \xi^a}{\partial x^\mu} dx^\mu, \tag{4.1}$$

- Proper element:

$$\begin{aligned} ds^2 &= d\xi_a d\xi^a & (4.2)\\ &= \eta_{ab} d\xi^a d\xi^b & (4.3)\\ &= \eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} dx^\mu dx^\nu & (4.4)\\ &= g_{\mu\nu} dx^\mu dx^\nu. & (4.5)\\ & & (4.6) \end{aligned}$$

- Metric: $\eta_{ab} =< \vec{e}_a, \vec{e}_b >$, $g_{\mu\nu} = \eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} =< \vec{e}_\mu, \vec{e}_\nu >$, are the metric of the manifold linear and curvilinear coordinates.

$$g = \det(g_{\mu\nu}).$$

- Volume element:

$$d^D \xi^a = \prod_a d\xi^a \wedge = \sqrt{g} \prod_\mu dx^\mu \wedge$$

Examples

$$\begin{aligned} \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z & (4.7)\\ &= \xi^1 \vec{e}_1 + \xi^2 \vec{e}_2 + \xi^3 \vec{e}_3 \qquad \text{(rectangular)} \\ &= r \sin\theta \cos\phi \vec{e}_1 + r \sin\theta \sin\phi \vec{e}_2 + r\cos\theta \vec{e}_3 \qquad \text{(spherical)} \\ &= \rho \cos\phi \vec{e}_1 + \rho \sin\phi \vec{e}_2 + z\vec{e}_3 \qquad \text{(cylindrical)} \end{aligned}$$

1. Spherical:

$$\begin{aligned}
\vec{e}_r &= \frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\vec{e}_1 + \sin\theta\sin\phi\vec{e}_2 + \cos\theta\vec{e}_3, \\
\vec{e}_\theta &= \frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\vec{e}_1 + r\cos\theta\sin\phi\vec{e}_2 - r\sin\theta\vec{e}_3, \\
\vec{e}_\phi &= \frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\vec{e}_1 + r\sin\theta\cos\phi\vec{e}_2,
\end{aligned}$$

$$\begin{aligned}
g_{rr} &= <\vec{e}_r, \vec{e}_r> = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1, \\
g_{r\theta} &= <\vec{e}_r, \vec{e}_\theta> = r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi - r\sin\theta\cos\theta = 0, \\
g_{\theta\theta} &= <\vec{e}_\theta, \vec{e}_\theta> = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2, \\
g_{\theta\phi} &= <\vec{e}_\theta, \vec{e}_\phi> = -r^2\sin\theta\cos\theta\sin\phi\cos\phi + r^2\sin\theta\cos\theta\sin\phi\cos\phi = 0, \\
g_{\phi\phi} &= <\vec{e}_\phi, \vec{e}_\phi> = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta, \\
g_{\phi r} &= <\vec{e}_\phi, \vec{e}_r> = -r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\sin\phi\cos\phi = 0.
\end{aligned}$$

Let $r = 1 \ \ \theta = 2 \ \ \phi = 3$, then

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad \det(g_{\mu\nu}) = r^4\sin^2\theta. \tag{4.8}$$

Hence

$$\begin{aligned}
d^3\vec{r} &= |d\xi^1 \wedge d\xi^2 \wedge d\xi^3| \qquad (4.9) \\
&= |\frac{\partial \xi^1}{\partial x^\mu} dx^\mu \wedge \frac{\partial \xi^2}{\partial x^\nu} dx^\nu \wedge \frac{\partial \xi^3}{\partial x^\lambda} dx^\lambda| \\
&= |\frac{\partial \xi^1}{\partial x^\mu}\frac{\partial \xi^2}{\partial x^\nu}\frac{\partial \xi^3}{\partial x^\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda| \\
&= |\epsilon^{\mu\nu\lambda}\frac{\partial \xi^1}{\partial x^\mu}\frac{\partial \xi^2}{\partial x^\nu}\frac{\partial \xi^3}{\partial x^\lambda} dx^1 \wedge dx^2 \wedge dx^3| \\
&= \sqrt{g}|dx^1 \wedge dx^2 \wedge dx^3|,
\end{aligned}$$

i.e.,

$$d^3\vec{r} = r^2\sin\theta dr d\theta d\phi. \tag{4.10}$$

2. Cylindrical

$$\begin{aligned}
\vec{e}_\rho &= \frac{\partial \vec{r}}{\partial \rho} = \cos\phi\vec{e}_1 + \sin\phi\vec{e}_2, \qquad (4.11) \\
\vec{e}_\phi &= \frac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\vec{e}_1 + \rho\cos\phi\vec{e}_2, \\
\vec{e}_z &= \frac{\partial \vec{r}}{\partial z} = \vec{e}_3,
\end{aligned}$$

$$
\begin{aligned}
g_{\rho\rho} &= <\vec{e}_\rho, \vec{e}_\rho> = \cos^2\phi + \sin^2\phi = 1, \\
g_{\phi\phi} &= <\vec{e}_\phi, \vec{e}_\phi> = \rho^2 \sin^2\phi + \rho^2\cos^2\phi = \rho^2, \\
g_{zz} &= <\vec{e}_z, \vec{e}_z> = 1, \\
g_{\rho\phi} &= <\vec{e}_\rho, \vec{e}_\phi> = -\rho\sin\phi\cos\phi + \rho\sin\phi\cos\phi = 0, \\
g_{\phi z} &= <\vec{e}_\phi, \vec{e}_z> = 0, \\
g_{z\rho} &= <\vec{e}_z, \vec{e}_\rho> = 0.
\end{aligned} \tag{4.12}
$$

So

$$
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \det(g_{\mu\nu}) = \rho^2. \tag{4.13}
$$

$$
d^3\vec{r} = \sqrt{g}\, d\rho d\phi dz = \rho d\rho d\phi dz. \tag{4.14}
$$

4.2 Fundamentals of relativistic theory

1. Coordinates:

$$
\xi^a = (\xi^0, \vec{\xi}) = (ct, \vec{\xi}), \qquad \xi_a = \eta_{ab}\xi^b = (\xi_0, \vec{\xi}) = (-ct, \vec{\xi}), \tag{4.15}
$$

where $\eta_{ab} = (-1, 1, 1, 1)$.

2. Momentum:

$$
p^a = (p^0, \vec{p}) = \left(\frac{E}{c}, \vec{p}\right), \qquad p_a = \eta_{ab}p^b = (p_0, \vec{p}) = \left(-\frac{E}{c}, \vec{p}\right). \tag{4.16}
$$

3. Energy:
From the definition of invariant proper time

$$
d\tau^2 = dt^2 - \frac{d\vec{\xi}^2}{c^2}, \tag{4.17}
$$

one has

$$
d\tau = dt\sqrt{1 - \frac{\vec{v}^2}{c^2}} = dt\gamma^{-1}, \tag{4.18}
$$

so

$$
\frac{dt}{d\tau} = \gamma = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}. \tag{4.19}
$$

Now, evaluate $p_a p^a$:

$$
\begin{aligned}
p_a p^a &= \eta_{ab} p_a p^a \\
&= -(p^0)^2 + \vec{p}^2 \\
&= -\frac{E^2}{c^2} + \vec{p}^2.
\end{aligned}
$$

On the other hand

$$p^a = m\frac{d\xi^a}{d\tau} = m\frac{d\xi^a}{d\xi^0}\frac{d\xi^0}{d\tau} = m\frac{d\xi^a}{dt}\frac{dt}{d\tau} = m\gamma\frac{d\xi^a}{dt} = mu^a, \tag{4.20}$$

$$u^a = \frac{d\xi^a}{d\tau} = \frac{d\xi^a}{dt}\frac{dt}{d\tau} = \gamma\frac{d\xi^a}{dt},$$

where

$$u^0 = \gamma\frac{d\xi^0}{dt} = \gamma c,$$
$$\vec{u} = \gamma\frac{d\vec{\xi}}{dt} = \gamma\vec{v}.$$

Hence

$$p^0 = m\gamma c, \qquad \vec{p} = m\gamma\vec{v}.$$

Therefore

$$p_a p^a = m^2 u_a u^a = -m^2\gamma^2(c^2 - \vec{v}^2) = -m^2c^2,$$
$$u_a u^a = -c^2.$$

Therefore, one has

$$-m^2c^2 = -\frac{E^2}{c^2} + \vec{p}^2,$$

i.e.,

$$E^2 = \vec{p}^2c^2 + m^2c^4,$$

where E is the total mass-energy, mc^2 is the rest mass-energy and $|\vec{p}c|$ is the kinetic energy of the particle.

4. Two invariant variables in Lorentz transform:

   (a) Four-dimensional Dirac-δ function:

   $$d^4\xi = |d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3| = d^4\xi' = |d\xi'^0 \wedge d\xi'^1 \wedge d\xi'^2 \wedge d\xi'^3|,$$

   where $\xi'^a = \Lambda^a_{.b}\xi^b$, $\xi'^a_\alpha = \Lambda^a_{.b}\xi^b_\alpha$, $\Lambda^a_{.b}$ is the matrix element of the Lorentz transform.

   (b) Four dimensional Volume element:

   $$\delta^4(\xi - \xi_\alpha) = \delta^4(\xi' - \xi'_\alpha),$$

[Proof]

(a)

$$\begin{aligned} d^4\xi' &= |d\xi'^0 \wedge d\xi'^1 \wedge d\xi'^2 \wedge d\xi'^3| \\ &= |\Lambda^0_{.a}\Lambda^1_{.b}\Lambda^2_{.c}\Lambda^3_{.d} d\xi^a \wedge d\xi^b \wedge d\xi^c \wedge d\xi^d| \\ &= \epsilon^{abcd}\Lambda^0_{.a}\Lambda^1_{.b}\Lambda^2_{.c}\Lambda^3_{.d}|d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3| \\ &= (\det\Lambda^a_{.b})d^4\xi \\ &= d^4\xi, \end{aligned}$$

where we have used the identify $\det(\Lambda^a_{.b}) = 1$.

(b)

$$\begin{aligned} \delta^4(\xi - \xi_\alpha) &= \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{ik_a(\xi^a - \xi^a_\alpha)} d^4k \\ &= \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{ik_a \bar{\wedge}^a_{b.}(\xi'^b - \xi'^b_\alpha)} d^4k \\ &= \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{ik'_a(\xi'^a - \xi'^a_\alpha)} \det(\bar{\wedge}^b_{a.}) d^4k' \\ &= \delta^4(\xi' - \xi'_\alpha), \end{aligned}$$

where we have used the identity $d^4k = \det(\bar{\wedge}^b_{a.})d^4k'$, and $\det(\bar{\wedge}^b_{a.}) = 1$.

[EOP]

5. Energy-momentum density field of a particle:

$$\begin{aligned} T^{ab}(\xi) &= c\int_{-\infty}^{+\infty} \delta^4(\xi - \xi_\alpha(\tau)) m \frac{d\xi^a_\alpha}{d\tau}\frac{d\xi^b_\alpha}{d\tau} d\tau \\ &= c^2\delta^3(\vec{\xi} - \vec{\xi}_\alpha(t))\frac{p^a(t)p^b(t)}{E(t)}, \end{aligned}$$

where T^{ab} is called the energy-momentum density tensor of a particle, which is a 2^{nd} order contravariant tensor, i.e., $T'^{ab}(\xi) = \Lambda^a_{.c}\Lambda^b_{.d}T^{cd}(\xi)$. Then, the energy-momentum 4-vector is

$$p^a(t) = \frac{1}{c}\int_{V_\infty(t)} T^{a0}(\xi) d^3\vec{\xi} = m\frac{d\xi^a_\alpha}{d\tau},$$

and $V_\infty(t)$ denotes the total space at $\xi^0 = ct$. $p^a(t)$ is a contravariant vector, i.e.,

$$p'^a(t') = \Lambda^a_{.b}p^b(t)$$

[Proof]

(a)

$$\begin{aligned} T^{ab}(\xi) &= c\int_{-\infty}^{+\infty} \delta^4(\xi - \xi_\alpha(\tau)) m \frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau}\frac{d\tau}{d\xi^0} d\xi^0 \\ &= cm\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau}\frac{d\tau}{d\xi^0}\delta^3(\vec{\xi} - \vec{\xi}_\alpha(t)) \\ &= m\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau}\gamma^{-1}\delta^3(\vec{\xi} - \vec{\xi}_\alpha(t)). \end{aligned}$$

At the same time,

$$p^a = m\frac{d\xi_\alpha^a}{d\tau}, \qquad p^b = m\frac{d\xi_\alpha^b}{d\tau}, \qquad E = p^0 c = mc\frac{d\xi_\alpha^0}{d\tau} = mc^2\gamma,$$

so

$$c^2\frac{p^a p^b}{E} = m\gamma^{-1}\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau},$$

hence

$$T^{ab}(\xi) = c^2\delta^3(\vec{\xi} - \vec{\xi}_\alpha(t))\frac{p^a(t)p^b(t)}{E(t)}.$$

(b)

$$p^a(t) = m\frac{d\xi_\alpha^a}{d\tau}, \tag{4.21}$$

$$\frac{1}{c}\int_{V_{\infty(t)}} T^{a0}(\xi)d^3\vec{\xi} = \frac{1}{c}\int_{V_{\infty(t)}} c^2\delta^3(\vec{\xi} - \vec{\xi}_\alpha(t))\frac{p^a(t)p^0(t)}{E(t)}d^3\vec{\xi}.$$

Since

$$p^0(t) = \frac{E(t)}{c}, \qquad p^a = m\frac{d\xi^a}{d\tau},$$

So

$$\frac{1}{c}\int_{V_{\infty(t)}} T^{a0}(\xi)d^3\vec{\xi} = \int_{V_{\infty(t)}} \delta^3(\vec{\xi} - \vec{\xi}_\alpha(t))p^a(t)d^3\vec{\xi} = p^a(t) = m\frac{d\xi_\alpha^a}{d\tau}.$$

(c) Since t is a fixed number, $d\xi^0 = 0$, so

$$\begin{aligned} p^a(t) &= \frac{1}{c}\int_{V_{\infty(t)}} T^{a0}(\xi)d^3\vec{\xi} \\ &= \frac{1}{c}\frac{1}{3!}\int_{V_{\infty(t)}} T^{ab}(\xi)\epsilon_{bcde}|d\xi^c \wedge d\xi^d \wedge d\xi^e|, \end{aligned}$$

therefore

$$\begin{aligned} p'^a(t') &= \frac{1}{c}\frac{1}{3!}\int_{V_\infty(t')} T'^{ab}(\xi')\epsilon_{bcde}|d\xi'^c \wedge d\xi'^d \wedge d\xi'^e| \\ &= \frac{1}{c}\frac{1}{3!}\int_{V_\infty(t)} \wedge^a_{.a1}\wedge^b_{.b1} T^{a_1b_1}(\xi')\cdot\epsilon_{bcde}\cdot\wedge^c_{.c1}\cdot\wedge^d_{.d1}\cdot\wedge^e_{.e1}|d\xi^{c_1}\wedge d\xi^{d_1}\wedge d\xi^{e_1}| \\ &= \frac{1}{c}\frac{1}{3!}\Lambda^a_{.a_1}\int_{V_\infty(t)} T^{a_1b_1}(\xi')(\det\Lambda^g_{.f})\epsilon_{b_1c_1d_1e_1}|d\xi^{c_1}\wedge d\xi^{d_1}\wedge d\xi^{e_1}| \\ &= \wedge^a_{.b}p^b(t). \end{aligned}$$

[EOP]

6. Force density $G(\xi)$:

$$\begin{aligned} G^a(\xi) &= c\int_{-\infty}^{+\infty}\delta^4(\xi-\xi_\alpha(\tau))f^a(\tau)d\tau \\ &= c\delta^3(\vec{\xi}-\vec{\xi}_\alpha(\xi^0))f^a(\xi^0)\frac{d\tau}{d\xi^0}. \end{aligned}$$

[Proof]

$$\begin{aligned} G^a(\xi) &= c\int_{-\infty}^{+\infty}\delta^4(\xi-\xi_\alpha(\tau))f^a(\tau)d\tau \\ &= c\int_{-\infty}^{+\infty}\delta^3(\vec{\xi}-\vec{\xi}_\alpha(\tau))\delta(\vec{\xi^0}-\vec{\xi}^0_\alpha(\tau))f^a(\tau)\frac{d\tau}{d\xi^0}d\xi^0 \\ &= c\delta^3(\vec{\xi}-\vec{\xi}_\alpha(\tau))f^a(\xi^0)\frac{d\tau}{d\xi^0}. \end{aligned}$$

[EOP]

7. Action law:

$$\partial_b T^{ab}(\xi) = G^a(\xi).$$

[Proof]

$$\begin{aligned} \partial_b T^{ab}(\xi) &= c\int_{-\infty}^{+\infty}\frac{\partial}{\partial\xi^b}\delta^4(\xi-\xi_\alpha(\tau))m\frac{d\xi^a_\alpha}{d\tau}\frac{d\xi^b_\alpha}{d\tau}d\tau \\ &= -c\int_{-\infty}^{+\infty}m\frac{d\xi^a_\alpha}{d\tau}\frac{d\xi^b_\alpha}{d\tau}\frac{\partial}{\partial\xi^b_\alpha}\delta^4(\xi-\xi_\alpha(\tau))d\tau \\ &= -c\left[m\frac{d\xi^a_\alpha}{d\tau}\delta^4(\vec{\xi}-\vec{\xi}_\alpha(\tau))\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}\delta^4(\xi-\xi_\alpha(\tau))m\frac{d^2\xi^a_\alpha}{d\tau^2}d\tau\right] \\ &= c\int_{-\infty}^{+\infty}\delta^4(\xi-\xi_\alpha(\tau))f^a(\tau)d\tau \\ &= G^a(\xi). \end{aligned}$$

[EOP]

8. If $\rho^a(\xi) = T^{a0}(\xi)$ is the energy momentum density field, $\vec{j}^a(\xi) = c(T^{a1}(\xi), T^{a2}(\xi), T^{a3}(\xi))$ is the current density of the energy momentum field, when $G^a(\xi) = 0$, one has,

$$\frac{\partial \rho^a}{\partial t} + \nabla \cdot \vec{j}^a = 0.$$

[Proof]
Since $G^a(\xi) = 0$, so $\partial_b T^{ab}(\xi) = 0$, therefore

$$\partial_0 T^{a0}(\xi) + \partial_k T^{ak}(\xi) = 0,$$

in which

$$\partial_0 T^{a0}(\xi) = \frac{1}{c}\frac{\partial}{\partial t} T^{a0} = \frac{1}{c}\frac{\partial}{\partial t}\rho^a,$$

$$\partial_k T^{ak}(\xi) = \frac{1}{c}\nabla \cdot \vec{j}^a(\xi),$$

hence

$$\frac{\partial \rho^a}{\partial t} + \nabla \cdot \vec{j}^a = 0.$$

[EOP]

9. When $G^a(\xi) = 0$, the 4-momentum is conserved, i.e.,

$$\frac{\partial p^a}{\partial t} = 0.$$

[Proof]

$$\begin{aligned}
\frac{\partial p^a}{\partial t} &= \frac{1}{c}\int_{V_{\infty(t)}} \frac{\partial}{\partial t} T^{a0}(\xi) d^3\vec{\xi} \\
&= \frac{1}{c}\int_{V_{\infty(t)}} \frac{\partial}{\partial t}\rho^a(\xi) d^3\vec{\xi} \\
&= -\int_{V_{\infty(t)}} \nabla \cdot \vec{j}^a(\xi) d^3\vec{\xi} \\
&= \oint_{\partial V_{\infty(t)}} \vec{j}^a(\xi) \cdot d^3\vec{S} \\
&= 0,
\end{aligned}$$

where $\partial V_{\infty(t)}$ is the boundary of $V_{\infty(t)}$ at $\xi^0 = ct$. At the boundary, $\vec{j}^a$ is zero. The reason is as follows:

$$\begin{aligned}
T^{ak}(\xi)|_{\xi\epsilon\partial V_{\infty(t)}} &= T^{ak}(t,\xi)|_{\xi\epsilon\partial V_{\infty(t)}} \\
&= c^2\delta^3(\vec{\xi} - \vec{\xi}_\alpha(t))\frac{p^a(t)p^k(t)}{E(t)}\Big|_{\xi\epsilon\partial V_{\infty(t)}}.
\end{aligned}$$

Since $|\xi_\alpha(\tau) < \infty|$, hence $\delta^3(\vec{\xi} - \vec{\xi}_\alpha(t))|_{\xi\epsilon\partial V_{\infty(t)}} = 0$ (i.e., $|\vec{\xi}| \to \infty$).

[EOP]

10. Angular momentum and spin:

    (a) The angular momentum density tensor is defined as

$$M^{abc}(\xi) = \xi^a T^{bc}(\xi) - \xi^b T^{ac}(\xi). \tag{4.22}$$

    (b) The angular momentum is defined as

$$J^{ab}(t) = \int_{V_{\infty(t)}} M^{ab0} d^3\vec{\xi}. \tag{4.23}$$

    (c) Spin is defined as

$$S_a = \frac{1}{2}\epsilon_{abcd} J^{bc} U^d, \tag{4.24}$$

    where $U^d = p^d/M$, M is the rest mass of the system and

$$M = [-\eta_{ab} p^a p^b]^{1/2}/c. \tag{4.25}$$

11. Mass system

$$T^{ab}(\xi) = c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} \delta^4(\xi - \xi_\alpha(\tau)) m \frac{d\xi^a_\alpha}{d\tau}\frac{d\xi^b_\alpha}{d\tau} d\tau, \tag{4.26}$$

$$G^a(\xi) = c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} \delta^4(\xi - \xi_\alpha(\tau)) f^a(\tau) d\tau, \tag{4.27}$$

$$p^a(\xi) = \frac{1}{c}\int_{V_\infty(t)} T^{a0}(\xi) d^3\vec{\xi}. \tag{4.28}$$

4.3 Electrodynamics in covariant forms

4.3.1 Electric current density 4-vector:

$$j^a = c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^4(\xi - \xi_\alpha(\tau)) \frac{d\xi^a_\alpha(\tau)}{d\tau} d\tau.$$

Notes:

1. $\vec{j}(\xi) = \sum\limits_{\alpha=1}^{N} \delta^3(\vec{\xi} - \vec{\xi}_\alpha(t)) q_\alpha \vec{v}_\alpha(t)$.

2. $j^0(\xi) = c\sum\limits_{\alpha=1}^{N} \delta^3(\vec{\xi} - \vec{\xi}_\alpha(\tau)) q_\alpha = \rho(\vec{\xi})c$.

3. Continuity equation (conservation of charge):

    (a) $\partial_a j^a(\xi) = 0$,

(b) $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j}(\xi) = 0;$

(c) $Q = \int_{V_\infty(t)} \rho(\xi) d^3\vec{\xi} = \sum_{\alpha=1}^{N} q_\alpha,$ and $\frac{dQ}{dt} = 0.$

[Proof]

1.

$$\begin{aligned}
\vec{j}(\xi) &= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^4(\xi - \xi_\alpha(\tau))\frac{d\vec{\xi}_\alpha(\tau)}{d\tau}\frac{d\tau}{d\xi^0}d\xi^0 \\
&= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^3(\vec{\xi} - \vec{\xi}_\alpha(\tau))\delta(\xi^0 - \xi^0_\alpha(\tau))\frac{d\vec{\xi}_\alpha(\xi^0)}{d\xi^0}d\xi^0 \\
&= \sum_{\alpha=1}^{N} q_\alpha \delta^3(\vec{\xi} - \vec{\xi}_\alpha(t))\vec{v}_\alpha(t);
\end{aligned}$$

2.

$$\begin{aligned}
j^0(\xi) &= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^4(\xi - \xi_\alpha(\tau))\frac{d\xi^0_\alpha(\tau)}{d\tau}\frac{d\tau}{d\xi^0}d\xi^0 \\
&= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^4(\vec{\xi} - \vec{\xi}_\alpha(\tau))\delta(\xi^0 - \xi^0_\alpha(\tau))\frac{d\xi^0_\alpha(\xi^0)}{d\xi^0}d\xi^0 \\
&= c\sum_{\alpha=1}^{N} q_\alpha \delta^3(\vec{\xi} - \vec{\xi}_\alpha(t)) \\
&= \rho(\xi)c;
\end{aligned}$$

3.

$$\begin{aligned}
\partial_a j^a &= \sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \partial_a \delta^4(\xi - \xi_\alpha(\tau))\frac{d\xi^a_\alpha(\tau)}{d\tau}d\tau \\
&= -\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \frac{\partial}{\partial \xi^a_\alpha}\delta^4(\xi - \xi_\alpha(\tau))\frac{d\xi^a_\alpha(\tau)}{d\tau}d\tau \\
&= -\sum_{\alpha=1}^{N} q_\alpha \delta^4(\xi - \xi_\alpha(t))\Big|_{-\infty}^{+\infty} \\
&= 0;
\end{aligned}$$

$$\partial_a j^a = \frac{\partial}{\partial ct}(\rho c) + \nabla \cdot \vec{j} = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0;$$

$$Q = \int d^3\vec{\xi}\rho(\xi);$$

$$\begin{aligned}\frac{dQ}{dt} &= \int d^3\vec{\xi}\frac{\partial\rho}{\partial t}\\ &= -\int d^3\vec{\xi}\nabla\cdot\vec{j}\\ &= -\oint \vec{j}\cdot d\vec{S}\\ &= 0.\end{aligned}$$

[EOP]

4.3.2 Maxwell's equations in covariant form:

1. Four-dimensional electromagnetic potential:

$$A^a = (A^0, \vec{A}) = (\phi, \vec{A}).$$

2. Electromagnetic field tensor:

$$F^{ab} = \partial^a A^b - \partial^b A^a.$$

3. Electric field and magnetic field written by the components of F^{ab}:

$$E^k = F^{0k}, \quad B^k = \frac{1}{2}\epsilon^{kij}F_{ij}.$$

4. Maxwell's equations:

$$\epsilon^{abcd}\partial_b F_{cd} = 0, \quad (\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}) = 0,$$

$$\partial_b F^{ab} = \frac{4\pi}{c} j^a.$$

[Proof]

1. Considering $\nabla\cdot\vec{B} = 0$, one has

$$\vec{B} = \nabla\times\vec{A},$$

i.e.,

$$\begin{aligned}B^k &= \epsilon^{kij}\partial_i A_j\\ &= \frac{1}{2}\epsilon^{kij}(\partial_i A_j - \partial_j A_i)\\ &= \frac{1}{2}\epsilon^{kij}F_{ij}.\end{aligned}$$

Further, from

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B},$$

one has

$$\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) = 0,$$

therefore

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \nabla A^0,$$

i.e.,

$$E^k = \partial^0 A^k - \partial^k A^0 = F^{0k}.$$

2. From $\nabla \cdot \vec{B} = 0$, one has

$$\partial_k B^k = \frac{1}{2}\epsilon^{kij}\partial_k F_{ij} = 0,$$

so that

$$\epsilon^{0abc}\partial_a F_{bc} = 0.$$

From

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B},$$

and

$$\begin{aligned} F_{0j} &= \eta_{0c}\eta_{jk}F^{ck} \quad (j = 1, 2, 3) \\ &= \eta_{00}\eta_{jk}F^{0k} \\ &= \eta_{00}\delta^j_k F^{0k} \\ &= \eta_{00}F^{0j} \\ &= -F^{0j}, \end{aligned}$$

$$E_j = \eta_{jl}E^l = \eta_{jl}F^{0l} = -\eta_{jl}F_{0l} = -F_{0j} = F_{j0},$$

one has

$$\begin{aligned} (\nabla \times \vec{E})^k &= \epsilon^{kij}\partial_i E_j \\ &= \epsilon^{kij}\partial_i F_{j0} \\ &= \frac{1}{2}\epsilon^{kij}(\partial_i F_{j0} - \partial_j F_{i0}). \end{aligned}$$

Because

$$\left(\frac{1}{c}\frac{\partial}{\partial t}\vec{B}\right)^k = \frac{1}{c}\frac{\partial}{\partial t}\cdot\frac{1}{2}\epsilon^{kij}F_{ij} = \frac{1}{2}\epsilon^{kij}\partial_0 F_{ij},$$

therefore

$$\begin{aligned} 0 &= \nabla \times \vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} \\ \Longrightarrow 0 &= \frac{1}{2}\epsilon^{0kij}(\partial_i F_{j0} - \partial_j F_{i0} + \partial_0 F_{ij}) \\ &= \frac{1}{2}\epsilon^{kij}(\partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0}) \\ &= -\frac{1}{4}\epsilon^{kabc}\partial_a F_{bc}. \end{aligned}$$

Therefore, one has

$$\begin{aligned}\epsilon^{0abc}\partial_a F_{bc} &= 0,\\ \epsilon^{kabc}\partial_a F_{bc} &= 0. \quad (k \neq 0)\end{aligned}$$

so that

$$\epsilon^{abcd}\partial_b F_{cd} = 0,$$

which is equivalent to

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0.$$

3. From

$$\nabla \cdot \vec{E} = \frac{4\pi}{c} j^0,$$

one has

$$\partial_k F^{0k} = \frac{4\pi}{c} j^0 \Longrightarrow \partial_b F^{0b} = \frac{4\pi}{c} j^0.$$

From

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E},$$

one has

$$\epsilon^{kij}\partial_i B_j = \frac{4\pi}{c} j^k + \partial_0 E^k.$$

Because

$$\begin{aligned}B_j &= \eta_{jl} B^l\\ &= \eta_{jl} \cdot \frac{1}{2}\epsilon^{lmn} F_{mn}\\ &= \frac{1}{2}\eta_{jl}\eta^{ll'}\eta^{mm'}\eta^{nn'}\epsilon_{l'm'n'}F^{m''n''}\eta_{mm''}\eta_{nn''}\\ &= \frac{1}{2}\eta_{jl}\eta^{ll'}\delta^{m'}_{m''}\delta^{n'}_{n''}\epsilon_{l'm'n'}F^{m''n''}\\ &= \frac{1}{2}\delta^{l'}_{j}\epsilon_{l'm'n'}F^{m'n'}\\ &= \frac{1}{2}\epsilon_{jmn}F^{mn},\end{aligned}$$

therefore

$$\begin{aligned}\text{l.h.s.} &= \epsilon^{kij}\partial_i \frac{1}{2}\epsilon_{jmn}F^{mn}\\ &= \frac{1}{2}(\delta^k_m\delta^i_n - \delta^k_n\delta^i_m)\partial_i F^{mn}\\ &= \frac{1}{2}(\partial_i F^{ki} - \partial_i F^{ik})\\ &= \frac{1}{2}(\partial_i F^{ki} + \partial_i F^{ki})\\ &= \partial_i F^{ki},\end{aligned}$$

$$\begin{aligned} \text{r.h.s.} &= \partial_0 E^k + \frac{4\pi}{c} j^k \\ &= \partial_0 F^{0k} + \frac{4\pi}{c} j^k \\ &= -\partial_0 F^{k0} + \frac{4\pi}{c} j^k, \end{aligned}$$

therefore

$$\partial_i F^{ki} + \partial_0 F^{k0} = \frac{4\pi}{c} j^k,$$

i.e.,

$$\partial_b F^{kb} = \frac{4\pi}{c} j^k. \quad (b = 0, 1, 2, 3)$$

In summary, since

$$\begin{aligned} \partial_b F^{0b} &= \frac{4\pi}{c} j^0, \\ \partial_b F^{kb} &= \frac{4\pi}{c} j^k, \end{aligned}$$

one have

$$\partial_b F^{ab} = \frac{4\pi}{c} j^a.$$

[EOP]

4.3.3 Lorentz force:

1.

$$f^a = qF^{ab} u_b / c,$$

$$\frac{dp^a}{d\tau} = f^a.$$

2.

$$G^a(\xi) = F^{ab}(\xi) j_b(\xi)/c.$$

[Proof]

1.

$$\begin{aligned} f^0 &= \frac{dp^0}{d\tau} \\ &= qF^{0b} u_b / c \\ &= qF^{0k} u_k / c + qF^{00} u_0 / c, \end{aligned}$$

$$u_k = \frac{d\xi_k}{d\tau} = \frac{d\xi_k}{dt}\frac{dt}{d\tau} = v_k \cdot \gamma,$$

$$u_0 = \frac{d\xi_0}{d\tau} = \frac{d\xi_0}{dt}\frac{dt}{d\tau} = -\gamma c,$$

therefore

$$\begin{aligned} f^0 &= qE^k \cdot \gamma v_k/c \\ &= q\vec{E} \cdot \vec{v} \cdot \gamma/c. \end{aligned}$$

Hence

$$\frac{dp^0}{dt} = \frac{dp^0}{d\tau}\frac{d\tau}{dt} = f^0 \cdot \gamma^{-1} = q(\vec{v} \cdot \vec{E}) \cdot \gamma \cdot \gamma^{-1}/c = q(\vec{v} \cdot \vec{E})/c$$

$$\begin{aligned} \frac{d\vec{p}}{dt} &= \frac{d\vec{p}}{d\tau}\frac{d\tau}{dt} \\ &= qF^{kl}u_l\frac{d\tau}{dt}\vec{e}_k/c \\ &= q(F^{k0}u_0 + F^{kj}u_j)\frac{d\tau}{dt}\vec{e}_k/c \\ &= q[-E^k \cdot (-\gamma c) + \epsilon^{kij}u_jB_i]\frac{d\tau}{dt}\vec{e}_k/c \\ &= q[E^k + \epsilon^{kij}v_iB_j]\gamma\frac{d\tau}{dt}\vec{e}_k \\ &= q\Big[\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\Big]. \end{aligned}$$

2.

$$\begin{aligned} G^a(\xi) &= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\xi - \xi_\alpha(\tau))f^a(\tau)d\tau \\ &= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\xi - \xi_\alpha(\tau))F^{ab}(\xi_\alpha(\tau))q_\alpha u_{\alpha b}d\tau/c \\ &= \sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\xi - \xi_\alpha(\tau))F^{ab}(\xi_\alpha(\tau))q_\alpha\frac{d\xi_{\alpha b}(\tau)}{d\tau}d\tau \\ &= \sum_{\alpha=1}^{N}F^{ab}(\xi)\eta_{bc}\int_{-\infty}^{+\infty}\delta^4(\xi - \xi_\alpha(\tau))q_\alpha\frac{d\xi_\alpha^c}{d\tau}d\tau \\ &= F^{ab}(\xi)\eta_{bc}j^c(\xi)/c \\ &= F^{ab}(\xi)j_b(\xi)/c. \end{aligned}$$

[EOP]

3. Motion equation of charged particles in electromagnetic field:

$$G^a(\xi) = \partial_b T^{ab}(\xi) = F^{ab}(\xi)j_b(\xi)/c,$$

$$T^{ab}_{tot} = T^{ab}(\xi) + T^{ab}_{em}(\xi),$$

$$T^{ab}(\xi) = \sum_{\alpha=1}^{n}T^{ab}_\alpha(\xi) = c \cdot \sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\xi - \xi_\alpha(\tau))m_\alpha\frac{d\xi^a_\alpha}{d\tau}\frac{d\xi^b_\alpha}{d\tau}d\tau,$$

$$T^{ab}_{em}(\xi) = \frac{1}{4\pi}\left[F^{a}_{\cdot c}(\xi)F^{bc}(\xi) - \frac{1}{4}\eta^{ab}F_{cd}(\xi)F^{cd}(\xi)\right].$$

$$\partial_b T^{ab}_{tot}(\xi) = 0.$$

[Proof]

$$\begin{aligned}\partial_b T^{ab}_{tot}(\xi) &= \partial_b T^{ab}(\xi) + \partial_b T^{ab}_{em}(\xi)\\ &= F^{ab}(\xi)j_b(\xi)/c + \partial_b T^{ab}_{em}(\xi);\end{aligned}$$

$$\begin{aligned}F^{ab}j_b/c &= F^{ab}\cdot\eta_{bc}j^c/c\\ &= F^{a}_{\cdot c}\cdot\frac{c}{4\pi}\partial_b F^{cb}/c\\ &= \frac{1}{4\pi}[-\partial_b F^{a}_{\cdot c}F^{bc} + (\partial^b F^{ac})F_{bc}],\end{aligned}$$

where

$$\begin{aligned}F_{bc}\partial^b F^{ac} &= \frac{1}{2}(F_{bc}\partial^b F^{ac} + F_{cb}\partial^c F^{ab})\\ &= \frac{1}{2}F_{bc}(\partial^b F^{ac} + \partial^c F^{ba})\\ &= \frac{1}{2}F_{bc}\partial^a F^{bc}\\ &= \frac{1}{4}\partial^a(F_{cd}F^{cd})\\ &= \frac{1}{4}\eta^{ab}\partial_b F_{cd}F^{cd}.\end{aligned}$$

Because

$$\partial^b F^{ac} + \partial^a F^{cb} + \partial^c F^{ba} = 0$$

so

$$\partial^b F^{ac} + \partial^c F^{ba} = -\partial^a F^{cb} = \partial^a F^{bc}$$

Therefore

$$\begin{aligned}F^{ab}j_b/c &= -\partial_b[F^{a}_{\cdot c}F^{bc} - \frac{1}{4}\eta^{ab}F_{cd}F^{cd}]\cdot\frac{1}{4\pi}\\ &= -\partial_b T^{ab}_{em}(\xi),\end{aligned}$$

therefore

$$\partial_b T^{ab}_{tot}(\xi) = 0.$$

[EOP]

4.4 Summary

4.4.1 Electromagnetic field tensor: F^{ab}

The electromagnetic field tensor is defined as

$$F^{ab} = \partial^a A^b - \partial^b A^a, \quad F_{ab} = \partial_a A_b - \partial_b A_a$$

Now we examine the properties and elements of this tensor, In the following discussions, we assume that $\mathrm{a, b, c, d, e} = 0, 1, 2, 3;\quad \mathrm{i, j, k, l, m, n} = 1, 2, 3$.

1. $F^{ab}(F_{ab})$ is an anti-symmetric, i.e., $F_{ab} = -F_{ba}$, $F_{ab} = -F_{ba}$.

2. On F^{aa} : $F^{aa} = F_{aa} = 0$.

3. $F^{0k} = -F_{0k} = -F^{0k}F_{k0}$:

$$F^{0k} = \partial^0 A^k - \partial^k A^0 + -\frac{1}{c}\frac{\partial a^k}{\partial t} - \partial_k A^0 = E^k.$$

   That is to say, F^{0k} plays the role of the three components of the electric field.

4. On F^{ij} :

$$B^k = (\nabla \times \vec{A})^k = \varepsilon^{kij}\partial_i A_j = \frac{1}{2}(\varepsilon^{kij}\partial_i A_j - \partial_j A_i) = \frac{1}{2}(\varepsilon^{kij}F_{ij}.$$

   So,

$$B^1 = \frac{1}{2}(\varepsilon^{1ij}F_{ij} = \frac{1}{2}(\varepsilon^{123}F_{23} + \varepsilon^{132}F_{32}) = \frac{1}{2}(F_{23} + F_{23}) = F_{23},$$
$$B^2 = \frac{1}{2}(\varepsilon^{2ij}F_{ij} = \frac{1}{2}(\varepsilon^{231}F_{31} + \varepsilon^{213}F_{13}) = \frac{1}{2}(F_{31} + F_{31}) = F_{31},$$
$$B^3 = \frac{1}{2}(\varepsilon^{3ij}F_{ij} = \frac{1}{2}(\varepsilon^{312}F_{12} + \varepsilon^{321}F_{21}) = \frac{1}{2}(F_{12} + F_{12}) = F_{12}.$$

5. Matric representation of F^{ab} and F_{ab}:

$$F^{ab} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^3 & -B^1 & 0 \end{bmatrix}, F^{ab} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^3 & -B^1 & 0 \end{bmatrix}, \quad (4.29)$$

4.4.2 Energy-momentum density tensor of the electromagnetic field: T^{ab}_{em}

The energy-momentum density tensor of the electromagnetic field:

$$T^{ab}_{em} = \frac{1}{4\pi}\left(F^a_{.c}F^{bc} - \frac{1}{4}\eta^{ab}F_{cd}F^{cd}\right) = \frac{1}{4\pi}F^a_{.c}F^{bc} + \eta^{ab}\mathcal{L}_{em} \quad (4.30)$$

Now we examine the properties and elements of this tensor, and we assume that $\mathrm{a, b, c, d, e} = 0, 1, 2, 3;\quad \mathrm{i, j, k, l, m, n} = 1, 2, 3$.

1. T^{ab}_{em} is a symmetric tensor, i.e., $T^{ab}_{em} = T^{ba}_{em}$.

2. On T^{00}_{em}:

$$\begin{aligned}
T^{00}_{em} &= \frac{1}{4\pi}\Big(F^{0}_{.c}F^{0c} - \frac{1}{4}\eta^{00}F_{cd}F^{cd}\Big) \\
&= \frac{1}{4\pi}\Big(F^{0c}F^{0c} + \frac{1}{4}F_{cd}F^{cd}\Big) \\
&= \frac{\vec{E}^2}{4\pi} - \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2) \\
&= \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \\
&= w_{em} \\
&= \mathcal{H}_{em}
\end{aligned}$$

So, $T^{00}_{em} = \mathcal{H}_{em}$ plays the role of the energy density, and also, the Hamiltonian density of the electromagnetic field, Also, it should be mentioned that

$$\mathcal{L}_{em} = -\frac{1}{16\pi}F_{cd}F^{cd} = \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2),$$

is the Lagrangian density of the free electromagnetic field.

3. On T^{0k}_{em}:

$$T^{0k}_{em} = \frac{1}{4\pi}\Big(F^{0}_{.c}F^{kc} - \frac{1}{4}\eta^{0k}F_{cd}F^{cd}\Big) = \frac{1}{4\pi}F^{0}_{.i}F^{ki} = \frac{1}{4\pi}F^{0i}F^{ki}.$$

On the other hand, since

$$(\vec{E}\times\vec{B})^k = \varepsilon^{kij}E_jB_j = \frac{1}{2}\varepsilon^{kij}F^{0i}\varepsilon_{jlm}F^{lm} = \frac{1}{2}(\delta^k_l\delta^i_m - \delta^k_m\delta^i_l)F^{0i}F^{lm} = F^{0i}F^{ki}.$$

So, one yields

$$T^{0k}_{em} = \frac{1}{4\pi}(\vec{E}\times\vec{B})^k = \frac{1}{c}(\vec{S})^k.$$

That is to say, T^{0k}_{em} plays the role of the k^{th} component of the Poynting vector (energy-flux vector).

4. On T^{ij}_{em}:

$$\begin{aligned}
T^{ij}_{em} &= \frac{1}{4\pi}\Big(F^{i}_{.c}F^{jc} - \frac{1}{4}\eta^{ij}F_{cd}F^{cd}\Big) \\
&= \frac{1}{4\pi}\Big(F_{ic}F^{jc} - \frac{1}{4}\delta^i_jF_{cd}F^{cd}\Big) \\
&= \frac{1}{4\pi}\Big(F_{i0}F^{j0} + F^{ik}F^{jk} - \frac{1}{4}\delta^i_jF_{cd}F^{cd}\Big) \\
&= \frac{1}{4\pi}\Big(- F^{i0}F^{j0} + F_{ik}F^{jk} - \frac{1}{4}\delta^i_jF_{cd}F^{cd}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi}(-E^iE^j + F_{ik}F^{jk}) - \frac{1}{16\pi}\delta^i_j F_{cd}F^{cd} \\
&= \frac{1}{4\pi}(-E^iE^j + F_{ik}F^{jk}) + \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2)\delta^i_j \\
&= \frac{1}{4\pi}(-E^iE^j - B^iB^j + \vec{B}^2\delta^i_j) + \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2)\delta^i_j \\
&= -\frac{1}{4\pi}(E^iE^j + B^iB^j - \frac{1}{2}\vec{E}^2\delta^i_j - \frac{1}{2}\vec{B}^2\delta^i_j)
\end{aligned}
$$

5. Matrix representation of T^{ab}_{em}:

$$
F^{ab} = \begin{bmatrix} \mathcal{H}_{em} & \frac{1}{c}S^1 & \frac{1}{c}S^2 & \frac{1}{c}S^3 \\ \frac{1}{c}S^1 & & & \\ \frac{1}{c}S^2 & & \mathcal{T} & \\ \frac{1}{c}S^3 & & & \end{bmatrix}. \tag{4.31}
$$

where

$$
\mathcal{T} = -\frac{1}{4\pi}(\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}\vec{E}^2\mathcal{I} - \frac{1}{2}\vec{B}^2\mathcal{I}).
$$

6. Energy-Momentum theorem: $\partial_b T^{ab}_{em} = -\frac{1}{c}F^{ab}j_b$.

   (a) For $a = 0$:

$$
\partial_b T^{0b}_{em} = \partial_0 T^{00}_{em} + \partial_k T^{0k}_{em} = \frac{1}{c}\frac{\partial}{\partial t}w_{em} + \frac{1}{c}\nabla\cdot\vec{S}_{em},
$$

$$
-\frac{1}{c}F^{0b}j_b = -\frac{1}{c}F^{0k}j_k = -\frac{1}{c}E^kj_k = -\frac{1}{c}\vec{j}\cdot\vec{E},
$$

   So one yields

$$
\frac{1}{c}\frac{\partial}{\partial t}w_{em} + \frac{1}{c}\nabla\cdot\vec{S}_{em} = -\vec{j}\cdot\vec{E},
$$

   where

$$
w_{em} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2), \qquad \vec{S}_{em} = \frac{c}{4\pi}(\vec{E}\times\vec{B}).
$$

   Hence,

$$
\frac{\partial}{\partial t}w_{em} + \nabla\cdot\vec{S}_{em} = -\sum_{\alpha=1}^{N} q_\alpha\vec{v}_\alpha(t)\delta(\vec{\xi} - \vec{\xi}_\alpha(t))\cdot\vec{E}(\vec{\xi}, t),
$$

therefore

$$\frac{d}{dt}E_{em} = -W,$$

where

$$E_{em}(t) = \int d^3\vec{\xi}\frac{1}{8\pi}[\vec{E}^2(\vec{\xi},t) + \vec{B}^2(\vec{\xi},t)],$$

$$\begin{aligned} W(t) &= \int d^3\vec{\xi}\sum_{\alpha=1}^{N} q_\alpha \vec{v}_\alpha(t)\delta(\vec{\xi}-\vec{\xi}_\alpha(t))\cdot\vec{E}(\vec{\xi},t) \\ &= \sum_{\alpha=1}^{N} q_\alpha \vec{v}_\alpha(t)\cdot\vec{E}(\vec{\xi}_\alpha(t),t), \end{aligned}$$

are the energy of the electromagnetic field and the work the electromagnetic field exerted on the charge particles.

(b) For $a = k$:

$$\begin{aligned} \partial_b T^{kb}_{em} &= \partial_0 T^{k0}_{em} + \partial_i T^{ki}_{em} \\ &= \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})^k + \partial_i(-\frac{1}{4\pi})[E^k E^i + B^k B^i - \frac{1}{2}\vec{E}^2\delta^k_i - \frac{1}{2}\vec{B}^2\delta^k_i] \\ &= \left\{\frac{\partial}{\partial t}\frac{1}{4\pi c}(\vec{E}\times\vec{B}) - \nabla\cdot\frac{1}{4\pi}\left[\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}\vec{E}^2\mathcal{I} - \frac{1}{2}\vec{B}^2\mathcal{I}\right]\right\}^k \\ &= (\frac{\partial}{\partial t}\vec{g}_{em} - \nabla\cdot\mathcal{T}_{em})^k, \end{aligned}$$

$$\begin{aligned} -\frac{1}{c}F^{kb}j_b &= -\frac{1}{c}F^{k0}j_0 - \frac{1}{c}F^{ki}j_i \\ &= -\frac{1}{c}E^k j^0 - \frac{1}{c}F^{ki}j^i \\ &= -\vec{E}^k(\vec{\xi},t)\sum_{\alpha=1}^{N} q_\alpha\delta(\vec{\xi}-\vec{\xi}_\alpha(t)) - \frac{1}{c}\sum_{\alpha=1}^{N} q_\alpha\left[\vec{v}_\alpha(t)\times\vec{B}(\vec{\xi},t)\right]^k\delta(\vec{\xi}-\vec{\xi}_\alpha(t)) \\ &= -\sum_{\alpha=1}^{N} q_\alpha\delta(\vec{\xi}-\vec{\xi}_\alpha(t))\left[\vec{E}(\vec{\xi},t) + \frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{\xi},t)}{c}\right]^k \\ &= -\left[\vec{f}_L(\vec{\xi},t)\right]^k, \end{aligned}$$

so one yields

$$\nabla\cdot\mathcal{T}_{em} = \frac{\partial}{\partial t}\vec{g}_{em} + \vec{f}_L(\vec{\xi},t),$$

where $\mathcal{T}_{em}$ is the stress tensor of the electromagnetic field, $\vec{g}_{em}$ is the momentum density of the electromagnetic field, and $\vec{f}_L(\vec{\xi},t)$ is the Lorentz force density. Hence, one derives

$$\frac{d}{dt}\vec{p}_{em}(t) + \vec{F}_L(\vec{\xi}_\alpha(t),t) = 0, \qquad \left(\frac{d\vec{p}_{em}(t)}{dt} = -\vec{F}_L(\vec{\xi}_\alpha(t),t)\right)$$

where

$$\vec{p}_{em}(t) = \int d^3\vec{\xi}\frac{1}{4\pi c}[\vec{E}(\vec{\xi},t)\times\vec{B}(\vec{\xi},t)],$$

$$\begin{aligned}\vec{F}_L(\vec{\xi}_\alpha(t),t) &= \int d^3\vec{\xi}\vec{f}_L(\vec{\xi},t)\\ &= \sum_{\alpha=1}^{N} q_\alpha\Big[\vec{E}(\vec{\xi}_\alpha(t),t)+\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{\xi}_\alpha(t),t)}{c}\Big],\end{aligned}$$

are the momentum of the electromagnetic field and the Lorentz exerted on the charged particles, also, $-\vec{F}_L(\vec{\xi}_\alpha,t)$ is the counter-force of the particles exerted on the electromagnetic field, respectively.

4.4.3 Energy-momentum density tensor of the charged particles: T_p^{ab}

The energy-momentum density tensor of the charged particles:

$$\begin{aligned}T_p^{ab} &= \sum_{\alpha=1}^{N} T_\alpha^{ab}\\ &= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau}\delta^4(\xi-\xi_\alpha)d\tau\\ &= c\sum_{\alpha=1}^{N}\delta^3(\vec{\xi}-\vec{\xi}_\alpha)\frac{p_\alpha^a p_\alpha^b}{E_\alpha},\end{aligned}$$

where α denotes the α^{th} charged particle and

$$T_\alpha^{ab} = c\int_{-\infty}^{+\infty} m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau}\delta^4(\xi-\xi_\alpha)d\tau,$$

$$\begin{aligned}T_\alpha^{a0} &= c\int_{-\infty}^{+\infty} m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^0}{d\tau}\delta^4(\xi-\xi_\alpha)d\tau\\ &= c\int_{-\infty}^{+\infty} m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^0}{d\tau}\delta^4(\xi-\xi_\alpha)\frac{d\tau}{d\xi^0}d\xi^0\\ &= c\int_{-\infty}^{+\infty} m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^0}{d\tau}\delta^3(\vec{\xi}-\vec{\xi}_\alpha)\delta(\xi^0-\xi_\alpha^0)d\xi^0\\ &= cm_\alpha\frac{d\xi_\alpha^a}{d\tau}\delta^3(\vec{\xi}-\vec{\xi}_\alpha),\end{aligned}$$

$$p_\alpha^a = \frac{1}{c}\int_{v_\infty(t)} T_\alpha^{a0}(\xi)d^3\vec{\xi} = m_\alpha\frac{d\xi_\alpha^a}{d\tau},$$

$$p_\alpha^0 = \frac{E_\alpha}{c}$$

1. T_p^{ab} is a symmetric, i.e., $T_p^{ab} = T_p^{ba}$.
2. T_p^{00}:

$$\begin{aligned}
T_p^{00} &= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} m_\alpha \frac{d\xi_\alpha^0}{d\tau}\frac{d\xi_\alpha^0}{d\tau}\delta^4(\xi-\xi_\alpha)d\tau \\
&= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} m_\alpha \frac{d\xi_\alpha^0}{d\tau}\frac{d\xi_\alpha^0}{d\xi^0}\delta^3(\vec{\xi}-\vec{\xi}_\alpha)\delta(\xi^0-\xi_\alpha^0)d\xi^0 \\
&= \sum_{\alpha=1}^{N} cm_\alpha \frac{d\xi_\alpha^0}{d\tau}\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) \\
&= \sum_{\alpha=1}^{N} m_\alpha c^2\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) \\
&= \sum_{\alpha=1}^{N} T_\alpha^{00},
\end{aligned}$$

where $\gamma_\alpha = \left(1-\frac{\vec{v}_\alpha^2}{c^2}\right)^{-1/2}$ is the Lorentz factor for the α^{th} charged particle.

3. T_p^{k0}:

$$\begin{aligned}
T_p^{k0} &= \sum_{\alpha=1}^{N} T_\alpha^{k0} \\
&= \sum_{\alpha=1}^{N} c\int_{-\infty}^{+\infty} m_\alpha \frac{d\xi_\alpha^k}{d\tau}\frac{d\xi_\alpha^0}{d\tau}\delta^4(\xi-\xi_\alpha)d\tau \\
&= \sum_{\alpha=1}^{N} c\int_{-\infty}^{+\infty} m_\alpha \frac{d\xi_\alpha^k}{d\tau}\frac{d\xi_\alpha^0}{d\tau}\delta^4(\xi-\xi_\alpha)\frac{d\tau}{d\xi^0}d\xi^0 \\
&= \sum_{\alpha=1}^{N} c\int_{-\infty}^{+\infty} m_\alpha \frac{d\xi_\alpha^k}{d\tau}\frac{d\xi_\alpha^0}{d\xi^0}\delta^3(\vec{\xi}-\vec{\xi}_\alpha)\delta(\xi^0-\xi_\alpha^0)d\xi^0 \\
&= \sum_{\alpha=1}^{N} cm_\alpha \frac{d\xi_\alpha^k}{d\tau}\delta^3(\vec{\xi}-\vec{\xi}_\alpha) \\
&= \sum_{\alpha=1}^{N} cm_\alpha v_\alpha^k\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha).
\end{aligned}$$

4. T_p^{ij}:

$$\begin{aligned}
T_p^{ij} &= \sum_{\alpha=1}^{N} T_\alpha^{ij} \\
&= \sum_{\alpha=1}^{N} c\int_{-\infty}^{+\infty} m_\alpha \frac{d\xi_\alpha^i}{d\tau}\frac{d\xi_\alpha^j}{d\tau}\delta^4(\xi-\xi_\alpha)d\tau \\
&= \sum_{\alpha=1}^{N} c\int_{-\infty}^{+\infty} m_\alpha \frac{d\xi_\alpha^i}{d\tau}\frac{d\xi_\alpha^j}{d\tau}\delta^4(\xi-\xi_\alpha)\frac{d\tau}{d\xi^0}d\xi^0
\end{aligned}$$

$$= \sum_{\alpha=1}^{N} m_\alpha \frac{d\xi^i_\alpha}{d\tau}\frac{d\xi^i_\alpha}{d\tau}\gamma_\alpha^{-1}\delta^3(\vec{\xi}-\vec{\xi}_\alpha)$$
$$= \sum_{\alpha=1}^{N} m_\alpha v^i_\alpha v^j_\alpha \gamma_\alpha \delta^3(\vec{\xi}-\vec{\xi}_\alpha).$$

$$T_p^{ab} = \begin{bmatrix} w_p & \frac{1}{c}S_p^1 & \frac{1}{c}S_p^2 & \frac{1}{c}S_p^3 \\ \frac{1}{c}S_p^1 & & & \\ \frac{1}{c}S_p^2 & & \mathcal{T}_p & \\ \frac{1}{c}S_p^3 & & & \end{bmatrix}.$$

where

$$w_p = \sum_{\alpha=1}^{N} m_\alpha c^2 \gamma_\alpha \delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)),$$

$$\mathcal{T}_p = \sum_{\alpha=1}^{N} m_\alpha \vec{v}_\alpha \vec{v}_\alpha \gamma_\alpha \delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)),$$

$$\vec{S}_p = \sum_{\alpha=1}^{N} m_\alpha c^2 \vec{v}_\alpha \gamma_\alpha \delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)).$$

5. Relation of the energy-momentum density tensor:

$$\partial_b T_p^{ab}(\xi) = G^a(\xi) = F^{ab}(\xi) j_b(\xi)$$

where $G^a(\xi)$ is the force density 4-vector exerted on the charged particle system and the electric current density 4-vector is defined as

$$j^a = c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^4(\xi-\xi_\alpha(\tau))\frac{d\xi^a_\alpha}{d\tau}d\tau,$$

$$j^0 = c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^4(\xi-\xi_\alpha(\tau))\frac{d\xi^0_\alpha}{d\tau}d\tau = c\sum_{\alpha=1}^{N} q_\alpha \delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) = \rho(\xi)c,$$

$$j^k = c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty} q_\alpha \delta^4(\xi-\xi_\alpha(\tau))\frac{d\xi^k_\alpha}{d\tau}d\tau = \sum_{\alpha=1}^{N} q_\alpha v^k_\alpha(t)\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)).$$

The reason is as follows:
First of all, the Lorentz force 4-vector exerted on a single charged particle is (denoted by α):

$$f^a_\alpha = \frac{1}{c} q_\alpha F^{ab}(\xi_\alpha) u_{\alpha b},$$

where

$$\begin{aligned} u_{\alpha b} &= \frac{d\xi_{\alpha b}}{d\tau}, \\ u_{\alpha 0} &= \frac{d\xi_{\alpha 0}}{d\tau} = -\frac{d\xi^0_\alpha}{d\tau} = -\gamma_\alpha c, \\ u_{\alpha k} &= \frac{d\xi_{\alpha k}}{d\tau} = \frac{d\xi^k_\alpha}{d\tau} = \gamma_\alpha v^k_\alpha. \end{aligned}$$

So

$$\begin{aligned} f^0_\alpha &= \frac{1}{c} q_\alpha F^{0b}(\xi_\alpha) u_{\alpha b} \\ &= \frac{1}{c} q_\alpha F^{0k}(\xi_\alpha) u_{\alpha k} \\ &= \frac{1}{c} q_\alpha E^k(\xi_\alpha) \gamma_\alpha v^k_\alpha \\ &= \frac{1}{c} q_\alpha \vec{v}_\alpha \cdot \vec{E}(\xi_\alpha) \gamma_\alpha, \end{aligned}$$

$$\begin{aligned} f^k_\alpha &= \frac{1}{c} q_\alpha F^{kb}(\xi_\alpha) u_{\alpha b} \\ &= \frac{1}{c} q_\alpha [F^{k0}(\xi_\alpha) u_{\alpha 0} + F^{kj}(\xi_\alpha) u_{\alpha j}] \\ &= \frac{1}{c} q_\alpha [-E^k(\xi_\alpha)(-\gamma_\alpha c) + \varepsilon^{kij} u_{\alpha j} B_i] \\ &= q_\alpha \gamma_\alpha [E^k(\xi_\alpha) + \frac{1}{c} \varepsilon^{kij} v_{\alpha j} B_i] \\ &= q_\alpha \gamma_\alpha \left[E(\xi_\alpha) + \frac{\vec{v}_\alpha(t) \times \vec{B}(\xi_\alpha)}{c} \right]^k. \end{aligned}$$

Since

$$\frac{dp^a_\alpha}{d\tau} = f^a_\alpha,$$

so

$$\frac{dp^0_\alpha}{dt} = \frac{dp^0_\alpha}{d\tau}\frac{d\tau}{dt} = \gamma_\alpha^{-1} \frac{dp^0_\alpha}{d\tau} = \gamma_\alpha^{-1} f^0_\alpha = \frac{1}{c} q_\alpha \vec{v}_\alpha \cdot \vec{E}(\xi_\alpha),$$

$$\frac{dp^k_\alpha}{dt} = \frac{dp^k_\alpha}{d\tau}\frac{d\tau}{dt} = \gamma_\alpha^{-1} \frac{dp^k_\alpha}{d\tau} = \gamma_\alpha^{-1} f^k_\alpha = q_\alpha \left[\vec{E}(\xi_\alpha) + \frac{\vec{v}_\alpha(t) \times \vec{B}(\xi_\alpha)}{c} \right]^k.$$

Therefore using $\delta^4(\xi-\xi_\alpha(\tau))$, we can naturally generalize the Lorentz force 4-vector into the force density 4-vector exerted on a system of charge particles as

$$\begin{aligned}
G^a(\xi) &= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\xi-\xi_\alpha)f_\alpha^a(\xi_\alpha)d\tau \\
&= \sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\xi-\xi_\alpha)q_\alpha F^{ab}(\xi_\alpha(\tau))u_{\alpha b}d\tau \\
&= F^{ab}(\xi)\eta_{bc}\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\xi-\xi_\alpha)q_\alpha\frac{d\xi_\alpha^c}{d\tau}d\tau \\
&= \frac{1}{c}F^{ab}(\xi)j_b(\xi).
\end{aligned}$$

Next, we prove that $\partial_b T_p^{ab}(\xi)=G^a(\xi)$.

$$\begin{aligned}
\partial_b T_p^{ab}(\xi) &= \partial_b\left[c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau}\delta^4(\xi-\xi_\alpha)d\tau\right] \\
&= -c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\xi_\alpha^b}{d\tau}\frac{\partial}{\partial\xi_\alpha^b}\delta^4(\xi-\xi_\alpha)d\tau \\
&= -c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}m_\alpha\frac{d\xi_\alpha^a}{d\tau}\frac{d\delta^4(\xi-\xi_\alpha)}{d\tau}d\tau \\
&= -c\sum_{\alpha=1}^{N}\left[m_\alpha\left.\frac{d\xi_\alpha^a}{d\tau}\right|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}m_\alpha\frac{d^2\xi_\alpha^a}{d\tau^2}\delta^3(\vec{\xi}-\vec{\xi}_\alpha)\delta(\xi^0-\xi_\alpha^0)d\tau\right] \\
&= c\sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}m_\alpha\frac{d^2\xi_\alpha^a}{d\tau^2}\delta^3(\vec{\xi}-\vec{\xi}_\alpha)\delta(\xi^0-\xi_\alpha^0)d\tau \\
&= \sum_{\alpha=1}^{N}m_\alpha\frac{d^2\xi_\alpha^a}{d\tau^2}\gamma_\alpha^{-1}\delta^3(\vec{\xi}-\vec{\xi}_\alpha) \\
&= \sum_{\alpha=1}^{N}f_\alpha^a(\xi_\alpha)\gamma_\alpha^{-1}\delta^3(\vec{\xi}-\vec{\xi}_\alpha) \\
&= \sum_{\alpha=1}^{N}\int_{-\infty}^{+\infty}\delta^4(\vec{\xi}-\vec{\xi}_\alpha)f_\alpha^a(\xi_\alpha)d\tau \\
&= G^a(\xi) \\
&= \frac{1}{c}F^{ab}(\xi)j_b(\xi).
\end{aligned}$$

6. Energy-momentum theorem of the charged particles:

$$\partial_b T_p^{ab}=\frac{1}{c}F^{ab}(\xi)j_b(\xi).$$

   (a) For $a=0$:

$$\partial_b T_p^{ab}=\partial_0 T_p^{00}+\partial_k T_p^{0k}=\frac{1}{c}\frac{\partial}{\partial t}w_p+\frac{1}{c}\nabla\cdot\vec{S}_p,$$

$$\frac{1}{c}F^{ab}j_b = \frac{1}{c}F^{0k}j_k = \frac{1}{c}E^k j_k = \frac{1}{c}\vec{j}\cdot\vec{E},$$

so one yields

$$\frac{\partial}{\partial t}w_p + \nabla\cdot\vec{S}_p = \vec{j}\cdot\vec{E},$$

where

$$w_p = \sum_{\alpha=1}^{N} m_\alpha c^2\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)),$$

$$\vec{S}_p = \sum_{\alpha=1}^{N} m_\alpha c^2\gamma_\alpha\vec{v}_\alpha(t)\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)).$$

Hence one derives

$$\frac{d}{dt}E_p = W,$$

where

$$\begin{aligned} E_p(t) &= \int d^3\vec{\xi}\sum_{\alpha=1}^{N} m_\alpha c^2\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) \\ &= \sum_{\alpha=1}^{N} m_\alpha c^2\gamma_\alpha \\ &\simeq \sum_{\alpha=1}^{N}[m_\alpha c^2 + \frac{1}{2}m_\alpha\vec{v}_\alpha^2(t)], \qquad (\frac{v_\alpha}{c}\ll 1) \end{aligned}$$

$$\begin{aligned} W &= \int d^3\vec{\xi}\sum_{\alpha=1}^{N} q_\alpha\vec{v}_\alpha(t)\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t))\cdot\vec{E}(\vec{\xi},t) \\ &= \sum_{\alpha=1}^{N} q_\alpha\vec{v}_\alpha(t)\cdot\vec{E}(\vec{\xi}_\alpha(t),t) \end{aligned}$$

are the energy of the charged particles and the work the electromagnetic field exerted on the charged particles.

(b) For $a = k$:

$$\begin{aligned} \partial_b T_p^{kb} &= \partial_0 T^{k0}\partial_i T^{ki} \\ &= \frac{\partial}{\partial t}\sum_{\alpha=1}^{N} m_\alpha v_\alpha^k\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) + \partial_i\sum_{\alpha=1}^{N} m_\alpha v_\alpha^k v_\alpha^i\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) \\ &= \left\{\frac{\partial}{\partial t}\sum_{\alpha=1}^{N} m_\alpha\vec{v}_\alpha(t)\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) + \nabla\cdot\sum_{\alpha=1}^{N} m_\alpha\vec{v}_\alpha\vec{v}_\alpha\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t))\right\}^k \\ &= \left[\frac{\partial}{\partial t}\vec{g}_p(\vec{\xi},t) + \nabla\cdot\mathcal{T}_p(\vec{\xi},t)\right]^k, \end{aligned}$$

$$\begin{aligned}\frac{1}{c}F^{kb}j_b &= \sum_{\alpha=1}^{N} q_\alpha \delta^3(\vec{\xi}-\vec{\xi}_\alpha(t))\left[\vec{E}(\vec{\xi},t)+\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{\xi},t)}{c}\right]^k \\ &= [\vec{f}_L(\vec{\xi},t)]^k,\end{aligned}$$

So we yields

$$\frac{\partial}{\partial t}\vec{g}_p(\vec{\xi},t)-\vec{f}_L(\vec{\xi},t) = -\nabla\cdot\mathcal{T}_p(\vec{\xi},t),$$

where $\mathcal{T}_p$ is the stress tensor of the charged particles, $\vec{g}_p$ is the momentum density of the charged particles, and $\vec{f}_L$ the Lorentz force density, Hence, one derives

$$\frac{d}{dt}\vec{P}_p(t) = \vec{F}_L(\vec{\xi}_\alpha(t),t),$$

where

$$\begin{aligned}\vec{P}_p(t) &= \int d^3\vec{\xi}\sum_{\alpha=1}^{N} m_\alpha\vec{v}_\alpha(t)\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t)) \\ &= \sum_{\alpha=1}^{N} m_\alpha\vec{v}_\alpha(t)\gamma_\alpha \\ &\simeq \sum_{\alpha=1}^{N} m_\alpha\vec{v}_\alpha(t), \qquad (\gamma_\alpha \ll 1)\end{aligned}$$

$$\vec{F}_L(\vec{\xi}_\alpha(t),t) = \sum_{\alpha=1}^{N} q_\alpha\left[\vec{E}(\vec{\xi}_\alpha(t),t)+\frac{\vec{v}_\alpha(t)\times\vec{B}(\vec{\xi}_\alpha(t),t)}{c}\right]$$

are the momentum of the charged particles and the Lorentz force exerted on the charged particles, respectively.

4.4.4 Total Energy-Momentum density tensor: T^{ab}_{tot}

The total .... is defined as:

$$T^{ab}_{tot} = T^{ab}_{em} + T^{ab}_{p}.$$

1. T^{ab}_{tot} is symmetric tensor, i.e., $T^{ab}_{tot} = T^{ba}_{tot}$.

2. $T^{00}_{tot} = w_{tot} = w_{em} + w_p = \frac{1}{8\pi}(\vec{E}^2+\vec{B}^2) + \sum_{\alpha=1}^{N} m_\alpha c^2\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t))$

3. $T^{0k}_{tot} = T^{k0}_{tot} = \frac{1}{c}(\vec{S}_{tot})^k = \frac{1}{c}(\vec{S}_{em}+\vec{S}_p)^k = \left[\frac{1}{4\pi}(\vec{E}\times\vec{B}) + \sum_{\alpha=1}^{N} m_\alpha c\vec{v}_\alpha\gamma_\alpha\delta^3(\vec{\xi}-\vec{\xi}_\alpha(t))\right]^k$

4.

$$
\begin{aligned}
T^{ij}_{tot} &= (\mathcal{T}_{tot})^{ij} \\
&= (\mathcal{T}_{em} + \mathcal{T}_p)^{ij} \\
&= \Big\{ -\frac{1}{4\pi}[\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}\vec{E}^2\mathcal{I} - \frac{1}{2}\vec{B}^2\mathcal{I}] + \sum_{\alpha=1}^{N} m_\alpha \vec{v}_\alpha \vec{v}_\alpha \gamma_\alpha \delta^3(\vec{\xi} - \vec{\xi}_\alpha(t)) \Big\}^{ij}.
\end{aligned}
$$

5. Energy-momentum conservation:

$$\partial_b T^{ab}_{tot} = 0.$$

(a) For $a = 0$:

$$\frac{\partial}{\partial t}(w_{em} + w_p) + \nabla \cdot (\vec{S}_{em} + \vec{S}_p) = 0,$$

So

$$\frac{\partial}{\partial t} w_{tot} + \nabla \cdot \vec{S}_{tot} = 0,$$

therefore

$$\frac{d}{dt} E_{tot} = 0,$$

where

$$
\begin{aligned}
E_{tot} &= \int d^3\vec{\xi}(w_{em} + w_p) \\
&= \sum_{\alpha=1}^{N} m_\alpha c^2 \gamma_\alpha + \frac{1}{8\pi}\int d^3\vec{\xi}[\vec{E}(\vec{\xi},t) \times \vec{B}(\vec{\xi},t)] \\
&\simeq \sum_{\alpha=1}^{N} [m_\alpha c^2 + \frac{1}{2} m_\alpha \vec{v}^2_\alpha(t)] + \frac{1}{8\pi}\int d^3\vec{\xi}[\vec{E}(\vec{\xi},t) \times \vec{B}(\vec{\xi},t)].
\end{aligned}
$$

(b) For $a = k$:

$$\frac{\partial}{\partial t}(\vec{g}_{em} + \vec{g}_p) + \nabla \cdot (\mathcal{T}_p - \mathcal{T}_{em}) = 0,$$

So

$$\frac{d}{dt}\vec{P}_{tot} = 0,$$

where

$$
\begin{aligned}
\vec{P}_{tot} &= \int d^3\vec{\xi}(\vec{g}_{em} + \vec{g}_p) \\
&= \sum_{\alpha=1}^{N} m_\alpha \vec{v}_\alpha(t)\gamma_\alpha + \frac{1}{4\pi c}\int d^3\vec{\xi}[\vec{E}(\vec{\xi},t) \times \vec{B}(\vec{\xi},t)] \\
&\simeq \sum_{\alpha=1}^{N} [m_\alpha \vec{v}_\alpha(t) + \frac{1}{4\pi c}\int d^3\vec{\xi}\vec{E}(\vec{\xi},t) \times \vec{B}(\vec{\xi},t)].
\end{aligned}
$$

Chapter 5

Classical Electromagnetic Field Theory

5.1 Variational methods for particles

1. Principle of least action:

   (a) Definitions:

   - Action: $S[x_l(t)] = \int_{t_1}^{t_2} L(x_l(t), \dot{x}_l(t), t)dt,$

   - Lagrangian: $L[x_l(t), \dot{x}_l(t), t] = T[\dot{x}_l(t), t] - V[x_l(t), t],$

   - Hamiltonian: $H[x_l(t), p_l(t), t] = \sum_l p_l(t)\dot{x}_l(t) - L[x_l(t), \dot{x}_l(t), t],$
     where x_l $(l = 1, 2, \cdots, 3N)$ is the generalized coordinates, p_l $(l = 1, 2, \cdots, 3N)$ is the generalized momentum, and satisfies

     $$p_l = \frac{\partial L}{\partial \dot{x}_l}.$$

   (b) Principle of least action: In all possible trajectories, the particles will take that the $\delta S = 0$ is satisfied.

2. Euler-Lagrangian equations:
   Since

   $$\delta S = \int_{t_1}^{t_2} \sum_l \Big(\frac{\partial L}{\partial x_l}\delta x_l + \frac{\partial L}{\partial \dot{x}_l}\delta \dot{x}_l\Big)dt = \int_{t_1}^{t_2} \sum_l \Big(\frac{\partial L}{\partial x_l} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_l}\Big)\delta x_l dt,$$

   where we have assumed that

   $$\delta x_l(t_2) = \delta x_l(t_1) = 0,$$

and the variation $\delta x_l(t)$ were assumed arbitrary, $\delta S = 0$ gives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_l} = \frac{\partial L}{\partial x_l}, \qquad (l = 1, 2, \cdots, 3N)$$

which are a set of second-order differential equations (totally $3N$ equations).

3. Equivalent Lagrangian

It can be verified that if $L \longrightarrow L' = L + \dfrac{df(x_l, t)}{dt}$, the equation of motion keeps invariant.

[Proof]

$$\begin{aligned} S[x_l(t)] \longrightarrow S'[x_l(t)] &= \int_{t_1}^{t_2} L'[x_l(t), \dot{x}_l(t), t]dt \\ &= S[x_l(t)] + \Big[f(x_l(t_2), t_2) - f(x_l(t_1), t_1)\Big] \\ &= S[x_l(t)] + \text{constant}, \end{aligned}$$

hence $\delta S = 0$ is equivalent to $\delta S' = 0$, and therefore, we call L' an equivalent Lagrangian of L.
[EOP]

4. Hamiltonian equation:

$$p_l = \frac{\partial L}{\partial \dot{x}_l},$$

$$H = \sum_l p_l \dot{x}_l - L,$$

Hence

(a)

$$\frac{\partial H}{\partial \dot{x}_l} = p_l - \frac{\partial L}{\partial \dot{x}_l} = 0,$$

i.e., $H(x_l(t), p_l(t), t)$ is $\dot{x}_l-$independent.

(b)

$$dH = \sum_l \left(\frac{\partial H}{\partial x_l}dx_l + \frac{\partial H}{\partial p_l}dp_l\right) + \frac{\partial H}{\partial t}dt,$$

also,

$$dH = \sum_l \left(\dot{x}_l dp_l - \frac{\partial L}{\partial x_l}dx_l\right) - \frac{\partial L}{\partial t}dt,$$

so one yields

$$\dot{x}_l = \frac{\partial H}{\partial p_l}, \qquad -\frac{\partial L}{\partial x_l} = \frac{\partial H}{\partial x_l}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Using Euler-Lagrangian equations and the definition of p_l, i.e.,

$$\frac{\partial L}{\partial x_l} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}_l}\Big) = \frac{d}{dt}p_l,$$

one has

$$\dot{x}_l = \frac{\partial H}{\partial p_l}, \qquad \dot{p}_l = -\frac{\partial H}{\partial x_l}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Further, it can be proved that

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} + \sum_l \Big(\frac{\partial H}{\partial x_l}\frac{dx_l}{dt} + \frac{\partial H}{\partial p_l}\frac{dp_l}{dt}\Big) \\ &= \frac{\partial H}{\partial t} + \sum_l \Big(\frac{\partial H}{\partial x_l}\frac{\partial H}{\partial p_l} - \frac{\partial H}{\partial p_l}\frac{\partial H}{\partial x_l}\Big) \\ &= \frac{\partial H}{\partial t}. \end{aligned}$$

So if H is not explicitly dependent on time, i.e., $\dfrac{\partial H}{\partial t} = 0$, then it is a constant of motion, i.e., $\dfrac{dH}{dt} = 0$.

5. Poisson brackets:
   For a variable $F(x_l, p_l, t)$,

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial F}{\partial t} + \sum_l \Big(\frac{\partial F}{\partial x_l}\dot{x}_l + \frac{\partial F}{\partial p_l}\dot{p}_l\Big) \\ &= \frac{\partial F}{\partial t} + \sum_l \Big(\frac{\partial F}{\partial x_l}\frac{\partial H}{\partial p_l} - \frac{\partial F}{\partial p_l}\frac{\partial H}{\partial x_l}\Big) \\ &= \frac{\partial F}{\partial t} + \big\{F, H\big\}. \end{aligned}$$

6. Example:
   The following gives an example of $N-$ charged particles moving in an electromagnetic field. (Only Coulomb potential is taken into account)
   The particles: $\vec{r}_{\alpha,i}(t)$ $(i = 1, 2, \cdots, 3N)$, α denotes the α^{th} particle.

   - Newton-Lorentz equation:

$$m_a\ddot{\vec{r}}_{\alpha,i}(t) = -q_\alpha \frac{\partial}{\partial \vec{r}_{\alpha,i}} U(\vec{r}_\alpha, t)$$

   - Lagrangian:

$$L(\vec{r}_\alpha, \dot{\vec{r}}_\alpha, t) = \sum_\alpha \Big[\Big(\frac{m_\alpha}{2}\dot{\vec{r}}^{\,2}_\alpha(t) - q_\alpha U(\vec{r}_\alpha, t)\Big)\Big].$$

   So Euler-Lagrangian equations give

$$\left.\begin{aligned} \frac{\partial L}{\partial \vec{r}_{\alpha,i}} &= -q_\alpha \frac{\partial}{\partial \vec{r}_{\alpha,i}} U(\vec{r}_\alpha, t) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_{\alpha,i}} &= m_\alpha \ddot{\vec{r}}_{\alpha,i} \end{aligned}\right\} \Longrightarrow m_a\ddot{\vec{r}}_{\alpha,i}(t) = -q_\alpha \frac{\partial}{\partial \vec{r}_{\alpha,i}} U(\vec{r}_\alpha, t).$$

- Hamiltonian:

$$\vec{p}_{\alpha,i} = \frac{\partial L}{\partial \dot{\vec{r}}_{\alpha,i}} = m_\alpha \dot{\vec{r}}_{\alpha,i},$$

$$H = \sum_\alpha \left[\frac{m_\alpha}{2} \dot{\vec{r}}_\alpha^2 + q_\alpha U(\vec{r}_\alpha, t) \right] = \sum_\alpha \left[\frac{p_\alpha^2}{2m_\alpha} + q_\alpha U(\vec{r}_\alpha, t) \right].$$

So Hamiltonian equations give

$$\frac{\partial H}{\partial \vec{r}_{\alpha,i}} = q_\alpha \frac{\partial}{\partial \vec{r}_{\alpha,i}} U(\vec{r}_\alpha, t), \qquad \frac{\partial H}{\partial \vec{p}_{\alpha,i}} = \frac{\vec{p}_{\alpha,i}}{m_\alpha},$$

$$\Rightarrow \dot{\vec{r}}_{\alpha,i} = \frac{\vec{p}_{\alpha,i}}{m_\alpha}, \qquad \dot{\vec{p}}_{\alpha,i} = -q_\alpha \frac{\partial}{\partial \vec{r}_{\alpha,i}} U(\vec{r}_\alpha, t). \tag{5.1}$$

5.2 Classical electromagnetic field theory (free field theory)

1. Definitions

   (a) Coordinates: $\Phi_i(\vec{r}, t), \quad (i = 1, 2, \cdots)$

   (b) Velocities: $\dot{\Phi}_i(\vec{r}, t) = \frac{\partial}{\partial t} \Phi_i(\vec{r}, t), \quad (i = 1, 2, \cdots)$

   (c) Momenta: $\pi_i(\vec{r}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i}, \quad (i = 1, 2, \cdots)$

   (d) Action:

$$\begin{aligned} S[\Phi_i] &= \int dt L(\Phi_i, \dot{\Phi}_i, \partial_j \Phi_i, t) \quad (j = x, y, z) \\ &= \int dt \int d^3\vec{r} \mathcal{L}(\Phi_i, \dot{\Phi}_i, \partial_j \Phi_i, t). \end{aligned}$$

   (e) Lagrangian density: $\mathcal{L}(\Phi_i, \dot{\Phi}_i, \partial_j \Phi_i, t)$.

   (f) Lagrangian: $L(\Phi_i, \dot{\Phi}_i, \partial_j \Phi_i, t) = \int d^3\vec{r} \mathcal{L}$.

   (g) Hamiltonian density: $\mathcal{H}(\pi_i, \Phi_i, \partial_j \Phi_i, t) = \sum_i \pi_i \dot{\Phi}_i - \mathcal{L}(\Phi_i, \dot{\Phi}_i, \partial_j \Phi_i, t)$.

   (h) Hamiltonian: $H = \int d^3\vec{r} \mathcal{H}$.

2. Principle of least action and Euler-Lagrangian equations

   (a) Principle of least action:

$$\delta S[\Phi_i] = 0.$$

(b) Euler-Lagrangian equations:

$$\begin{aligned}\delta S[\Phi_i] &= \int dt \int d^3\vec{r} \sum_i \left(\frac{\partial \mathcal{L}}{\partial \Phi_i} \delta\Phi_i + \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} \delta\dot{\Phi}_i + \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_j \Phi_i)} \delta(\partial_j \Phi_i) \right) \\ &= \int dt \int d^3\vec{r} \sum_i \left(\frac{\partial \mathcal{L}}{\partial \Phi_i} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} - \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \Phi_i)} \right) \delta\Phi_i(\vec{r}, t) \\ &= 0.\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} = \frac{\partial \mathcal{L}}{\partial \Phi_i} - \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \Phi_i)}.$$

To be more compact, using Einstein convention, one has

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} = \partial_a \frac{\partial \mathcal{L}}{\partial(\partial_a \Phi_i)}. \qquad (a = 0, 1, 2, 3)$$

(c) Equivalent Lagrangian:

$$L \longrightarrow L' = L + \frac{d}{dt} f(\Phi_i, \vec{r}, t)$$

does not change the state of motion.

(d) Hamiltonian equations:

$$\pi_i(\vec{r}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i}.$$

$$\mathcal{H}(\pi_i, \Phi_i, \partial_j \Phi_i) = \sum_i \pi_i \dot{\Phi}_i - \mathcal{L}(\Phi_i, \dot{\Phi}_i, \partial_j \Phi_i).$$

$$\begin{aligned}d\mathcal{H} &= \frac{\partial \mathcal{H}}{\partial t} dt + \sum_i \left(\frac{\partial \mathcal{H}}{\partial \pi_i} d\pi_i + \frac{\partial \mathcal{H}}{\partial \Phi_i} d\Phi_i + \frac{\partial \mathcal{H}}{\partial \partial_j \Phi_i} d\partial_j \Phi_i \right) \\ &= \sum_i \left(\dot{\Phi}_i d\pi_i + \pi_i d\dot{\Phi}_i - \frac{\partial \mathcal{L}}{\partial \Phi_i} d\Phi_i - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} d\dot{\Phi}_i - \frac{\partial \mathcal{L}}{\partial \partial_j \Phi_i} d\partial_j \Phi_i \right) \\ &= \sum_i \left(\dot{\Phi}_i d\pi_i - \frac{\partial \mathcal{L}}{\partial \Phi_i} d\Phi_i - \frac{\partial \mathcal{L}}{\partial \partial_j \Phi_i} d\partial_j \Phi_i \right).\end{aligned}$$

So one has

$$\dot{\Phi}_i = \frac{\partial \mathcal{H}}{\partial \pi_i}, \qquad -\frac{\partial \mathcal{L}}{\partial \Phi_i} = \frac{\partial \mathcal{H}}{\partial \Phi_i}, \qquad \frac{\partial \mathcal{H}}{\partial \partial_j \Phi_i} = -\frac{\partial \mathcal{L}}{\partial \partial_j \Phi_i}.$$

Using Euler-Lagrangian equations,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \Phi_i} &= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} + \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \Phi_i)} \\ &= \frac{\partial}{\partial t}\pi_i - \sum_j \partial_j \frac{\partial \mathcal{H}}{\partial(\partial_j \Phi_i)},\end{aligned}$$

one has

$$\frac{\partial}{\partial t}\pi_i - \sum_j \partial_j \frac{\partial \mathcal{H}}{\partial(\partial_j \Phi_i)} = -\frac{\partial \mathcal{H}}{\partial \Phi_i}.$$

(e) Examples −1:

Consider an electromagnetic field in vacuum:

$$H = \frac{1}{8\pi}\int d^3\vec{r}\Big[\vec{E}^2(\vec{r},t) + \vec{B}^2(\vec{r},t)\Big],$$

$$L = \frac{1}{8\pi}\int d^3\vec{r}\Big[\vec{E}^2(\vec{r},t) - \vec{B}^2(\vec{r},t)\Big],$$

$$\mathcal{L} = \frac{1}{8\pi}\Big[\vec{E}^2(\vec{r},t) - \vec{B}^2(\vec{r},t)\Big],$$

$$\mathcal{H} = \frac{1}{8\pi}\Big[\vec{E}^2(\vec{r},t) + \vec{B}^2(\vec{r},t)\Big],$$

$$S = \frac{1}{8\pi}\int dt\int d^3\vec{r}\Big[\vec{E}^2(\vec{r},t) - \vec{B}^2(\vec{r},t)\Big].$$

To proceed, using scalar and vector potential

$$\vec{E} = -\nabla U - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}, \qquad \vec{B} = \nabla\times\vec{A}.$$

So,

$$\mathcal{L}(U,\dot{U},\partial_j U; A_i,\dot{A}_i,\partial_j A_i) = \frac{1}{8\pi}\Big[\Big(\frac{1}{c}\dot{\vec{A}} + \nabla U\Big)^2 - \Big(\nabla\times\vec{A}\Big)^2\Big].$$

Now, we evaluate $\mathcal{L}$, firstly,

$$\begin{aligned}
\Big(\nabla\times\vec{A}\Big)\cdot(\nabla\times\vec{A}) &= \varepsilon_{ijk}\partial_j A_k \varepsilon_{ilm}\partial_l A_m \\
&= \Big(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\Big)\partial_j A_k\partial_l A_m \\
&= \partial_j A_k\partial_j A_k - \partial_j A_k\partial_k A_j \\
&= \sum_{i,j}(\partial_j A_i)^2 - \Big(\sum_i \partial_i A_i\Big)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{L}(U,\dot{U},\partial_j U; A_i,\dot{A}_i,\partial_j A_i) &= \frac{1}{8\pi}\sum_i\Big[\frac{1}{c^2}\dot{A}_i^2 + \frac{2}{c}\dot{A}_i\partial_i U + (\partial_i U)^2\Big] \\
&- \frac{1}{8\pi}\Big[\sum_{i,j}(\partial_j A_i)^2 - \Big(\sum_i \partial_i A_i\Big)^2\Big].
\end{aligned}$$

Now, evaluate Euler-Lagrangian equations,

(a) Set $\Phi_i = U$, then from Euler-Lagrangian equation, one has

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{U}} = 0, \qquad \frac{\partial \mathcal{L}}{\partial U} = 0.$$

$$\begin{aligned}
\sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j U)} &= \frac{1}{8\pi}\sum_j \partial_j\left(\frac{2}{c}\dot{A}_j + 2\partial_j U\right) \\
&= \frac{1}{4\pi}\sum_j \partial_j\left(\frac{1}{c}\dot{A}_j + \partial_j U\right) \\
&= -\frac{1}{4\pi}\sum_j \partial_j E_j \\
&= -\frac{1}{4\pi}\nabla\cdot\vec{E}.
\end{aligned}$$

Hence, one has

$$\nabla\cdot\vec{E} = 0.$$

(b) Set $\Phi_i = A_i$, then

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{A}_i} &= \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\frac{2}{c^2}\dot{A}_i + \frac{2}{c}\partial_i U\right) \\
&= \frac{1}{4\pi}\frac{1}{c}\left(\frac{1}{c}\ddot{A}_i + \partial_i \dot{U}\right) \\
&= -\frac{1}{4\pi c}\dot{E}_i.
\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial A_i} = 0,$$

$$\begin{aligned}
\sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j A_i)} &= -\frac{2}{8\pi}\sum_j\left[\partial_j\partial_j A_i - \delta_{ij}\partial_j\nabla\cdot\vec{A}\right] \\
&= -\frac{2}{8\pi}\left[\nabla^2\vec{A} - \nabla(\nabla\cdot\vec{A})\right]_i \\
&= \frac{2}{8\pi}\left[\nabla\times(\nabla\times\vec{A})\right]_i \\
&= \frac{1}{4\pi}(\nabla\times\vec{B})_i.
\end{aligned}$$

Therefore, from Euler-Lagrangian equations, i.e.,

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} = \frac{\partial \mathcal{L}}{\partial \Phi_i} - \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \Phi_i)},$$

one has

$$-\frac{1}{4\pi c}\dot{\vec{E}} = -\frac{1}{4\pi}\nabla\times\vec{B},$$

i.e.,

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}.$$

(c) $\nabla \times \vec{E}$ and $\nabla \cdot \vec{B}$ equations can be directly derived by writing the potentials, i.e.,

$$\begin{aligned} \nabla \times \vec{E} &= \nabla \times \Big(- \nabla U - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\Big) \\ &= -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{A} \\ &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \end{aligned}$$

$$\nabla \cdot \vec{B} = \nabla \cdot \nabla \times \vec{A} = 0.$$

(d) π_i:
Set $\Phi_i = U$, then

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{U}} = 0,$$

Set $\Phi_i = A_i$, then

$$\begin{aligned} \pi_i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} \\ &= \frac{1}{8\pi}\Big[\frac{2}{c^2}\dot{A}_i + \frac{2}{c}\partial_j U\Big] \\ &= -\frac{1}{4\pi c}E_i, \end{aligned}$$

So

$$\begin{aligned} \mathcal{H} &= \sum_i \pi_i \dot{\Phi}_i - \mathcal{L} \\ &= \sum_i -\frac{1}{4\pi c}E_i \dot{A}_i - \mathcal{L} \\ &= \sum_i \frac{1}{4\pi c}\Big(\frac{1}{c}\dot{A}_i + \partial_i U\Big)\dot{A}_i - \frac{1}{8\pi}\sum_i \Big[\frac{1}{c^2}\dot{A}_i^2 + \frac{2}{c}\dot{A}_i \partial_i U + (\partial_i U)^2\Big] \\ &+ \frac{1}{8\pi}\Big[\sum_{i,j}(\partial_j A_i)^2 - \Big(\sum_i \partial_i A_i\Big)^2\Big] \\ &= \sum_i \frac{1}{8\pi}\Big[\frac{1}{c^2}\dot{A}_i^2 - (\partial_i U)^2\Big] + \frac{1}{8\pi}\Big[\sum_{i,j}(\partial_j A_i)^2 - \Big(\sum_i \partial_i A_i\Big)^2\Big] \\ &= \frac{1}{8\pi}\Big(\vec{E}^2 + \vec{B}^2\Big), \end{aligned}$$

(e) Example-2:

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)U(\vec{r},t) = 0.$$

$$\mathcal{L}(U,\dot{U},\partial_j U) = \frac{1}{c^2}\Big(\frac{\partial U}{\partial t}\Big)^2 - (\nabla U)^2 \quad = \quad \frac{1}{c^2}\dot{U}^2 - (\sum_j \partial_j U)^2.$$

$$\begin{aligned}
\Phi_i &= U, \\
\frac{\partial \mathcal{L}}{\partial U} &= 0, \\
\frac{\partial \mathcal{L}}{\partial \dot{U}} &= \frac{2}{c^2}\dot{U}, \\
\frac{\partial \mathcal{L}}{\partial(\partial_j U)} &= -2\partial_j U.
\end{aligned}$$

Therefore

$$\sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j U)} = -\sum_j 2\partial_j\partial_j U = -2\nabla^2 U.$$

Since

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{U}} = \frac{2}{c^2}\ddot{U},$$

from Euler-Lagrangian equations, i.e.,

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{U}} = \frac{\partial \mathcal{L}}{\partial U} - \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j U)},$$

one yields

$$\frac{2}{c^2}\frac{\partial^2}{\partial t^2}U = 2\nabla^2 U,$$

So that

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)U(\vec{r},t) = 0.$$

Further,

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{U}} = \frac{2}{c^2}\dot{U}.$$

$$\begin{aligned}
\mathcal{H} &= \sum_i \pi_i \dot{\Phi}_i - \mathcal{L} \\
&= \frac{2}{c^2}\dot{U}^2 - \Big[\frac{1}{c^2}\dot{U}^2 - (\nabla U)^2\Big] \\
&= \frac{1}{c^2}\Big(\frac{\partial U}{\partial t}\Big)^2 + (\nabla U)^2.
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} &= \frac{1}{8\pi}\Big(\vec{E}^2 - \vec{B}^2\Big) \\
&= \frac{1}{8\pi}\sum_i \Big[\frac{1}{c^2}\dot{A}_i^2 + \frac{2}{c}\dot{A}_i\partial_i U + (\partial_i U)^2\Big] - \frac{1}{8\pi}\Big[\sum_{i,j}(\partial_j A_i)^2 - \Big(\sum_i \partial_i A_i\Big)^2\Big],
\end{aligned}$$

$$\pi_U = \frac{\partial \mathcal{L}}{\partial \dot{U}} = 0, \Longrightarrow \pi_U \dot{U} = 0.$$

$$\pi_{A_i} = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \frac{1}{8\pi}\Big[\frac{2}{c^2}\dot{A}_i + \frac{2}{c}\partial_i U\Big].$$

$$\Longrightarrow \sum_i \pi_{A_i}\dot{A}_i = \frac{1}{8\pi}\sum_i \Big[\frac{2}{c^2}\dot{A}_i^2 + \frac{2}{c}A_i\partial_i U\Big].$$

So

$$\begin{aligned}
\mathcal{H} &= \sum_i \pi_i \dot{\Phi}_i - \mathcal{L} \\
&= \sum_i \pi_i \dot{A}_i - \mathcal{L} \\
&= \frac{1}{8\pi}\sum_i \Big[\frac{2}{c^2}\dot{A}_i^2 - \Big(\partial_i U\Big)^2\Big] + \frac{1}{8\pi}\Big[\sum_{i,j}(\partial_j A_i)^2 - \Big(\sum_i \partial_i A_i\Big)^2\Big] \\
&= \frac{1}{8\pi}\vec{E}^2 + \frac{1}{8\pi}|\nabla \times \vec{A}|^2 \\
&= \frac{1}{8\pi}\vec{E}^2 + \frac{1}{8\pi}\vec{B}^2.
\end{aligned}$$

Also

$$\mathcal{L} = -\frac{1}{16\pi}F_{ab}F^{ab} \tag{5.2}$$

[Proof]

$$\begin{aligned}
\mathcal{L} &= \frac{1}{8\pi}[\vec{E}^2 - \vec{B}^2] \\
&= \frac{1}{8\pi}[E_k E^k - B_k B^k],
\end{aligned} \tag{5.3}$$

$$E^k = F^{0k}, \quad E_k = E^k = F^{0k} = -F_{0k} \tag{5.4}$$

So

$$\begin{aligned}
E_k E^k &= -F_{0k}F^{0k} \\
&= -\frac{1}{2}(F_{0k}F^{0k} + F_{k0}F^{k0}).
\end{aligned} \tag{5.5}$$

$$B^k = \frac{1}{2}\epsilon^{kij}F_{ij}, \quad B_k = \frac{1}{2}\epsilon_{ki'j'}F^{i'j'}. \tag{5.6}$$

So

$$\begin{aligned} B_k B^k &= \frac{1}{4}\epsilon^{kij}\epsilon_{ki'j'}F_{ij}F^{i'j'} \\ &= \frac{1}{4}(\delta^i_{.i'}\delta^j_{.j'} - \delta^i_{.j'}\delta^j_{.i'})F_{ij}F^{i'j'} \\ &= \frac{1}{4}F_{ij}(F^{ij} - F^{ji}) \\ &= \frac{1}{2}F_{ij}F^{ij}, \end{aligned} \tag{5.7}$$

So

$$\begin{aligned} \mathcal{L} &= \frac{1}{8\pi}\left\{-\frac{1}{2}[F_{0k}F^{0k} + F_{k0}F^{k0}] - \frac{1}{2}F_{ij}F^{ij}\right\} \\ &= -\frac{1}{16\pi}\left\{F_{0k}F^{0k} + F_{k0}F^{k0} + F_{ij}F^{ij}\right\} \\ &= -\frac{1}{16\pi}F_{ab}F^{ab}. \end{aligned} \tag{5.8}$$

5.3 Lagrangian and Hamiltonian of free classical electromagnetic field

The Lagrangian density of the free electromagnetic field is:

$$\begin{aligned} \mathcal{L}_{em} &= -\frac{1}{16\pi}F_{cd}F^{cd} \\ &= -\frac{1}{16\pi}(\partial_c A_d - \partial_d A_c)(\partial^c A^d - \partial^d A^c) \\ &= -\frac{1}{16\pi}(\partial_c A_d\partial^c A^d + \partial_d A_c\partial^d A^c - \partial_c A_d\partial^d A^c - \partial_d A_c\partial^c A^d) \\ &= -\frac{1}{8\pi}(\partial_c A_d\partial^c A^d - \partial_c A_d\partial^d A^c), \end{aligned}$$

where

$$\begin{aligned} \partial_c A_d\partial^c A^d &= \partial_0 A_d\partial^0 A^d + \partial_k A_d\partial^k A^d \\ &= (\partial_0 A_0\partial^0 A^0 + \partial_0 A_k\partial^0 A^k) + (\partial_k A_0\partial^k A^0 + \partial_k A_i\partial^k A^i) \\ &= \left[\frac{1}{c^2}(\dot{A}^0)^2 - \frac{1}{c^2}\sum_{k=1}^{3}(\dot{A}^k)^2\right] + \left[\sum_{i,k=1}^{3}(\partial_k A^i)^2 - \sum_{k=1}^{3}(\partial_k A^0)^2\right], \end{aligned}$$

and

$$\begin{aligned} \partial_c A_d\partial^d A^c &= \partial_0 A_d\partial^d A^0 + \partial_k A_d\partial^d A^k \\ &= (\partial_0 A_0\partial^0 A^0 + \partial_0 A_k\partial^k A^0) + (\partial_k A_0\partial^0 A^k + \partial_k A_i\partial^i A^k) \\ &= \left[\frac{1}{c^2}(\dot{A}^0)^2 + \frac{1}{c}\sum_{k=1}^{3}\dot{A}^k\partial_k A^0\right] + \left[\frac{1}{c}\sum_{k=1}^{3}\dot{A}^k\partial_k A^0 + \sum_{i,k=1}^{3}\partial_k A^i\partial_i A^k\right] \\ &= \frac{1}{c^2}(\dot{A}^0)^2 + \frac{2}{c}\sum_{k=1}^{3}\dot{A}^k\partial_k A^0 + \sum_{i,k=1}^{3}\partial_k A^i\partial_i A^k. \end{aligned}$$

So

$$
\begin{aligned}
\mathcal{L}_{em} &= -\frac{1}{8\pi}\left[\frac{1}{c^2}(\dot{A}^0)^2 - \frac{1}{c^2}\sum_{k=1}^{3}(\dot{A}^k)^2\right] + \left[\sum_{i,k=1}^{3}(\partial_k A^i)^2 - \sum_{k=1}^{3}(\partial_k A^0)^2\right] \\
&\quad + \frac{1}{8\pi}\left[\frac{1}{c^2}(\dot{A}^0)^2 + \frac{2}{c}\sum_{k=1}^{3}\dot{A}^k\partial_k A^0 + \sum_{i,k=1}^{3}\partial_k A^i \partial_i A^k\right] \\
&= \frac{1}{8\pi}\left\{\frac{1}{c^2}\sum_{k=1}^{3}(\dot{A}^k)^2 + \frac{2}{c}\sum_{k=1}^{3}\dot{A}^k\partial_k A^0 + \sum_{k=1}^{3}(\partial_k A^0)^2 + \sum_{i,k=1}^{3}\left[\partial_k A^i\partial_i A^k - (\partial_k A^i)^2\right]\right\} \\
&= \frac{1}{8\pi}\left[\left(-\nabla A^0 - \frac{1}{c}\vec{A}\right)^2 - (\nabla\times\vec{A})^2\right] \\
&= \frac{1}{8\pi}\left[\vec{E}^2 - \vec{B}^2\right],
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
(\nabla\times\vec{A})^2 &= \varepsilon_{ijk}\partial^j A^k \varepsilon^{ij'k'}\partial_{j'}A_{k'} \\
&= (\delta_j^{j'}\delta_k^{k'} - \delta_k^{j'}\delta_j^{k'})\partial^j A^k \partial_{j'}A_{k'} \\
&= \partial^j A^k(\partial_j A_k - \partial_k A_j) \\
&= \sum_{i,k=1}^{3}\left[(\partial_k A^i)^2 - \partial_k A^i \partial_i A^k\right].
\end{aligned}
$$

Then, the Lagrangian density of the free electromagnetic field is

$$
\begin{aligned}
\mathcal{L}_{em} &= \frac{1}{8\pi}\left[\vec{E}^2 - \vec{B}^2\right] \\
&= -\frac{1}{16\pi}F_{cd}F^{cd} \\
&= \frac{1}{8\pi}\left\{\frac{1}{c^2}\sum_{k=1}^{3}(\dot{A}^k)^2 + \frac{2}{c}\sum_{k=1}^{3}\dot{A}^k\partial_k A^0 + \sum_{k=1}^{3}(\partial_k A^0)^2 + \sum_{i,k=1}^{3}\left[\partial_k A^i\partial_i A^k - (\partial_k A^i)^2\right]\right\}.
\end{aligned}
$$

Now, evaluate the Hamiltonian density of the free electromagnetic field. First of all, the canonical momentum is

$$\pi_a = \frac{\partial\mathcal{L}_{em}}{\partial\dot{A}^a}.$$

So,

$$\pi_0 = \frac{\partial\mathcal{L}_{em}}{\partial\dot{A}^0} = 0.$$

So, the Hamiltonian density is

$$
\begin{aligned}
\mathcal{H}_{em} &= \pi_a\dot{A}^a - \mathcal{L}_{em} \\
&= \pi_k\dot{A}^k - \mathcal{L}_{em}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4\pi c}\left[\frac{1}{c}\dot{A}^k + \partial_k A^0\right]\dot{A}^k - \mathcal{L}_{em} \\
&= \frac{1}{8\pi}\left[\vec{E}^2 + \vec{B}^2\right] + \frac{1}{4\pi}\nabla A^0 \cdot \vec{E} \\
&= \frac{1}{8\pi}\left[\vec{E}^2 + \vec{B}^2\right] + \frac{1}{4\pi}\nabla\phi \cdot \vec{E}.
\end{aligned}
$$

Hence, the Hamiltonian of the free electromagnetic field is:

$$
\begin{aligned}
H_{em} &= \int d^3\vec{x}\mathcal{H}_{em} \\
&= \int d^3\vec{x}\left\{\frac{1}{8\pi}\left[\vec{E}^2 + \vec{B}^2\right] + \frac{1}{4\pi}\nabla\phi \cdot \vec{E}\right\} \\
&= \frac{1}{8\pi}\int (\vec{E}^2 + \vec{B}^2) d^3\vec{x},
\end{aligned}
$$

where we have used Gaussian theorem, i.e., $\nabla \cdot \vec{E} = 0$, and the fact that

$$
\begin{aligned}
\int d^3\vec{x}\nabla\phi \cdot \vec{E} &= \int d^3\vec{x}\left[\nabla \cdot (\phi\vec{E}) - \phi\nabla \cdot \vec{E}\right] \\
&= \oiint \phi\vec{E} \cdot d\vec{\sigma} - \int d^3\vec{x}\phi(\nabla \cdot \vec{E}) \\
&= 0.
\end{aligned}
$$

So, we may also rewrite the Hamiltonian density as

$$
\mathcal{H}_{em} = \frac{1}{8\pi}\left[\vec{E}^2 + \vec{B}^2\right].
$$

Appendix A

Appendix

A.1 Some identities, theorems, and equations related to Fourier transform

1. Parseval identity:

$$\int_{-\infty}^{+\infty} d^3\vec{r} F^*(\vec{r}) G(\vec{r}) = \int_{-\infty}^{+\infty} d^3\vec{k} \tilde{F}^*(\vec{k}) \tilde{G}(\vec{k}). \tag{A.1}$$

   [Proof]
   Since

$$F^*(\vec{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} d^3\vec{k} \tilde{F}^*(\vec{k}) \exp(-i\vec{k} \cdot \vec{r}),$$

$$G(\vec{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} d^3\vec{k}' G(\vec{k}') \exp(i\vec{k}' \cdot \vec{r}),$$

   therefore

$$\begin{aligned} \int_{-\infty}^{+\infty} d^3\vec{r} F^*(\vec{r}) G(\vec{r}) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3\vec{k} d^3\vec{k}' \tilde{F}^*(\vec{k}) \tilde{G}(\vec{k}') \exp[-i(\vec{k} - \vec{k}') \cdot \vec{r}] d^3\vec{r} \\ &= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3\vec{k} d^3\vec{k}' \tilde{F}^*(\vec{k}) \tilde{G}(\vec{k}') (2\pi)^3 \delta(\vec{k} - \vec{k}') \\ &= \int_{-\infty}^{+\infty} \tilde{F}^*(\vec{k}) \tilde{G}(\vec{k}) d^3\vec{k}, \end{aligned}$$

   where we have used the identity

$$\int_{-\infty}^{+\infty} \exp[-i(\vec{k} - \vec{k}') \cdot \vec{r}] d^3\vec{r} = (2\pi)^3 \delta(\vec{k} - \vec{k}'),$$

   and

$$\int_{-\infty}^{+\infty} \delta(\vec{r}) f(\vec{r}) d^3\vec{r} = f(0).$$

   [EOP]

2. Convolution theorem:

$$\int_{-\infty}^{+\infty} d^3\vec{r}' F(\vec{r}')G(\vec{r}-\vec{r}') = \int_{-\infty}^{+\infty} \tilde{F}(\vec{k})\tilde{G}(\vec{k})\exp(i\vec{k}\cdot\vec{r})d^3\vec{k}.$$

[Proof]
Let

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{+\infty} d^3\vec{r}' F(\vec{r}')G(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{+\infty} d^3\vec{k}\tilde{V}(\vec{k})\exp(i\vec{k}\cdot\vec{r}),$$

$$\tilde{V}(\vec{k}) = (2\pi)^{\frac{3}{2}}\tilde{F}(\vec{k})\tilde{G}(\vec{k}).$$

Since

$$F(\vec{r}') = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \tilde{F}(\vec{k})\exp(i\vec{k}\cdot\vec{r}')d^3\vec{k},$$

$$G(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \tilde{G}(\vec{k}')\exp[i\vec{k}'\cdot(\vec{r}-\vec{r}')]d^3\vec{k}',$$

therefore

$$\begin{aligned}
\text{l.h.s.} &= \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2\pi)^3}\int \tilde{F}(\vec{k})\tilde{G}(\vec{k}')\exp[-i(\vec{k}'-\vec{k})\cdot\vec{r}']\exp(i\vec{k}'\cdot\vec{r})d^3\vec{r}'d^3\vec{k}d^3\vec{k}' \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2\pi)^3}\int \tilde{F}(\vec{k})\tilde{G}(\vec{k}')(2\pi)^3\delta(\vec{k}'-\vec{k})\exp(i\vec{k}'\cdot\vec{r})d^3\vec{k}d^3\vec{k}' \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \tilde{F}(\vec{k})\tilde{G}(\vec{k})\exp(i\vec{k}\cdot\vec{r})d^3\vec{k} \\
&= \text{r.h.s.}
\end{aligned}$$

So

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\vec{r}' F(\vec{r}')G(\vec{r}-\vec{r}') \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad \tilde{F}(\vec{k})\tilde{G}(\vec{k}).$$

[EOP]

3. Some important relations of Fourier transform:

   (a)

$$\frac{1}{4\pi r} \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{k^2}.$$

[Proof]
On the one hand,

$$\begin{aligned}
\mathcal{F}\left\{\frac{1}{4\pi r}\right\} &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\frac{1}{4\pi r}\exp(-i\vec{k}\cdot\vec{r})d^3\vec{r} \\
&= \frac{1}{(2\pi)^{3/2}}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\underset{a\to 0}{\text{limit}}\left[\frac{1}{4\pi\sqrt{r^2+a^2}}\exp(-ikr\cos\theta)r^2\sin\theta\right]d\phi d\theta dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{(2\pi)^{3/2}} \lim_{a\to 0} \left[\int_0^{+\infty} \int_0^{\pi} \frac{1}{2\sqrt{r^2+a^2}} \exp(-ikr\cos\theta) r^2 d\cos\theta dr \right] \\
&= \frac{1}{k}\frac{1}{(2\pi)^{3/2}} \operatorname*{Limit}_{a\to 0} \left[\int_0^{+\infty} \frac{r\sin kr}{\sqrt{r^2+a^2}} dr \right] \\
&= \frac{1}{(2\pi)^{3/2}k} \lim_{a\to 0} \left[\sqrt{a^2}\mathrm{sgn}(k) K_1\Big(k\mathrm{sgn}(k)\sqrt{a^2}\Big) \right] \\
&= \frac{1}{(2\pi)^{3/2}} \frac{1}{k^2},
\end{aligned}
$$

where $K_1\left(k\mathrm{sgn}\,(k)\,\sqrt{a^2}\right)$ is the second distortion Bessel function. On the other hand,

$$
\begin{aligned}
\mathcal{F}^{-1}\Big\{\frac{1}{(2\pi)^{3/2}}\frac{1}{k^2}\Big\} &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{3/2}} \frac{1}{k^2} \exp(i\vec{k}\cdot\vec{r}) d^3\vec{k} \\
&= \frac{1}{(2\pi)^3} \int_0^{+\infty} \int_0^{\pi} \int_0^{2\pi} \exp(ikr\cos\theta)\sin\theta d\phi d\theta dk \\
&= \frac{-2\pi}{(2\pi)^3} \int_0^{+\infty} \int_0^{\pi} \exp(ikr\cos\theta) d\cos\theta dk \\
&= \frac{1}{r}\frac{4\pi}{(2\pi)^3} \int_0^{+\infty} \frac{\sin kr}{k} dk \\
&= \frac{1}{r}\frac{4\pi}{(2\pi)^3}\frac{\pi}{2} \\
&= \frac{1}{4\pi r}.
\end{aligned}
$$

[EOP]

(b)

$$
\frac{\vec{r}}{4\pi r^3} \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad \frac{1}{(2\pi)^{3/2}} \frac{-i\vec{k}}{k^2}.
$$

[Proof]
Since

$$
\nabla\frac{1}{r} = -\frac{\vec{r}}{r^3}, \qquad \nabla \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad i\vec{k},
$$

so the Fourier transform of $\frac{\vec{r}}{r^3}$ is that of $-\nabla\frac{1}{r}$. Also, since

$$
\frac{1}{4\pi r} \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{k^2}
$$

so

$$
\frac{\vec{r}}{4\pi r^3} = -\nabla\frac{1}{4\pi r} \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{k^2}(-i\vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{-i\vec{k}}{k^2}.
$$

[EOP]

(c)

$$\delta(\vec{r}-\vec{r}_a) \quad \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \quad \frac{1}{(2\pi)^{\frac{3}{2}}}\exp(-i\vec{k}\cdot\vec{r}_a).$$

[Proof]

$$\begin{aligned}\mathcal{F}\{\delta(\vec{r}-\vec{r}_a)\} &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{+\infty}\delta(\vec{r}-\vec{r}_a)\exp[-i\vec{k}\cdot\vec{r}]d^3\vec{r}\\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\exp(-i\vec{k}\cdot\vec{r}_a).\end{aligned}$$

[EOP]

4. For a transverse field $\vec{F}(\vec{r})$ and a longitudinal field $\vec{G}(\vec{r})$, one has

$$\int d^3\vec{r}\vec{F}(\vec{r})\cdot\vec{G}(\vec{r}) = 0.$$

[Proof]
From Parseval identity, we have

$$\int d^3\vec{r}\vec{F}(\vec{r})\cdot\vec{G}(\vec{r}) = \int d^3\vec{k}\tilde{\vec{F}}(\vec{k})\cdot\tilde{\vec{G}}(\vec{k}),$$

in which

$$\vec{k}\cdot\tilde{\vec{F}}(\vec{k}) = 0,$$

and

$$\vec{k}\times\tilde{\vec{G}}(\vec{k}) = 0.$$

So

$$\tilde{\vec{F}}(\vec{k}) = (I-\vec{k}^0\vec{k}^0)\cdot\tilde{\vec{F}}_{total}(\vec{k}),$$

$$\tilde{\vec{G}}(\vec{k}) = \vec{k}^0[\vec{k}^0\cdot\tilde{\vec{G}}_{total}(\vec{k})].$$

Then,

$$\begin{aligned}\tilde{\vec{F}}(\vec{k})\cdot\tilde{\vec{G}}(\vec{k}) &= \Big[(I-\vec{k}^0\vec{k}^0)\cdot\tilde{\vec{F}}_{total}(\vec{k})\Big]\Big[\cdot\vec{k}^0\vec{k}^0\cdot\tilde{\vec{G}}_{total}(\vec{k})\Big]\\ &= \tilde{\vec{F}}_{total}(\vec{k})\cdot\vec{k}^0\vec{k}^0\cdot\tilde{\vec{G}}_{total}(\vec{k}) - \vec{k}^0\vec{k}^0\cdot\tilde{\vec{F}}_{total}(\vec{k})\cdot\vec{k}^0\vec{k}^0\cdot\tilde{\vec{G}}_{total}(\vec{k})\\ &= \tilde{\vec{F}}_{total}(\vec{k})\cdot\tilde{\vec{G}}_{total}(\vec{k}) - \tilde{\vec{F}}_{total}(\vec{k})\cdot\tilde{\vec{G}}_{total}(\vec{k})\\ &= 0,\end{aligned}$$

$$\int d^3\vec{k}\tilde{\vec{F}}(\vec{k})\cdot\tilde{\vec{G}}(\vec{k}) = 0,$$

therefore

$$\int d^3\vec{r}\vec{F}(\vec{r})\cdot\vec{G}(\vec{r}) = 0.$$

[EOP]

A.2 Preliminaries of tensor and vector analysis and some important formulae

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