Research Statement

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My principal interest lies in studies strongly motivated by practical applications. I am particularly attracted to interdisciplinary projects which require knowledge of many fields in order to produce results that can be applied in real life.

My current research focuses on inverse problems for partial differential equations, in particular inverse acoustic and electromagnetic scattering theory. I have endeavored to uniqueness in the inverse scattering problems. Now we give a brief description of my works I have done under the supervision of Prof. Bo Zhang during my Ph.D. period. These works can be summarized in two themes as follows:

- **Unique determination of a ball by a finite number of far field data:**
  Since a ball is uniquely determined by its radius and center, it seems that only a finite number of far field data is enough to identify the ball. Under the assumption that the radius is small enough, we proved that a perfectly conducting ball is uniquely determined by at most four far field data with a single incident direction and polarization in [11] and the shape of a sound-soft ball is uniquely determined by the modulus of a single far field datum measured at a fixed spot corresponding to a single incident plane wave in [22].
- **Uniqueness results in inverse scattering problem in a piecewise homogeneous medium:**
  Inverse scattering problem in a layered medium is a subject of great practical importance. The main objective is to show that the penetrable interface between layers and the obstacle with its properties can be uniquely determined by the far field patterns for incident plane waves or near fields for point sources. We have established such uniqueness results with the help of some generalized reciprocity relations in a series of papers [23, 24, 25, 26, 27, 28].

In what follows, I will describe each of these areas and also my future plans in details.

1 Unique determination of a ball by a finite number of far field data

1.1 A brief description of inverse scattering problem

Broadly speaking, the scattering problems are concerned with the effect an inhomogeneity (scatterer) has on an incident (acoustic or electromagnetic) wave. In particular, if the total field is viewed as the sum of an incident field u^i and a scattered field u^s then the *direct scattering problem* is to study the (far-field or near-field) behavior of u^s from a knowledge of u^i and the differential equation governing the wave motion. Of possibly even more interest is the *inverse* *scattering problem* of extracting the nature of the inhomogeneity from a knowledge of the (far-field or near-field) behavior of u^s, i.e., to reconstruct the differential equation and/or its domain of definition from the behavior of its solution(s). Here the governing PDE is scalar wave equation for acoustic scattering and Maxwell's equations for electromagnetic scattering. Such inverse problem has been playing an indispensable role in real life, and forms the basis of many areas of science and engineering, such as radar and sonar (e.g., mine or submarine detection), geophysical exploration (e.g., oil and gas exploration), non-destructive testing (e.g., crack detection) and medical imaging (e.g., breast cancer detection) [5, 12, 15].

The basic problem in classical scattering theory (as opposed to quantum scattering theory) is the scattering of time-harmonic acoustic or electromagnetic waves by an inhomogeneity. In the following, we take the inverse acoustic scattering by a sound-soft obstacle as an example for illustration. The incident wave is given by the time-harmonic acoustic plane wave

$$u^i(x,d) = e^{ikx\cdot d}$$

where $k = \omega/c$ is the wave number, ω is the frequency, c is the speed of sound in the background medium and d is the direction of propagation. The impenetrable obstacle D will be assumed to be a compact set in $\mathbb{R}^n (n = 2, 3)$ with connected complement $G = \mathbb{R}^n \backslash D$, and the total field will be expressed as the sum of the incident field and the scattered field. Then, the total wave $u = u^i + u^s$ in G satisfies the reduced wave equation or Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } G. \tag{1.1}$$

For a *sound soft* obstacle the pressure of the total wave vanishes on the boundary, so a Dirichlet boundary condition

$$u = 0 \qquad \text{on } \partial D \tag{1.2}$$

is imposed. In order to obtain the well-posedness of the direct problem, the scattered wave $u^s(x)$ is required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial r} - ik_0 u^s\right) = 0 \tag{1.3}$$

uniformly in all directions $x/|x|$, where $r = |x|$. Moreover, it is known that $u^s(x)$ has the following asymptotic representation

$$u^s(x,d) = \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} \left\{ u^\infty(\widehat{x}, d) + O\left(\frac{1}{|x|}\right) \right\} \text{ as } |x| \to \infty \tag{1.4}$$

uniformly for all directions $\widehat{x} := x/|x|$, where the function $u^\infty(\widehat{x}, d)$ defined on the unit sphere S^{n-1} is known as the far field pattern with $\widehat{x}$ and d denoting, respectively, the observation direction and the incident direction.

Now, the *direct problem*, given information on the incident wave and the obstacle, is to find such far field pattern. Whereas the *inverse problem* taking the reverse is to determine the shape and location of the obstacle from the measurements of far field pattern.

As usual in most of the inverse problems, the first question to ask in this context is the identifiability, that is, whether an obstacle can be identified from a knowledge of the far field pattern. Mathematically, the identifiability is the uniqueness issue which is of theoretical interest and is required in order to proceed to efficient numerical methods of solutions. There is an interesting and well-known conjecture as follows:

One incoming plane wave for one single direction and one single wave number completely determines the scatterer (without any additional a priori information).

As it was remarked in [12] that this is a well-known question that supposedly can be solved by elementary means. However, it has been open for thirty to forty years and there is no idea how to attack it. There is much progress in this inverse problem, but more remains challengingly unsolved (see [12, 29]).

1.2 Unique determination of a ball by at most four far field data

Explicit representations of the far field patterns are only available for sound-soft ball and sound-hard ball. Based on Colton's theorem [3], its proved that a ball is uniquely determined from the far field pattern corresponding to a single incident direction by C. Liu [20] for sound-soft ball and by K. Yun [34] for sound-hard ball. However, it is noted that a ball is uniquely determined by its radius and center. Based on an investigation of the properties of the Bessel and Neumann functions, H. Liu & J. Zou [21] proved that the radius R of the underlying ball (centered at origin) is uniquely determined by the single far field datum. If the location, that is its center, was not given as a prior information, three more measurement data must be added to uniquely determine its center. In cooperation with Prof. Bo Zhang and Dr. Guanghui Hu, this has been proved and extended to electromagnetic scattering in [11].

1.3 Unique determination of a sound-soft ball by the modulus of a single far field datum

In practical applications it is not always the case that information about the full far field pattern is known, but instead only its modulus might be given. Therefore, we are interested in the inverse problem described as follows:

Given the modulus of the far field pattern $|u^\infty|$, for one single incident plane wave u^i, determine the boundary of the sound soft obstacle D.

For the shifted domain $D_h := \{x + h : x \in D\}$ with $h \in \mathbb{R}^n$ a fixed unit vector, the far field pattern u_h^∞ satisfies the equality [18, 19]

$$u_h^\infty(\widehat{x}) = e^{ikh\cdot(d-\widehat{x})}u^\infty(\widehat{x}), \tag{1.1}$$

that is, the modulus of the far field pattern is invariant under translation. Therefore only the shape but not the location may be uniquely determined by the modulus of the far field pattern. It was pointed out in [18] that it is a very difficult problem to obtain an analogue for the Schiffer's uniqueness result [5], since its proof heavily relies on the fact that, by Rellich's lemma, the far field pattern u^∞ uniquely determines the scattered wave u^s. Also it was remarked in [13], a corresponding result is not available for the modulus of the far field pattern, even with the translation invariance taken into account. Recently, there are some efficient numerical implementations [13, 14, 18] imply that shape reconstruction from the modulus of the far field pattern is possible.

Collaborating with Prof. Bo Zhang, with the help of some ideas from Liu & Zou [21], I made a first step in this direction. We proved that the shape of a sound-soft ball is uniquely determined by the modulus of a single far field datum in [22]. We want remark is that all the results in

[21, 11, 22] has restriction on the size of kR. A recent effort is to drop this restriction, extend the result [22] to more general setting and to inverse electromagnetic scattering problems. We have a conjecture that the modulus of the far field pattern is is strictly monotonically increasing for the radius of the ball. Thus a lipschitz stability can be obtained.

2 Uniqueness results in inverse scattering problem in a piece-wise homogeneous medium

This section provides brief descriptions of my main works I have done under the supervision of Prof. Bo Zhang during my Ph.D. period.

2.1 The first kind of layered medium (see Fig. 1)

In practical applications, the background might not be homogeneous and then may be modeled as a layered medium. A medium of this type that is a nested body consisting of a finite number of homogeneous layers occurs in various areas of applications such as non-destructive testing, biomedical imaging and geophysical explorations. In non-destructive testing, for example, the conducting wire can be modeled in terms of an inhomogeneous impedance boundary condition while the coating can be characterized as an arbitrarily shaped lossy dielectric layer.

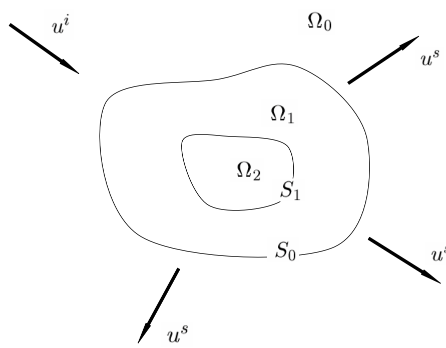

Figure 1: Scattering in a two-layered background medium

For simplicity, and without loss of generality, we restrict our presentation to the case where the obstacle is buried in a two-layered background medium in $\mathbb{R}^3$, as shown in Figure 1. In particular let $\Omega_2 \subset \mathbb{R}^3$ be an open bounded region with a C^2 boundary S_1 such that the background $\mathbb{R}^3\backslash\overline{\Omega_2}$ is divided by means of a closed C^2 surface S_0 into two connected domains Ω_0 and Ω_1. Here, Ω_0 is the unbounded homogeneous media and Ω_1 is the bounded homogeneous media. We assume that the boundary S_1 of the obstacle Ω_2 has a dissection $S_1 = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 and Γ_1 are two disjoint, relatively open subsets of S_1. Furthermore, boundary conditions of Dirichlet and impedance type with the surface impedance a nonnegative continuous function

$\lambda \in C(\Gamma_1)$ are specified on Γ_0 and Γ_1, respectively. Note that the case $\Gamma_1 = \emptyset$ corresponds to a *sound-soft* obstacle, and the case $\Gamma_0 = \emptyset$, $\lambda = 0$ leads to a Neumann boundary condition which corresponds to a *sound-hard* obstacle.

Now the propagation of time-harmonic acoustic waves in a two-layered medium in $\mathbb{R}^3$ is modeled by the Helmholtz equation with boundary conditions on the interface S_0 and boundary S_1:

$$\Delta u + k_0^2 u = 0 \qquad \text{in } \Omega_0, \tag{2.1}$$

$$\Delta v + k_1^2 v = 0 \qquad \text{in } \Omega_1, \tag{2.2}$$

$$u - v = 0, \ \frac{\partial u}{\partial \nu} - \lambda_0 \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } S_0, \tag{2.3}$$

$$\mathscr{B}(v) = 0 \qquad \text{on } S_1, \tag{2.4}$$

$$\lim_{r\to\infty} r\Big(\frac{\partial u^s}{\partial r} - ik_0 u^s\Big) = 0 \qquad r = |x| \tag{2.5}$$

where ν is the unit outward normal to the interface S_0 and boundary S_1, λ_0 is a positive constant. Here, the total field $u = u^s + u^i$ given as the sum of the unknown scattered wave u^s which is required to satisfy the Sommerfeld radiation condition (2.5) and incident plane wave $u^i = e^{ik_0 x\cdot d}$, k_j is the positive wave number given by $k_j = \omega_j / c_j$ in terms of the frequency ω_j and the sound speed c_j in the corresponding region Ω_j $(j = 0, 1)$. The distinct wave numbers k_j $(j = 0, 1)$ correspond to the fact that the background medium consists of several physically different materials. On the interface S_0, the so-called "transmission condition" (2.3) is imposed, which represents the continuity of the medium and equilibrium of the forces acting on it. The boundary condition $\mathscr{B}(v) = 0$ on S_0 is understood as follows

$$v = 0 \qquad \text{on } \Gamma_0, \tag{2.6}$$

$$\frac{\partial v}{\partial \nu} + i\lambda v = 0 \qquad \text{on } \Gamma_1. \tag{2.7}$$

Thus, the boundary condition (2.4) is a general and realistic boundary allowing the pressure of the total wave v vanishes on Γ_0 and the normal velocity is proportional to the excess pressure on the coated part Γ_1.

The *direct problem* is looking for a pair of functions $u \in C^2(\Omega_0) \cap C^{1,\alpha}(\overline{\Omega_0})$ and $v \in C^2(\Omega_1) \cap C^{1,\alpha}(\overline{\Omega_1})$ satisfying (2.1)-(2.5). By the variational method, the well-posedness (existence, uniqueness and stability) of the direct problem has been studied by Athanasiadis and Stratis [2] for Direchlet boundary condtion and by Liu, Zhang and Hu [27] for a general mixed boundary condition (2.4). Recently, in cooperation with Prof. Bo Zhang[24], an integral equation method is employed to establish the well-posedness of the direct problem. The result is then used, in conjunction with the representation in a combination of layer potentials of the solution, to prove a priorii estimates of the solution on some part of the interface S_0. The a priori estimates will play an important role in the proof of the inverse problem.

Moreover, it is known that $u^s(x)$ has the following asymptotic representation

$$u^s(x, d) = \frac{e^{ik_0|x|}}{|x|}\left\{u^\infty(\widehat{x}, d) + O\Big(\frac{1}{|x|}\Big)\right\} \text{ as } |x| \to \infty \tag{2.8}$$

uniformly for all directions $\widehat{x} := x/|x|$, where the function $u^\infty(\widehat{x}, d)$ defined on the unit sphere S is known as the far field pattern with $\widehat{x}$ and d denoting, respectively, the observation direction and the incident direction.

The *inverse problem* we consider is, given the wave numbers k_j $(j = 0, 1)$, the positive constant λ_0 and the far field pattern $u^\infty(\widehat{x}, d)$ for all incident plane waves with incident direction $d \in S$, to determine the obstacle Ω_2 with its physical property $\mathscr{B}$ and the interface S_0.

However, to the authors' knowledge, there are few uniqueness results for inverse obstacle scattering in a piecewise homogeneous medium. In particular, for the case of the scattering in a known piecewise homogeneous medium. Yan and Pang [33] gave a proof of uniqueness of the *sound-soft* obstacle based on Schiffer's idea. But their method can not be extended to other boundary conditions. They also gave a result for the case of a *sound-hard* obstacle in a two-layered background medium in [31] using a generalization of Schiffer's method. However, their method is hard to be extended to the case of a multilayered background medium and seems unreasonable to require the interior wave number to be in an interval. Recently, based on a generalization of the mixed reciprocity relation, we [27] showed that both the obstacle Ω_2 and its physical property $\mathscr{B}$ can be uniquely recovered from a knowledge of the far field pattern for incident plane waves. This seems to be appropriate for a number of applications where the physical nature of the obstacle is unknown. The tools and uniqueness result developed in [27] can also be extended to inverse electromagnetic scattering problem [23]. For the case of the scattering in an unknown piecewise homogeneous medium. Athanasiadis, Ramm and Stratis [1] and Yan [32] proved that the interfaces between layers are determined uniquely by the corresponding far field pattern which is a special case in the sense that the impenetrable obstacle does not exist.

In [24], we have proved that both the inaccessible obstacle Ω_2 with its physical property $\mathscr{B}$ and the interface S_0 can be uniquely determined by a knowledge of the far-field pattern. The uniqueness result has also been extended to the scattering of an inhomogeneous penetrable obstacle [25] and inverse electromagnetic scattering [26] in a piecewise homogeneous medium. These results obtained in these paper are also available for the case of multilayered medium and can be proved similarly.

2.2 The second kind of layered medium (see Fig. 2)

In practical applications, such as a mine buried in the soil, the domain surrounding the obstacle (mine) consists of two half-spaces (air and soil) with different electromagnetic coefficients, separated by a flat infinite interface. Moving an electronic device parallel to the flat infinite interface to generate a time-harmonic field, the induced field is measured within the same device. The goal is to retrieve information from these data to detect and identify buried obstacles. For more information, especially on mine detection, the reader is referred to [7] and the many references therein.

The ***inverse problem*** we consider is, taken measurements in the upper half-space Ω_1, to recover the obstacle D and its physical property. In recent years, many numerical reconstruction methods have been proposed to solve the above inverse problem; see, e.g. the linear sampling method proposed by Gebauer et al. [8] and by Cakoni, Fares and Haddar [6], an iterative method proposed by Delbary et al. [7], and an asymptotic factorization method studied by Griesmaier [9] and by Griesmaier and Hanke [10].

However, to our knowledge, no uniqueness result is available for the above inverse scattering problem. In our recent paper [28], we have proved that both the obstacle and its physical property can be uniquely determined from the measurements of the tangential component of the electric fields on Σ^m corresponding to all incident electric dipoles with sources on Σ^i and

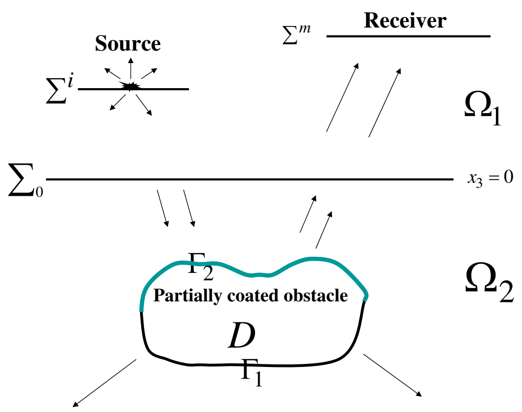

Figure 2: Scattering from partially coated obstacles in a two-layered medium

two polarizations $p = e_1 := (1, 0, 0)$ and $p = e_2 := (0, 1, 0)$. The key ingredient of our proof is a novel reciprocity relation established in [28] for the solutions of the scattering problem of the electro dipole located at two different source points.

A recent effort is to the investigation of the numerical simulation and stability behavior of the inverse process.

3 Conclusions

The inverse problems are the longest continuous subjects that I have been working since my graduate study at Institute of Applied Mathematics, Chinese Academy of Sciences. These research topics are not only interesting to me but have also provided me with valuable background that I need as a mathematician. In studying these topics for a long time, I have been exposed to a wide range of inverse problems.

During my Ph.D. study, I studied also many qualitative methods in inverse scattering theory such as the Linear Sampling Method proposed by D. Colton & A. Kirsch, the Singular Source Method developed by R. Potthast [30] and the Factorization Method studied by A. Kirsch [16]. Therefore, a recent effort is, with the help of some tools I have developed during the proof of the uniqueness results, to make insightful investigation on numerically test.

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