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EPL, **90** (2010) 68002

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

June 2010

EPL, **90** (2010) 68002 www.epljournal.org
doi: 10.1209/0295-5075/90/68002

Enumeration of spanning trees in a pseudofractal scale-free web

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received 11 May 2010; accepted in final form 9 June 2010
published online 13 July 2010

PACS 89.75.Hc – Networks and genealogical trees
PACS 05.50.+q – Lattice theory and statistics (Ising, Potts, etc.)
PACS 05.20.-y – Classical statistical mechanics

Abstract – Spanning trees are an important quantity characterizing the reliability of a network, however, explicitly determining the number of spanning trees in networks is a theoretical challenge. In this paper, we study the number of spanning trees in a small-world scale-free network and obtain the exact expressions. We find that the entropy of spanning trees in the studied network is less than 1, which is in sharp contrast to previous result for the regular lattice with the same average degree, the entropy of which is higher than 1. Thus, the number of spanning trees in the scale-free network is much less than that of the corresponding regular lattice. We present that this difference lies in disparate structure of the two networks. Since scale-free networks are more robust than regular networks under random attack, our result can lead to the counterintuitive conclusion that a network with more spanning trees may be relatively unreliable.

Introduction. – The enumeration of spanning trees in networks (graphs) is a fundamental issue in mathematics [1–3], physics [4,5], and other disciplines [6]. A spanning tree of any connected network is defined as a minimal set of edges that connect every node. The problem of spanning trees is relevant to various aspects of networks, such as reliability [7,8], optimal synchronization [9], standard random walks [10], and loop-erased random walks [11]. In particular, the number of spanning trees corresponds to the partition function of the q-state Potts model [12] in the limit of q approaching zero, which in turn closely relates to the sandpile model [13].

Because of the diverse applications in a number of fields [14], a lot of efforts have been devoted to the study of spanning trees. For example, the exact number of spanning trees in regular lattices [4,15] and Sierpinski gaskets [5] has been explicitly determined in previous studies. However, regular lattices and fractals cannot well mimic the real-life networks, which have been recently found to synchronously exhibit two striking properties: scale-free behavior [16] and small-world effects [17] that has a strong impact on the enumeration problems on networks. For example, previous work on counting subgraphs, such as cliques [18], loops and Hamiltonian cycles [19], has shown that scale-free degree distribution implies a very non-trivial structure of subgraphs. However, so far the investigation on the number of spanning trees in scale-free small-world networks is still missing. In view of the distinct structure, as compared to regular lattices, it is of great interest to examine spanning trees in scale-free small-world networks.

In this paper, we intend to fill this gap by providing a first analytical research of spanning trees in a small-world network with inhomogeneous connectivity. In order to exactly obtain the number of spanning trees, by using a renormalization group method [20], we consider a deterministically growing scale-free network with small-world effect. We find that the entropy of its spanning trees is smaller than 1, which is a striking result that is qualitatively different from that of two-dimensional regular lattices with identical average degree, in which the entropy is higher than 1. Thus, the number of its spanning trees is much lower than that of its corresponding regular lattice. We show that this difference can be accounted for by the heterogeneous structure of scale-free networks. Since the network under study is much robust to random deletion of edges, as opposed to regular lattice, our result suggests that networks with more spanning trees are not always more stable to random breakdown of edges, compared with those networks with less spanning trees.

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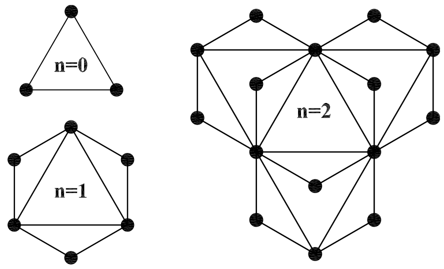

Fig. 1: The first three generations of the iterative scale-free network.

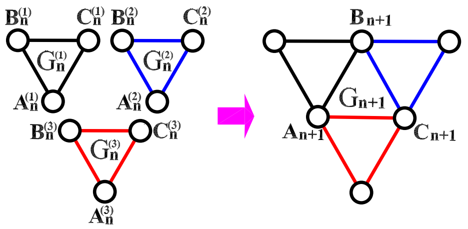

Fig. 2: (Color online) Second construction method of the network. G_{n+1} can be obtained by joining three copies of G_n denoted as $G_n^{(\eta)}$ ($\eta = 1, 2, 3$), the three hubs of which are represented by $A_n^{(\eta)}$, $B_n^{(\eta)}$, and $C_n^{(\eta)}$. In the merging process, hubs $A_n^{(1)}$ (respectively, $C_n^{(1)}$, $A_n^{(2)}$) and $B_n^{(3)}$ (respectively, $B_n^{(2)}$, $C_n^{(3)}$) are identified as a hub node A_{n+1} (respectively, B_{n+1}, C_{n+1}) in G_{n+1}.

Pseudofractal scale-free web. – The studied scale-free network [21,22], denoted by G_n after n ($n \geqslant 0$) generations, is constructed as follows: For $n = 0$, G_0 is a triangle. For $n \geqslant 1$, G_n is obtained from G_{n-1}: every existing edge in G_{n-1} introduces a new node connected to both ends of the edge. Figure 1 illustrates the construction process for the first three generations. The network exhibits some typical properties of real networks. Its degree distribution $P(k)$ obeys a power law $P(k) \sim k^{1+\ln 3/\ln 2}$, the average distance scales logarithmically with network order (number of nodes) [23], and the clustering coefficient is $\frac{4}{5}$. Alternatively, the network can be also created in another method [23,24]. Given the generation n, G_{n+1} may be obtained by joining at the hubs (the most connected nodes) three copies of G_n, see fig. 2. According to the latter construction algorithm, we can easily compute the network order of G_n is $V_n = \frac{3^{n+1}+3}{2}$. In G_n, there are three hubs denoted by A_n, B_n, and C_n, respectively.

Number of spanning trees. – After introducing the network construction and its properties, next we will study both numerically and analytically spanning trees in this scale-free network.

Numerical solution. According to the well-known result [25], we can obtain numerically but exactly the number of spanning trees, $N_{\mathrm{ST}}(n)$, by computing the non-zero eigenvalues of the Laplacian matrix associated

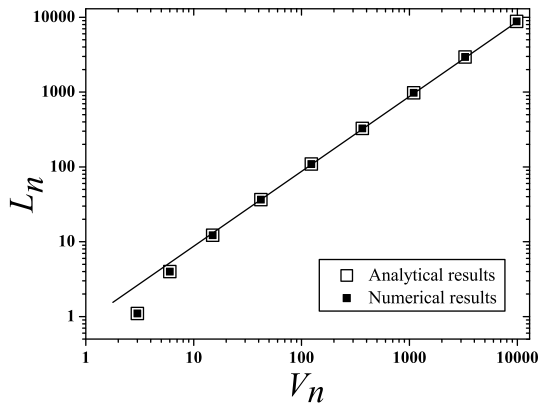

Fig. 3: Logarithm of the number of spanning trees $N_{\mathrm{ST}}(n)$ in network G_n as a function of network order V_n on a log-log scale. In the figure, $L_n = \ln N_{\mathrm{ST}}(n)$; the filled symbols are the numerical results obtained from eq. (1), while the empty symbols correspond to the exact values from eq. (11), both of which completely agree with each other.

with G_n as

$$N_{\mathrm{ST}}(n) = \frac{1}{V_n} \prod_{i=1}^{i=V_n-1} \lambda_i(n), \tag{1}$$

where $\lambda_i(n)$ ($i = 1, 2, \ldots, V_n - 1$) are the $V_n - 1$ nonzero eigenvalues of the Laplacian matrix for G_n. For a network, the non-diagonal element l_{ij} ($i \neq j$) of its Laplacian matrix is -1 (or 0) if nodes i and j are (or not) directly connected, while the diagonal entry l_{ii} equals the degree of node i. Using eq. (1), we can calculate directly the number of spanning trees $N_{\mathrm{ST}}(n)$ of G_n (see fig. 3). From fig. 3, we can see that $N_{\mathrm{ST}}(n)$ approximately grows exponentially in V_n. This allows to define the entropy of spanning trees for G_n as the limiting value [1–3]

$$E_{G_n} = \lim_{V_n \to \infty} \frac{\ln N_{\mathrm{ST}}(n)}{V_n}, \tag{2}$$

which is a finite number and a very interesting quantity characterizing the network structure.

It should be mentioned that although the expression of eq. (1) seems compact, the computation of eigenvalues of a matrix of order $V_n \times V_n$ makes heavy demands on time and computational resources for large networks. Thus, one can count the number of spanning trees by directly calculating the eigenvalues only for the first several iterations, which is not acceptable for large graphs. Particularly, by using the eigenvalue method it is difficult and even impossible to obtain the entropy E_{G_n}. It is thus of significant practical importance to develop a computationally cheaper method for enumerating spanning trees that is devoid of calculating eigenvalues. Fortunately, the iterative network construction permits to calculate recursively $N_{\mathrm{ST}}(n)$ and E_{G_n} to obtain exact solutions.

Closed-form formula. To get around the difficulties of the eigenvalue method, we use an analytic technique based on a decimation procedure [20]. For simplicity, we use t_n

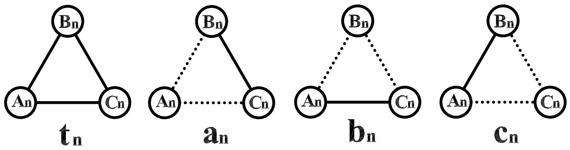

Fig. 4: Illustrative definition for the spanning subgraphs of G_n. The two hub nodes connected by a solid line are in one tree, and the two hub nodes linked by a dotted line belong to different trees.

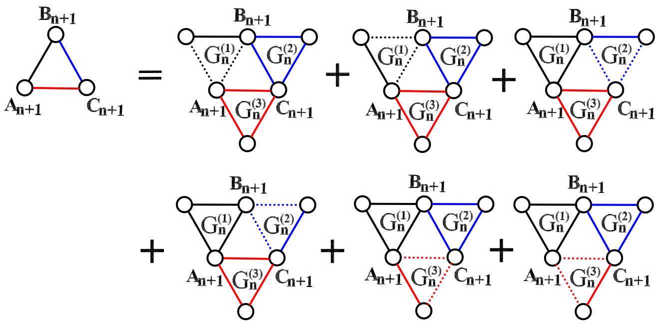

Fig. 5: (Color online) Illustration for the recursion expression for the number of spanning trees t_{n+1} in G_{n+1}. The two nodes at both ends of a solid line are in one tree, while the two nodes at both ends of a dotted line are in separate trees.

to express $N_{\rm ST}(n)$. Moreover, let a_n denote the number of spanning subgraphs of G_n consisting of two trees such that the hub node A_n belongs to one tree and the two other hubs (B_n and C_n) are in the other tree. Analogously, we can define quantities b_n and c_n, see fig. 4. By symmetry, we have $a_n = b_n = c_n$. Thus, in the following computation, we will replace b_n and c_n by a_n.

Considering the self-similar network structure, the following fundamental relations can be established:

$$t_{n+1} = (t_n)^2(a_n + c_n + a_n + b_n + c_n + b_n) = 6a_n(t_n)^2 \quad (3)$$

and

$$a_{n+1} = t_n[(c_n)^2 + a_n b_n + b_n c_n + a_n c_n] = 4t_n(a_n)^2\,. \quad (4)$$

Equation (3) can be explained as follows. Since G_{n+1} is obtained via merging three G_n by identifying three couples of hub nodes, to get the number of spanning trees t_{n+1} for G_{n+1}, one of the copies of G_n must be spanned by two trees. There are six possibilities as shown in fig. 5, from which it is easy to derive eq. (3). Analogously, eq. (4) can be understood based on fig. 6.

To obtain t_n, we define an intermediary variable $h_n = \frac{t_n}{a_n}$ that obeys the following recursive relation:

$$h_{n+1} = \frac{t_{n+1}}{a_{n+1}} = \frac{3t_n}{2a_n} = \frac{3}{2}h_n\,. \quad (5)$$

With the initial condition $t_0 = 3$ and $a_0 = 1$, we have $h_0 = 3$. Hence, eq. (5) is solved to yield

$$h_n = \frac{3^{n+1}}{2^n}\,. \quad (6)$$

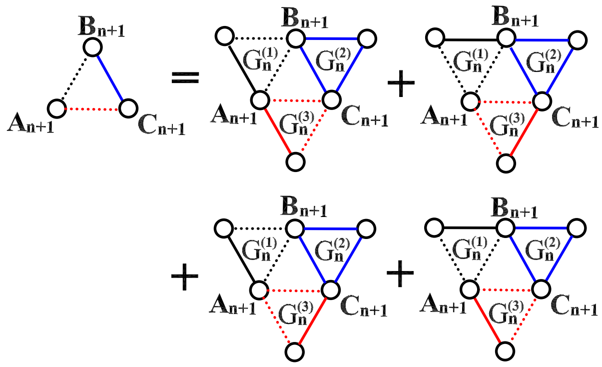

Fig. 6: (Color online) Illustration for the recursive expression for the number of spanning subgraphs a_{n+1} corresponding to network G_{n+1}. The two nodes at both ends of a solid line (dotted line) are in one tree (two trees).

Then,

$$a_n = \frac{2^n}{3^{n+1}}t_n\,. \quad (7)$$

Plugging this expression into eq. (3) leads to

$$t_{n+1} = \frac{2^{n+1}}{3^n}(t_n)^3\,. \quad (8)$$

Considering the initial value $t_0 = 3$, we can solve eq. (8) to obtain the explicit solution

$$N_{\rm ST}(n) = t_n = 2^{(3^{n+1}-2n-3)/4}3^{(3^{n+1}+2n+1)/4}\,. \quad (9)$$

Analogously, we can derive the exact formula for a_n as

$$a_n = 2^{(3^{n+1}+2n-3)/4}3^{(3^{n+1}-2n-3)/4}\,. \quad (10)$$

It not difficult to represent $N_{\rm ST}(n)$ as a function of the network order V_n, with the aim to obtain the relation between the two quantities. Recalling $V_n = \frac{3^{n+1}+3}{2}$, we have $3^{n+1} = 2V_n - 3$ and $n+1 = \ln(2V_n - 3)/\ln 3$. These relations enable one to write $N_{\rm ST}(n)$ in terms of V_n as

$$N_{\rm ST}(n) = 2^{[V_n - \ln(2V_n-3)/\ln 3 - 2]/2}3^{[V_n + \ln(2V_n-3)/\ln 3 - 2]/2}\,. \quad (11)$$

We have confirmed the closed-form expressions for $N_{\rm ST}(n)$ against direct computation from eq. (1). In the full range of $0 \leqslant n \leqslant 8$, they are perfectly consistent with each other, which shows that the analytical formulas provided by eqs. (9) and (11) are right. Figure 3 shows the comparison between the numerical and analytical results.

Equation (11) unveils the explicit dependence relation of $N_{\rm ST}(n)$ on the network order V_n. Inserting eq. (11) into eq. (2), it is easy to obtain the entropy of spanning trees for G_n given by

$$E_{G_n} = \lim_{V_n\to\infty} \frac{\ln N_{\rm ST}(n)}{V_n} = \frac{1}{2}(\ln 2 + \ln 3) \simeq 0.89588. \quad (12)$$

This obtained asymptotic value is the smallest entropy (lower than 1) that has not been reported earlier for other networks with an average degree of 4. For example, the entropy for spanning trees in the square lattice is 1.16624 [4], a value larger than 1. Thus, the number of spanning trees in G_n is much less than that in the square lattice with the same average degree of nodes.

From the result obtained above, we can conclude that the pseudofractal scale-free network has less spanning trees than the regular lattice with the same average degree. The difference can be attributed to the structural characteristics of the two classes of networks. In scale-free networks, nodes have a heterogeneous connectivity, which leads to an inhomogeneous distribution of Laplacian spectra [21,26,27]. On the contrary, in regular lattices, since all nodes have approximately the same degree, their Laplacian spectra have a homogenous distribution. Thus, for two given scale-free and regular networks with the same order and average node degree, the sum of the eigenvalues of their Laplacian matrices are the same, but the product of non-zero Laplacian spectra of the scale-free network is smaller than its counterpart of the regular network, because of the different distributions of the Laplacian spectra resulting from their distinct connectivity distribution. Hence, the heterogeneous structure is responsible for the difference of number of spanning trees in scale-free networks and regular lattices. It should be stressed that although we only study a specific deterministic scale-free network, we expect to find a qualitatively similar result about spanning trees in real-world scale-free networks, since they have similar structural characteristics as that discussed above.

As an important invariant of a network, the number of spanning trees is a relevant measure of the reliability of the network. Intuitively, among all connected graphs with the same numbers of nodes and edges, networks having more spanning trees are more resilient (reliable) to the random removal of edges, compared with those with less spanning trees. That is to say, the former has a larger threshold of bond percolation than that of the latter. However, recent works [28–30] have shown that inhomogeneous networks, such as scale-free networks, are impressively more robust than homogeneous networks (*e.g.*, exponential networks and regular networks) with respect to random deletion of the edges. Thus, combining with our above result, we can reach the following counterintuitive conclusion that networks (*e.g.*, scale-free networks) with less spanning trees do not mean more vulnerable to random breakdown of links than those (*e.g.*, regular lattices) with more spanning trees.

Conclusions. – In summary, diverse real-life networks possess power-law degree distribution and small-world effect. In this paper, we have studied and enumerated explicitly the number of spanning trees in a scale-free network with small-world behavior. The exact solution was obtained on the basis of some precise recursion relations derived from the iterative construction of the network addressed. It was demonstrated that scale-free network has much less spanning trees compared to the regular lattice with the same number of nodes and edges. It was shown that this difference is rooted in the inherent architecture of the two types of networks. Although it is generally thought that increasing the number of spanning trees over all networks with identical number of nodes and edges can lead to a less fragile network, our results strikingly indicate otherwise. Our work may be helpful for designing and improving the reliability of networks.

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The authors would like to thank Yichao Zhang and Yuan Lin for their assistance. This research was supported by the National Natural Science Foundation of China under Grants No. 60704044 and No. 60873070, the National Basic Research Program of China under Grant No. 2007CB310806, and Shanghai Leading Academic Discipline Project No. B114.

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