

# Complex Singularities and the Lorenz Attractor\*

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**Abstract.** The Lorenz attractor is one of the best-known examples of applied mathematics. However, much of what is known about it is a result of numerical calculations and not of mathematical analysis. As a step toward mathematical analysis, we allow the time variable in the three-dimensional Lorenz system to be complex, hoping that solutions that have resisted analysis on the real line will give up their secrets in the complex plane. Knowledge of singularities being fundamental to any investigation in the complex plane, we build upon earlier work and give a complete and consistent formal development of complex singularities of the Lorenz system using the *psi series*. The psi series contain two undetermined constants. In addition, the location of the singularity is undetermined as a consequence of the autonomous nature of the Lorenz system. We prove that the psi series converge, using a technique that is simpler and more powerful than that of Hille, thus implying a two-parameter family of singular solutions of the Lorenz system. We pose three questions, answers to which may bring us closer to understanding the connection of complex singularities to Lorenz dynamics.

**Key words.** Lorenz attractor, psi series, complex singularities

**AMS subject classifications.** 34M35, 37D45

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**1. Introduction.** The nonlinear system of equations

$$(1.1) \quad \begin{aligned} \frac{dx}{dt} &= 10(y - x), \\ \frac{dy}{dt} &= 28x - y - xz, \\ \frac{dz}{dt} &= -8z/3 + xy, \end{aligned}$$

which is named after Lorenz, gives the best-known example of a strange attractor. Lorenz [21, 22] derived this system to argue that the unpredictability of weather is due to the nature of the solutions of the Navier–Stokes equations and not due to stochastic terms of unknown origin, his point being that a deterministic system could possess an attracting and invariant set on which the dynamics is bounded and linearly unstable. When such strange attractors exist, trajectories are chaotic and appear random.

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While Lorenz [21, p. 141, 1963] could write that the atmosphere was not normally regarded as deterministic, we now know that the incompressible Navier–Stokes equations by themselves explain a remarkable wealth of turbulence phenomena including coherent motions in the near-wall region, the law of the wall, intermittency, and vortex structures in fully developed turbulence [2]. The density and temperature of the atmosphere vary with altitude, and there is significant electrical activity in the atmosphere that is sustained by about 40,000 thunderstorms that occur around the world in any single day [6, Chapter 9]. If we nevertheless think that the physics of the atmosphere is deterministic, Lorenz and his system are partly responsible.

Lorenz’s point of view was dynamical. He viewed the state of (1.1) as a point in  $R^3$  and its solutions as trajectories in  $R^3$ . The dynamical point of view has overwhelmingly dominated work on the Lorenz system, and Lorenz’s original paper [21] has remained an outstanding introduction to dynamics. In it, a careful reader can find discussions of numerical errors, of concepts of stability, of symbolic dynamics (aspects of which Lorenz seems to have rediscovered for himself), of the density of periodic solutions on the Lorenz attractor, and of the fractal nature of the Lorenz attractor.

The point of view in this paper, unlike Lorenz’s, will be mainly function theoretic. We view  $t$  in (1.1) as a complex variable and  $x, y, z$  as analytic functions of a complex variable. Our interest is in triples of analytic functions which satisfy (1.1). Our hope is that an investigation in the complex plane will open a route to the mathematical analysis of the Lorenz system.

For the most part, we deal with certain singular solutions of the Lorenz system, which will be introduced momentarily. As the right-hand side of the Lorenz system (1.1) is analytic, every solution of the Lorenz system admits analytic continuation to the complex plane. For some solutions, the analytic continuations have singularities of the form we deal with, as indicated by numerical results summarized in section 5. In the second part of this introduction, we pose three questions to help connect the complex singularities with Lorenz dynamics.

From residue integration, the method of steepest descent, and the use of deformation of contours to effect analytic continuation of certain special functions, we know that knowledge of singularities is often useful to investigations in the complex plane. This observation explains our focus on singular solutions of the Lorenz system.

**1.1. Psi Series Solutions of the Lorenz System.** The most common types of singularities are poles, algebraic branch points, and logarithmic branch points. The singularities of the Lorenz system that we examine are of none of these types but are given by psi series representations.

DEFINITION 1.1. *A logarithmic psi series centered at  $t_0$  is a series of the form  $\sum_{n=-N}^{\infty} p_n(\eta)(t - t_0)^n$ , where  $N$  is an integer,  $\eta = \log(b(t - t_0))$ , and each  $p_n$  is a polynomial in  $\eta$ . In the definition of  $\eta$ ,  $b$  is a complex number with  $|b| = 1$ , with  $b = \pm i$  often being convenient choices.*

Throughout this paper,  $\log$  will denote the principal branch of  $\log$ . The choice of the branch is ultimately immaterial but taking  $\eta = \log(-i(t - t_0))$  instead of  $\eta = \log(t - t_0)$  leads to more convenient branch cuts if  $\Im(t_0) < 0$ , as we explain in section 3. For a slightly different definition of logarithmic psi series, along with definitions of psi series of other types, see [13, Chapter 7.1]. The only type of psi series that arises in this paper is the type given by Definition 1.1, and by psi series we refer to that definition only.

The psi series of Definition 1.1 are like the Laurent series, except that the coefficients are polynomials in  $\eta$  instead of being constants. For that reason, the psi

series singularities were called pseudopoles by Hille [11]. Even though the coefficients are polynomials in  $\eta$ , each nonzero term of the logarithmic psi series dominates the following term in magnitude in the limit  $t \rightarrow t_0$ .

In an intriguing and original pair of papers, Tabor and Weiss [32] and Levine and Tabor [19] considered psi series solutions of the Lorenz system (1.1). The psi series they used were expressed as a double sum. Below we give the psi series in a different form:

$$\begin{aligned}
 x(t) &= \frac{P_{-1}(\eta)}{t-t_0} + P_0(\eta) + P_1(\eta)(t-t_0) + P_2(\eta)(t-t_0)^2 + \cdots, \\
 y(t) &= \frac{Q_{-2}(\eta)}{(t-t_0)^2} + \frac{Q_{-1}(\eta)}{t-t_0} + Q_0(\eta) + Q_1(\eta)(t-t_0) + Q_2(\eta)(t-t_0)^2 + \cdots, \\
 (1.2) \quad z(t) &= \frac{R_{-2}(\eta)}{(t-t_0)^2} + \frac{R_{-1}(\eta)}{t-t_0} + R_0(\eta) + R_1(\eta)(t-t_0) + R_2(\eta)(t-t_0)^2 + \cdots.
 \end{aligned}$$

Here the  $P_i$ ,  $Q_i$ , and  $R_i$  are polynomials in  $\eta$ , where  $\eta = \log(b(t-t_0))$  as in Definition 1.1. As the Lorenz system is autonomous,  $t_0$  is an arbitrary complex number. The fact that the leading powers of  $(t-t_0)$  in the three series in (1.2) are  $-1$ ,  $-2$ , and  $-2$  may be guessed by substituting poles  $(t-t_0)^{-\alpha}$ ,  $(t-t_0)^{-\beta}$ ,  $(t-t_0)^{-\gamma}$  for  $x$ ,  $y$ ,  $z$  into the Lorenz system and then solving for  $\alpha$ ,  $\beta$ ,  $\gamma$  by matching the order of the left- and right-hand sides [32]. This test-power method [13, p. 90] does not always work and can be tricked into failing for the Lorenz system with a linear change of variables.

Melkonian and Zypchen [24] have recast the psi series of Tabor and Weiss [32] into the formalism of Hille [11]. The formal development of psi series that we give in section 3 is similar to that of Melkonian and Zypchen [24] but improves that of Melkonian and Zypchen in two respects. First, the development in section 3 shows the dependence on undetermined constants  $C$  and  $D$  explicitly, pointing out the occurrence of  $\eta$  and  $C$  in the group  $(\eta+C)$ . Second, we prove that the degrees of  $P_{m+1}$ ,  $Q_m$ ,  $R_m$  are given by  $\lfloor \frac{m+2}{2} \rfloor$  for  $m = 0, 1, \dots$ . The proof hinges on a surprising cancellation for  $m = 2$ . It is important to get such details fully right if a mathematical theory is to be set up. As Hille [13, p. 68] pointed out, “constants of integration play a remarkable role in the advanced theory of nonlinear DEs.” In addition, a complete formal calculation is essential for a fully correct convergence proof.

The first few coefficients of the psi series (1.2) are listed in Table 1.1. It is evident that  $\eta$  and  $C$  always occur in the group  $(\eta+C)$ . If  $D$  were real, the coefficients of the polynomials in  $(\eta+C)$  listed in that table all would be either pure imaginary or real.

The following is one of our main theorems. It reappears in a more specific form in section 4, where it is proved.

**THEOREM 1.1.** *The psi series (1.2), some of whose coefficients are listed in Table 1.1, satisfy the Lorenz system (1.1) in the disc  $|t-t_0| \leq r$  for some  $r > 0$  but with the singular point  $t = t_0$  and a branch cut deleted from the disc. The constants  $C$  and  $D$  are undetermined.*

The proof of this theorem is valid for any choice of the undetermined constants  $C$  and  $D$ , but the estimate for  $r$  depends upon the choice. A key step in its proof is to show the convergence of the psi series.

An important aspect of the convergence of the Lorenz psi series is not brought out in Theorem 1.1. As evident from the appearance of  $\eta$  in Definition 1.1, a typical psi series will have logarithmic branch points in the  $t$ -plane. To get around the multiple-valuedness, Theorem 1.1 fixes a branch cut in the  $t$ -plane. The branch

**Table 1.1** *Coefficients of the psi series of (1.2) with  $\eta$  as in Definition 1.1. Evidently, the degrees of  $P_{m+1}, Q_m, R_m$  are  $\lfloor \frac{m+2}{2} \rfloor$  for  $m = 0, 1, 2, 3$ . The other valid choice for coefficients of (1.2) is obtained by changing the signs of all the  $P_i$ 's and  $Q_i$ 's, while leaving the  $R_i$ 's unchanged. The constants  $C$  and  $D$  are both undetermined.*

$Q_{-2}, R_{-2}$		$-\frac{1}{5}i$	$-\frac{1}{5}$
$P_{-1}, Q_{-1}, R_{-1}$	$2i$	$2i$	$\frac{17}{9}$
$P_0, Q_0, R_0$	$\frac{71}{9}i$	$-\frac{349}{81}i - \frac{988}{81}i(\eta + C)$	$\frac{1385}{54} - \frac{988}{81}(\eta + C)$
$P_1, Q_1, R_1$	$-\frac{9880}{81}i(\eta + C)$	$-\frac{25991}{108}i + \frac{64220}{243}i(\eta + C)$	$-\frac{211189}{972} + \frac{167960}{729}(\eta + C)$
$P_2$		$-\frac{2108195}{972}i + \frac{469300}{243}i(\eta + C)$	
$Q_2$	$\frac{3}{10}iD - \frac{477319147}{131220}i - \frac{167831753}{65610}i(\eta + C) - \frac{273676}{2187}i(\eta + C)^2$		
$R_2$	$-\frac{1}{5}D + \frac{138959125}{17496} - \frac{58846039}{32805}(\eta + C) - \frac{1444456}{2187}(\eta + C)^2$		
$P_3$		$iD - \frac{96356411}{6561}i(\eta + C) - \frac{2736760}{6561}i(\eta + C)^2$	
$Q_3$	$-\frac{25925844899}{708588}i + \frac{32}{27}iD - \frac{516846814}{59049}i(\eta + C) + \frac{26636480}{2187}i(\eta + C)^2$		
$R_3$	$-\frac{55}{27}D + \frac{64036692917}{3542940} - \frac{2458513}{2187}(\eta + C) + \frac{813193160}{59049}(\eta + C)^2$		
$P_4$		$\frac{25}{54}iD - \frac{64653009635}{708588}i - \frac{107735075}{118098}i(\eta + C) + \frac{206615500}{6561}i(\eta + C)^2$	

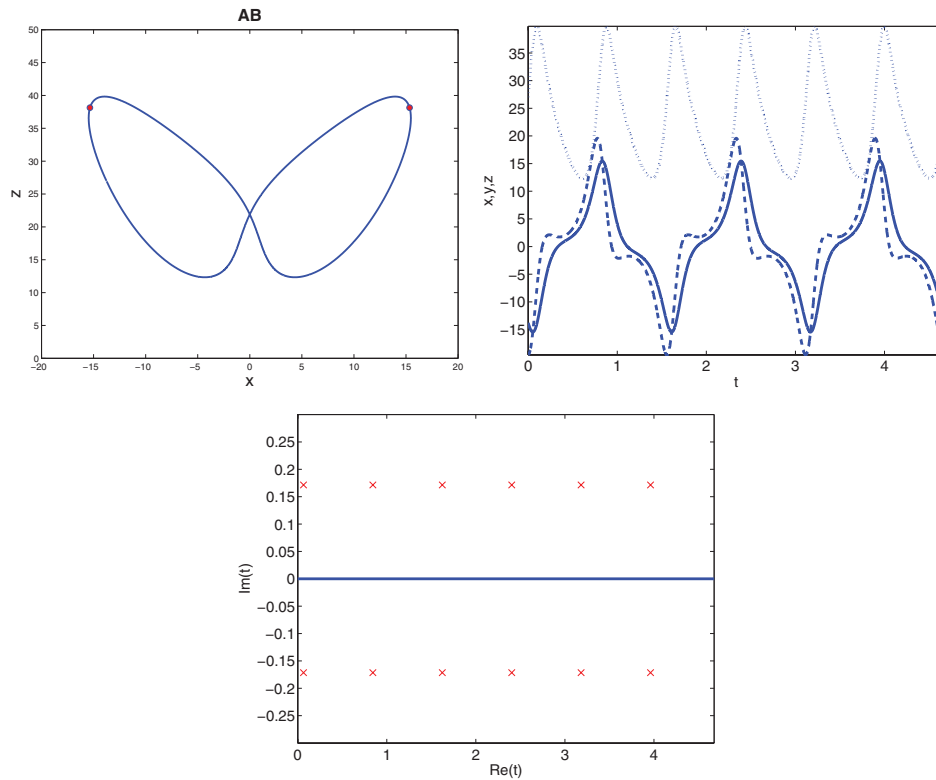
cut can be dispensed with by parameterizing the Riemann surface using  $\eta$ . A discussion of convergence in the  $\eta$ -plane is found in section 4 (see Figure 4.1 in particular).

Hille’s “frontal attack” to prove convergence of psi series can be modified to apply to the Lorenz system [11, 24]. In an appendix, Hille [12] pointed out that his technique could handle only the Emden–Fowler system (see section 2) with  $p = 2$ , while a more complicated technique due to Smith [30] could handle  $p = 2, 3, \dots$ . The technique we use in section 4.1 is also a frontal attack, but it is a good deal more transparent than Hille’s approach. In place of an elaborate analytic setup and an inductive hypothesis to bound the coefficients of the psi series, we use the Laplace transform, elementary combinatorics, and an elementary implicit function theorem. Our technique seems to extend to all the cases handled by Smith [30]. Detailed comments on this point are found in section 4.2.

**1.2. Complex Singularities and Lorenz Dynamics: Three Questions.** From Theorem 1.1 we get a two-parameter family of singular solutions of the Lorenz system (1.1). The form of the singular solutions is given by the psi series (1.2), and the two undetermined constants  $C$  and  $D$  are shown in Table 1.1. The location  $t_0$  of the singularity can be anywhere in the complex  $t$ -plane.

For some definite integrals, the singularities of the integrand and Cauchy’s residue theorem imply the value of the integral. So we ask, what do the singular solutions of the Lorenz system tell us about the dynamics in  $R^3$  for real time? As the analytic theory of solutions of the Lorenz system is still in its infancy, a complete answer to the question cannot be given. Nevertheless, the question merits a thorough discussion.

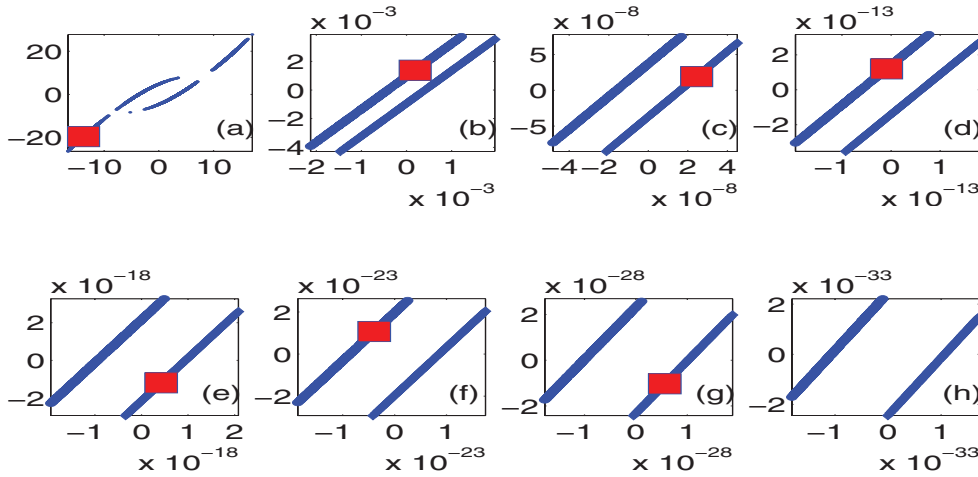
Many beautiful visualizations of the Lorenz attractor are found on the Internet. The visualizations originally offered by Lorenz [21] are packed with information and are models of concision. The Lorenz attractor is a butterfly-like subset of  $R^3$ . Except for the fixed points, all trajectories either approach the attractor as  $t \rightarrow \infty$  or are already on it.



**Fig. 1.1** *The periodic orbit in the first plot is labeled  $AB$  to indicate the sequence in which it moves between the  $A$  quadrant (with  $x < -16.432$ ,  $y < -16.432$ ,  $z = 27$ ) and the  $B$  quadrant (with  $x > 16.432$ ,  $y > 16.432$ ,  $z = 27$ ). Each filled circle is directly below a singularity in the complex  $t$ -plane. To the right are plots of  $x(t)$  (solid),  $y(t)$  (dashed),  $z(t)$  (dotted) against real  $t$ . In the bottom plot, the location of the complex singularities of  $AB$  that are closest to the real line are marked as crosses. The orbit  $AB$  is computed with 547 digits of precision.*

Figure 1.1 shows the periodic orbit labeled  $AB$ , which resides on the attractor. A great advantage of computing such orbits, as opposed to arbitrary trajectories, is that the computations take on a definite character that makes it possible to report them precisely. As already mentioned at the beginning of this introduction, periodic orbits are believed to be dense in the Lorenz attractor. Such orbits can be computed with great precision. The locations of the complex singularities shown in the rightmost plot of Figure 1.1 were obtained by computing the orbit  $AB$  with more than 500 digits of precision.

A worthy goal for the analytic theory of the Lorenz system is a proof of existence of periodic solutions  $(x(t), y(t), z(t))$  of the Lorenz system (1.1), where we seek a proof that is based solely on mental conceptions. There is a definiteness to seeking periodic solutions, as already pointed out. In addition, periodic orbits are key to extracting order from chaos, to borrow an expression from Strogatz [31]. For instance, Figure 1.2, which illustrates the fractal property of the Lorenz attractor, was obtained by computing periodic orbits. The plots were computed in parallel on a machine with two quadcore 2.33 GHz Xeon processors. The plots took a day or two of computing. For the theory behind such computations, see [33] and [34].



**Fig. 1.2** *Fractal property of the Lorenz attractor. (a) The intersection of an arbitrary trajectory on the Lorenz attractor with the section  $z = 27$ . The plot shows a rectangle in the  $x$ - $y$  plane. All later plots (b) and above) zoom in on a tiny region (too small to be seen by the unaided eye) at the center of the red rectangle of the preceding plot to show that what appears to be a line is in fact not a line. These plots and the plots of [33, 34], of which these plots are a refinement, appear to be the only plots made of the fractal structure of the Lorenz attractor.*

A proof of existence of periodic solutions of the Lorenz system (1.1) appears to be far away. We formulate three questions to serve as more immediate goals for the development of the analytic theory of the Lorenz system.

**QUESTION 1.1.** *Are all singular solutions of the Lorenz system given by psi series expansions (1.2) with suitable choice of the undetermined constants  $C$  and  $D$ ?*

The role of the undetermined constants  $C$  and  $D$  is partly shown in Table 1.1. Their role in the psi series is clarified further in sections 3 and 4. Lorenz [21] gave arguments that partially imply that a real solution of the Lorenz system cannot become singular in finite time. The implication covers both increasing and decreasing time. In section 5, we give a complete proof of that result. Thus for solutions of the Lorenz system that are real for real  $t$ , the locations  $t_0$  of the complex singularities must have a nonzero imaginary part. In fact, Foias and others [8, Theorem 2.3] have proved that for solutions on the Lorenz attractor, the imaginary part of the location of the singularity in the complex  $t$ -plane must exceed 0.037 in magnitude. For an investigation of the backward-in-time behavior of the Lorenz system (for real data), see the paper by Foias and Jolly [7].

The techniques used to deduce psi series solutions of the Lorenz system are not of much use for answering Question 1.1. However, if  $t_0$  is any singular point of the Lorenz system, then  $|x(t)| + |y(t)| + |z(t)| \rightarrow \infty$  as  $t \rightarrow t_0$ , as implied by a slightly stronger theorem proved in section 5.

For analytic functions such as the gamma and zeta functions, analytic continuation into the complex plane is an important step in understanding the true nature of those functions [25]. The question of analytic continuation is important in the theory of differential equations in the complex plane as well [13]. These observations motivate us to ask the following question.

QUESTION 1.2. *Do solutions of the initial value problem for the Lorenz system with  $(x(0), y(0), z(0))$  being finite (but possibly complex) admit of analytic continuation to the entire complex  $t$ -plane except for branch points?*

An affirmative answer to Question 1.1 appears to imply an affirmative answer to Question 1.2. The process of analytic continuation can be blocked by singularities. But if all singularities are given by psi series of the form (1.2), Theorem 1.1 implies that we can continue around any such singularity into a disc of finite radius around that singularity (radius is  $r$  in the theorem). The possibility where a succession of psi series singular solutions of decreasing radii of convergence accumulate on another singular point is easily ruled out if the answer to Question 1.1 is yes.

Singular solutions given by psi series representations exist for plane quadratic systems as well as plane polynomial systems [12, 30]. Such planar systems certainly cannot exhibit chaos [31]. The dynamics of planar systems is tightly circumscribed by results such as the Poincaré–Bendixson theorem. Unlike the Lorenz system, the planar systems considered by Hille [12] and Smith [30] can have real solutions that develop singularities in finite time. Yet one is probably justified in thinking the mere existence of singular solutions represented by psi series is unlikely to tell us anything about the chaotic nature of the Lorenz system.

This is perhaps the place to comment on the three free parameters with which the Lorenz system is usually written but which are given the values used by Lorenz [21] in (1.1). The three parameters correspond to the Rayleigh number, the Prandtl number, and the system size for the convection PDE from which the Lorenz system was derived. With regard to the choice of these parameters, there are three cases for which the Lorenz system admits a Laurent series as a solution [29, 32]. There are five other cases, due to Segur [29] and Kuš [18], for which time-dependent integrals of motion are known. In their pioneering work, Tabor and Weiss [32] considered the connection between integrability and the type of the singularities. For another discussion of the connection between psi series and integrability, see [3].

In addition to the integrable cases, there are a number of other regions in parameter space where the Lorenz system has nonchaotic dynamics yet admits singular solutions with psi series representation. In these instances, it is quite possible that even though the real-valued dynamics is nonchaotic, more varied solutions exist when complex numbers are allowed. In the case of plane polynomial systems, although the differential equations cannot have chaotic solutions that are real [1, 31], the equations may have chaotic solutions that are complex.

It is not entirely clear how the nature of the singularity can be connected to chaotic dynamics. It is perhaps significant that only real solutions have a bearing on dynamics. Therefore we ask the following question.

QUESTION 1.3. *If a psi series solution of the Lorenz system (1.1) of the form (1.2) is obtained by analytic continuation of a solution that is real for real  $t$ , what constraints must  $C$ ,  $D$ , and  $t_0$  satisfy?*

The detailed development of psi series found in section 3 and partly shown in Table 1.1 could help answer this question. Numerical computations are also likely to be useful. A suspicion of ours is that the undetermined constant  $D$  is real for the psi series singularities of Question 1.3.

**2. A Brief History of Early Work on Psi Series.** The equation of Briot and Bouquet

$$(2.1) \quad t \frac{dw}{dt} = pt + w + F(t, w),$$

where  $F$  is a polynomial with quadratic and higher terms, seems to be the simplest differential equation whose singularities are given by psi series. Dulac [4, p. 368, 1912] and Malmquist [23, p. 19, 1921] (also see Theorem 11.3.1 of [13]) proved that the general solution of (2.1) around  $t = 0$  is given by a convergent psi series if  $p$  is a positive integer. For generalizations to higher order Briot–Bouquet equations, see [17].

In the last decade of his life, Einar Hille [10, 11, 12, 13] grew interested in the Emden–Fowler equation  $d^2y/dt^2 = t^{-2/p}y^{1+2/p}$  with  $p > 1$  being a positive integer. The Emden–Fowler equation originally arose in cosmology. The special case  $p = 2$  is the Thomas–Fermi equation, which arose in atomic physics. After 60 years of encounters with differential equations, Hille wrote a splendid book on ordinary differential equations in the complex plane [13, 1976]. The last chapter of that book gives an outline of the work of Hille and Russell A. Smith [30] on psi series singularities of the Emden–Fowler equation. The techniques involved are highly relevant to the Lorenz system. In section 4, we point out that some of the theorems of Hille and Smith admit simpler proofs using an approach introduced in that section.

From Hille’s illuminating bibliographic discussions [13], it is clear that Dulac [5, 1934] was a central figure with regard to psi series, with Horn [14, 1905] being another early contributor. Hille does not mention Dulac’s claim about one of the Hilbert problems, however, and indeed that claim was mistaken [15]. It appears that the error was related to a subtlety in the interpretation of psi series in the complex plane [15].

**3. Formal Development.** The formal development of psi series has a history that goes back a hundred years or more. All formal developments proceed in a similar way—one begins with psi series and then determines their coefficients using a recursion. In two of his papers, Hille [11, 12] gave clear and detailed formal developments. Our derivation is quite similar but is more careful about subtleties such as the choice of the branch of log, the degrees of the polynomials  $P_i$ ,  $Q_i$ , and  $R_i$  in (1.2), and the role of the undetermined constants ( $C$  and  $D$  in Table 1.1).

Since the Lorenz system (1.1) is autonomous, the choice of the location  $t_0$  of the singularity is arbitrary. For the sake of definiteness and because the primary interest is in solutions that are real for real  $t$ , we may assume  $\Re(t_0) < 0$  and take  $\eta = \log(-i(t - t_0))$  to obtain a branch cut that does not intersect the real axis. However, nothing changes if  $t_0$  is arbitrary and some other branch cut is chosen for defining  $\eta$ . The choice of branch cut is equivalent to the choice of  $b$  in Definition 1.1.

The form of the singularity is assumed to be given by (1.2):

$$(3.1) \quad x(t) = \sum_{m=-1}^{\infty} P_m(\eta)(t-t_0)^m, \quad y(t) = \sum_{m=-2}^{\infty} Q_m(\eta)(t-t_0)^m, \quad z(t) = \sum_{m=-2}^{\infty} R_m(\eta)(t-t_0)^m,$$

where  $P_m$ ,  $Q_m$ , and  $R_m$  are polynomials in  $\eta$ . We arrived at this form based on numerical work summarized in section 5. However, the credit for discovering the form of the psi series singularities of the Lorenz system belongs for the most part to Tabor and Weiss [32].

Substituting (3.1) into (1.1) and denoting derivatives with respect to  $\eta$  by a prime, we get

$$(3.2a) \quad \sum_{m=-1}^{\infty} (P'_m(\eta) + mP_m(\eta))(t-t_0)^{m-1} = 10Q_{-2}(t-t_0)^{-2} + \sum_{m=-1}^{\infty} (10Q_m(\eta) - 10P_m(\eta))(t-t_0)^m,$$



$$(3.2b) \quad \sum_{m=-2}^{\infty} (Q'_m(\eta) + mQ_m(\eta))(t-t_0)^{m-1} = 28 \sum_{m=-1}^{\infty} P_m(\eta)(t-t_0)^m - \sum_{m=-2}^{\infty} Q_m(\eta)(t-t_0)^m \\ - \sum_{m=-3}^{\infty} \left( \sum_{j=-1}^{m+2} P_j(\eta)R_{m-j}(\eta) \right) (t-t_0)^m,$$

$$(3.2c) \quad \sum_{m=-2}^{\infty} (R'_m(\eta) + mR_m(\eta))(t-t_0)^{m-1} = -\frac{8}{3} \sum_{m=-2}^{\infty} R_m(\eta)(t-t_0)^m \\ + \sum_{m=-3}^{\infty} \left( \sum_{j=-1}^{m+2} P_j(\eta)Q_{m-j}(\eta) \right) (t-t_0)^m.$$

For the psi series on either side of (3.2), a nonzero term with  $m = m_1$  is greater in magnitude than an  $m = m_2$  term in the limit  $t \rightarrow t_0$  if  $m_1 < m_2$ . Therefore it is formally consistent to equate powers of  $(t - t_0)$  in increasing order.

Equating coefficients of  $(t - t_0)^{-2}$  in (3.2a) and of  $(t - t_0)^{-3}$  in (3.2b) and (3.2c), we get  $P'_{-1} - P_{-1} = 10Q_{-2}$ ,  $Q'_{-2} - 2Q_{-2} = -P_{-1}R_{-2}$ , and  $R'_{-2} - 2R_{-2} = P_{-1}Q_{-2}$ . The degree of  $P_{-1}$  and  $Q_{-2}$  in  $\eta$  must be the same, while the degree of  $R_{-2}$  must be twice that degree, and the degree of  $Q_{-2}$  must be the sum of the degrees of the other two. The only possibility is for all the degrees to be zero. We get

$$(3.3) \quad (P_{-1}, Q_{-2}, R_{-2}) = (2i, -i/5, -1/5) \quad \text{or} \quad (-2i, i/5, -1/5).$$

We consider only the first possibility for now but will account for the second possibility in Lemma 3.2.

The next set of equations is  $P'_0 = 10(Q_{-1} - P_{-1})$ ,  $Q'_{-1} = Q_{-1} - 2iR_{-1} + P_0/5 + i/5$ , and  $R'_{-1} = R_{-1} + 2iQ_{-1} - iP_0/5 + 8/15$ . The only solution polynomial in  $\eta$  is given by

$$(3.4) \quad (P_0, Q_{-1}, R_{-1}) = (71i/9, 2i, 17/9).$$

For  $m = 0, 1, 2, \dots$ , we equate powers of  $(t - t_0)^m$  in (3.2a) and powers of  $(t - t_0)^{m-1}$  in (3.2b) and (3.2c) to get

$$(3.5) \quad X'_m = A_m X_m + F_m(\eta),$$

where

$$(3.6) \quad X_m = \begin{pmatrix} P_{m+1} \\ Q_m \\ R_m \end{pmatrix}, \quad A_m = \begin{pmatrix} -m-1 & 10 & 0 \\ \frac{1}{5} & -m & -2i \\ -\frac{i}{5} & 2i & -m \end{pmatrix}, \\ F_m = \begin{pmatrix} -10P_m \\ 28P_{m-1} - Q_{m-1} - \sum_{j=0}^m P_j R_{m-j-1} \\ -\frac{8}{3}R_{m-1} + \sum_{j=0}^m P_j Q_{m-j-1} \end{pmatrix}.$$

The eigenvalues of  $A_m$  are  $-m + 2$ ,  $-m$ , and  $-m - 3$ . If the linear system (3.5) is diagonalized using the eigenvectors of  $A_m$  as a basis, it turns into three scalar equations of the form  $d\xi/d\eta = \alpha\xi + f(\eta)$  with  $\alpha$  being  $-m + 2$  or  $-m$  or  $-m - 3$  and with  $f$  being a polynomial in each case. If  $\alpha \neq 0$ , we have a unique polynomial solution for  $\xi(\eta)$  whose degree is the same as that of  $f$ .

We can have  $\alpha = 0$  if and only if  $m = 0$  or  $m = 2$ . Thus if  $m \neq 0$  and  $m \neq 2$ , we can assert that (3.5) has a unique polynomial solution  $X_m$  and the degree of that solution in  $\eta$  is the same as that of  $F_m$ .

In the case  $m = 0$ ,  $F_m$  is a constant and the three scalar equations are of the form  $d\xi/d\eta = 2\xi + \beta_1$ ,  $d\xi/d\eta = -3\xi + \beta_2$ , and  $d\xi/d\eta = \beta_3$ , where the  $\beta_i$  are known constants. The only admissible solution of either of the first two equations is a constant. The last equation, however, has the solution  $\beta_3(\eta + C)$ , where  $C$  is an undetermined constant. If the eigenvectors of  $A_0$  are multiplied by the respective solutions and summed, we get

$$(3.7) \quad \begin{pmatrix} P_1 \\ Q_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} -9880i/81 \\ -988i/81 \\ -988/81 \end{pmatrix} (\eta + C) + \begin{pmatrix} 0 \\ -349i/81 \\ 1385/54 \end{pmatrix},$$

where the factor multiplying  $(\eta + C)$  is the eigenvector of  $A_0$  that corresponds to the eigenvalue  $-m = 0$ .

The matrix  $A_m$  has a zero eigenvalue again when  $m = 2$ . In this case, the degree of  $F_m$  in  $\eta$  is 2. We would expect the polynomial solution  $X_m$  of (3.6) to be cubic. However, the component of  $F_m$  along the eigenvector of  $A_m$  corresponding to the eigenvalue  $-m + 2 = 0$  is zero (with regard to this point, compare (2.9) of [32]). Therefore  $P_3$ ,  $Q_2$ , and  $R_2$ , which make up  $X_2$ , are all quadratic in  $\eta$  as shown in Table 1.1. A new undetermined constant  $D$  enters at this stage. If  $P_3$ ,  $Q_2$ , and  $R_2$  were cubic and not quadratic,  $\lfloor \frac{m+2}{2} \rfloor$  in the lemma below would be replaced by  $\lfloor \frac{m+2}{2} \rfloor + \lfloor \frac{m+2}{4} \rfloor$ .

LEMMA 3.1. *The degrees of the polynomials  $P_{m+1}(\eta)$ ,  $Q_m(\eta)$ , and  $R_m(\eta)$  are at most  $\lfloor \frac{m+2}{2} \rfloor$  for  $m = 0, 1, 2, \dots$*

*Proof.* For  $m = 0, 1, 2$ , the lemma can be verified explicitly using Table 1.1. If the maximum degree of a component of  $X_k$  is  $d_k$  for  $0 \leq k < m$ , (3.4) and (3.6) imply that the degree of  $F_m$  is at most

$$\max_{0 \leq j \leq m} (d_{j-1} + d_{m-j-1}),$$

where we assume  $m \geq 3$  and take  $d_{-1} = 0$ . We use the inductive hypothesis and note

$$d_{j-1} + d_{m-j-1} \leq \left\lfloor \frac{j+1}{2} \right\rfloor + \left\lfloor \frac{m-j+1}{2} \right\rfloor \leq \left\lfloor \frac{m+2}{2} \right\rfloor$$

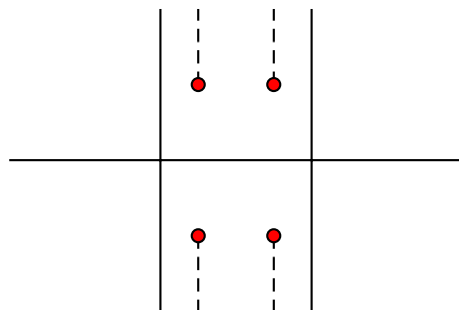
for  $0 \leq j \leq m$  to complete the proof. The second inequality above is an equality for odd  $j$ .  $\square$

It appears as if the degrees in Lemma 3.1 are actually equal to  $\lfloor \frac{m+2}{2} \rfloor$ . To prove as much, one has to rule out cancellations that can happen in a variety of ways, which may or may not be worth the trouble. Below we give a formula for the polynomial solution  $X_m$  of (3.5) that is easily derived using the variation of constants formula and integration by parts:

$$(3.8) \quad X_m = - \sum_{j=0}^{\lfloor \frac{m+2}{2} \rfloor} A_m^{-j-1} \frac{d^j F_m}{d\eta^j}$$

for  $m \geq 3$ . The correctness of (3.8) can be verified by direct substitution into (3.5).

The lemma below summarizes the discussion in this section.



**Fig. 3.1** Schematic plot of the location of the singularities in the  $t$ -plane for an orbit such as  $AB$ . The singularities are shown as red spots and the branch cuts are dashed. Only singularities within a single period are shown in the  $t$ -plane (compare Figure 1.1).

LEMMA 3.2.

- (i) For the coefficients  $P_{m+1}$ ,  $Q_m$ ,  $R_m$  shown in Table 1.1 for  $-2 \leq m \leq 3$  and defined for  $m \geq 3$  by (3.6) and (3.8), the psi series (3.1) (or (1.2)) satisfy the Lorenz system (1.1) formally. The location of the singularity  $t_0$  is arbitrary and two undetermined constants,  $C$  and  $D$ , occur in the psi series. The constant  $C$  and  $\eta$  always occur in the group  $(\eta + C)$ .
- (ii) Another formal solution is obtained by flipping the signs of all the  $P$ 's and the  $Q$ 's while leaving the  $R$ 's unchanged.
- (iii) For the solution to be formally valid,  $\eta$  can be defined as  $\log(b(t - t_0))$  for any complex number  $b$  with  $|b| = 1$ .

*Proof.* For the part about flipping signs, note that the Lorenz system is unchanged by the transformation  $(x, y, z) \rightarrow (-x, -y, z)$ . More specifically, note that flipping signs of the  $P$ 's and the  $Q$ 's changes the sign of the first two components of  $F_m$  in (3.6) but not that of the third component.

This other formal solution accounts for the second possibility in (3.3).  $\square$

If the psi series singularity is an analytic continuation of a solution that is real for real  $t$ , the location  $t_0$  of the singularity must be off the real line (see section 5). Accordingly, as  $\Im(t_0) < 0$  or  $\Im(t_0) > 0$ , the choices  $b = -i$  or  $b = i$  give branch cuts that do not intersect the real line, as shown in Figure 3.1.

**4. Proof of Convergence.** Hille's [11] proof of the convergence of psi series solutions relies on the formula

$$X_m(\eta) = \int_{-\infty}^{\eta} e^{(\eta-s)A_m} F_m(s) ds$$

for the solution  $X_m$  of (3.5) which is polynomial in  $\eta$ . A similar formula is fundamental to the approximation of strange attractors, including Lorenz's, by algebraic sets in the work of Foias, Temam, and others [8, 9].

Our proof of convergence does not use Hille's formula but instead relies on the Laplace transform and other devices. In the second part of this section, we remark that our technique will likely give simpler proofs for certain theorems of Hille and Smith. In one instance, our technique can probably be used to prove a theorem that has been stated but not proved completely.

**4.1. Psi Series Solutions of the Lorenz System.** If  $p$  is a polynomial in  $(\eta + C)$ , we define  $|p|$  as the sum of the absolute values of its coefficients. Since  $\eta$  and  $C$  always occur in the group  $(\eta + C)$ , we can think of  $C$  as being subsumed by  $\eta$ .  $|X_m|$  is defined as the maximum of  $|P_{m+1}|$ ,  $|Q_m|$ , and  $|R_m|$ . For  $m \geq 2$ ,  $|X_m|$  will depend upon the undetermined constant  $D$ . The key to the proof of convergence of the psi series (3.1) is a bound of the form  $|X_m| < K_1 K_2^m$ , where  $K_1$  and  $K_2$  are positive constants that depend upon the undetermined parameter  $D$ .

For  $F_m$  defined by (3.6),  $|F_m|$  is the maximum of  $|\cdot|$  over its three components, each of which is a polynomial in  $(\eta + C)$ . We begin with the following easy lemma.

LEMMA 4.1. For  $m \geq 3$ ,

$$|F_m| \leq 30 |X_{m-1}| + 28 |X_{m-2}| + \sum_{j=1}^{m-1} |X_{m-j-1}| |X_{j-1}|.$$

*Proof.* If  $p$  and  $q$  are polynomials in  $\eta + C$ , then  $|pq| \leq |p| |q|$  and  $|p + q| \leq |p| + |q|$ . Repeated use of those inequalities with the definition (3.6) of  $X_m$  and  $F_m$  gives

$$|F_m| \leq 10 |X_{m-1}| + 28 |X_{m-2}| + \sum_{j=0}^m |X_{m-j-1}| |X_{j-1}|.$$

The lemma results when the  $j = 0$  and  $j = m$  terms are moved out of the summation while using Table 1.1 to note that  $|X_{-1}| < 10$ .  $\square$

For matters related to the existence and uniqueness of the Laplace transform that arise implicitly in the proof below, see [35]. In the lemma below, we treat only polynomials in  $\eta$  (assuming  $C = 0$ ), but the lemma still applies when  $\eta$  and  $C$  occur in the group  $(\eta + C)$  and  $C \neq 0$ .

LEMMA 4.2. Let  $\alpha$  be a complex number with  $|\alpha| > 1$  and let  $f(\eta)$  be a polynomial in  $\eta$ . Let  $\xi(\eta)$  be the polynomial solution of the differential equation

$$(4.1) \quad \frac{d\xi}{d\eta} = \alpha\xi + f(\eta).$$

If the polynomial  $f(\eta)$  is of degree  $n$ , assume  $|\alpha| \geq a(n + 1/2)$  for some  $a > 1$ . Then

$$(4.2) \quad |\xi| \leq \frac{1}{|\alpha|} \frac{a}{a-1} |f|.$$

*Proof.* Let  $f(\eta) = f_0 + f_1\eta + \dots + f_n\eta^n$ . To take the Laplace transform of (4.1), we multiply (4.1) by  $e^{-\eta s}$  and integrate from  $\eta = 0$  to  $\eta = \infty$ . We get

$$s\hat{\xi}(s) - \alpha\hat{\xi}(s) = \xi(0) + \frac{f_0}{s} + \frac{1!f_1}{s^2} + \frac{2!f_2}{s^3} + \dots + \frac{n!f_n}{s^{n+1}}.$$

Rearranging, we have

$$\hat{\xi}(s) = \frac{\xi(0)}{s - \alpha} + \frac{f_0}{(s - \alpha)s} + \frac{1!f_1}{(s - \alpha)s^2} + \dots + \frac{n!f_n}{(s - \alpha)s^{n+1}}.$$

All terms on the right-hand side above except the first are rewritten using the identity

$$\frac{1}{(s - \alpha)s^k} = \frac{1}{\alpha^k(s - \alpha)} - \frac{1}{\alpha^k s} - \frac{1}{\alpha^{k-1}s^2} - \dots - \frac{1}{\alpha s^k}.$$

In the resulting expression,  $\xi(0)$  is chosen to cancel all the  $1/(s - \alpha)$  terms to get a polynomial solution. We then have

$$(4.3) \quad \hat{\xi}(s) = - \sum_{k=0}^n k! f_k \left( \frac{1}{\alpha^{k+1} s} + \frac{1}{\alpha^k s^2} + \cdots + \frac{1}{\alpha s^{k+1}} \right)$$

$$(4.4) \quad = - \sum_{k=1}^{n+1} \frac{1}{s^k} \left( \frac{(k-1)! f_{k-1}}{\alpha} + \frac{k! f_k}{\alpha^2} + \cdots + \frac{n! f_n}{\alpha^{n+2-k}} \right).$$

The coefficients of  $\xi(\eta)$  are evident from inspecting the summations (4.3) and (4.4). From the summation (4.3) and the inverse Laplace transform, we get

$$(4.5) \quad |\xi| \leq \sum_{k=0}^n |f_k| \left( \frac{k!}{0! |\alpha^{k+1}|} + \frac{k!}{1! |\alpha^k|} + \cdots + \frac{k!}{(k-1)! |\alpha^2|} + \frac{k!}{k! |\alpha|} \right) = \sum_{k=0}^n \frac{|f_k|}{|\alpha|} \left( \sum_{j=0}^k \frac{k!}{j! |\alpha^{k-j}|} \right).$$

To clarify the calculation that gives (4.5), let us consider the special case  $d\xi/d\eta = \alpha\xi + \eta^k$ . Its unique polynomial solution is  $\xi = \eta^k/\alpha - k\eta^{k-1}/\alpha^2 - \cdots - k!/ \alpha^{k+1}$  and this  $|\xi|$  corresponds to the  $k$ th term in (4.5).

Next we bound  $k!/j! |\alpha^{k-j}|$  for  $0 \leq k \leq n$  and  $0 \leq j \leq k$ :

$$\begin{aligned} \frac{k!}{j! |\alpha^{k-j}|} &= |\alpha^{j-k}| k(k-1)\cdots(j+1) \\ &= |\alpha^{j-k}| (k(j+1))((k-1)(j+2))((k-2)(j+3))\cdots L \\ &\leq \left( \frac{k+j+1}{2|\alpha|} \right)^{k-j} \leq \left( \frac{k+1/2}{|\alpha|} \right)^{k-j} \leq \left( \frac{n+1/2}{|\alpha|} \right)^{k-j} \\ &\leq 1/a^{k-j}. \end{aligned}$$

In the second line above, the last factor  $L$  is either  $(k+j+1)/2$  or  $((k+j)(k+j+2)/4)$ . The first inequality in the third line is obtained by applying the inequality  $xy \leq ((x+y)/2)^2$  repeatedly. The inequality in the last line uses the assumption  $|\alpha| \geq a(n+1/2)$  made in the statement of the lemma.

Returning to (4.5), we have

$$|\xi| \leq \frac{|f|}{|\alpha|} (1 + 1/a + 1/a^2 + \cdots),$$

which completes the proof.  $\square$

The inequality in the lemma below is not strict mainly because  $|F_m| = 0$  is not ruled out.

LEMMA 4.3. For  $m \geq 8$ ,  $|X_m| \leq 192 |F_m| / (m-2)$ .

*Proof.* We take the matrix of eigenvectors of  $A_m$  defined in (3.6) to be

$$V = \begin{pmatrix} -5i & 10i & -5i \\ -3i/2 & i & i \\ 1 & 1 & 1 \end{pmatrix},$$

where the columns are ordered to correspond to the eigenvalues  $-m+2$ ,  $-m$ , and  $-m-3$ , respectively.

If (3.5) is rewritten using a similarity transformation that turns  $A_m$  into a diagonal matrix, we get three scalar equations

$$\frac{d\xi_i}{d\eta} = \alpha_i \xi_i + f_i$$

for  $i = 1, 2, 3$ , where  $(\alpha_1, \alpha_2, \alpha_3) = (-m + 2, -m, -m - 3)$ ,  $(f_1, f_2, f_3)' = V^{-1}F_m$ , and  $X_m = V(\xi_1, \xi_2, \xi_3)'$  (the prime denotes transpose).

To apply Lemma 4.2 to each of the scalar equations, we use Lemma 3.1 and take  $n = \lfloor (m + 2)/2 \rfloor$ . In addition, we choose an  $a > 1$  such that

$$|\alpha_1| = |m - 2| \geq a(m + 3)/2 \geq a(n + 1/2).$$

The choice  $a = 12/11$  works for  $m \geq 8$ . Thus we get  $|\xi_i| \leq 12|f_i|/|\alpha_i| \leq 12|f_i|/(m - 2)$  for  $i = 1, 2, 3$ .

We have  $|f_i| \leq \|V^{-1}\|_\infty |F_m|$  for  $i = 1, 2, 3$  and  $|X_m| \leq \|V\|_\infty \max(|\xi_1|, |\xi_2|, |\xi_3|)$ . Combining the inequalities, we get

$$|X_m| \leq \frac{12}{m - 2} \|V\|_\infty \|V^{-1}\|_\infty |F_m|.$$

The proof is completed by verifying that  $\|V\|_\infty \|V^{-1}\|_\infty = 16$ . □

The lemma below is crucial to showing that the psi series expansions which formally satisfy the Lorenz system by Lemma 3.2 are convergent. Its proof is structured to be transparent but does not give the best constants.

LEMMA 4.4. *For positive constants  $K_1$  and  $K_2$  which depend upon the undetermined constant  $D$  of Lemma 3.2,  $|X_m| < K_1 K_2^m$  for  $m = 0, 1, 2, \dots$*

*Proof.* By Lemmas 4.1 and 4.3, we have

$$|X_m| \leq \frac{30 \times 192}{m - 2} |X_{m-1}| + \frac{28 \times 192}{m - 2} |X_{m-2}| + \frac{192}{m - 2} \sum_{j=1}^{m-1} |X_{m-j-1}| |X_{j-1}|$$

for  $m \geq 8$ . If we define  $x_m = |X_m|$  for  $m = 0, 1, \dots, 7$  and, for  $m \geq 8$ ,

$$(4.6) \quad x_m = 960x_{m-1} + 896x_{m-2} + 32 \sum_{j=1}^{m-1} x_{m-j-1} x_{j-1},$$

then  $|X_m| \leq x_m$  (after noting  $192/6 = 32$  and so on).

Let  $f(Z) = \sum_{m=0}^\infty x_m Z^m$  be the generating function of the  $x_m$  sequence. Using (4.6), we get

$$(4.7) \quad f(Z) - (c_0 + c_1 Z + \dots + c_7 Z^7) = 960Zf(Z) + 896Z^2f(Z) + 32Z^2f(Z)^2.$$

In (4.7), the constants  $c_0, \dots, c_7$  account for the fact that the recurrence (4.6) is valid only for  $m \geq 8$ . They are put in to get  $x_0, x_1, \dots, x_7$  as the coefficients of  $Z^0, Z^1, \dots, Z^7$ , respectively. They can be determined explicitly (compare Table 1.1); for instance,  $c_0 = x_0 = |X_0|$  and  $c_1 = x_1 - 960x_0 = |X_1| - 960|X_0|$ . Because  $|X_2|, \dots, |X_7|$  depend upon  $D$ , so will  $c_2, \dots, c_7$ .

The implicit function theorem implies the existence of a unique analytic function with  $f(0) = x_0$  that satisfies (4.7)—if all terms of (4.7) are moved to the left and  $f(Z)$  is treated as a variable, the partial derivative of the left-hand side with respect to  $f$  is 1

when  $Z = 0$ , thus verifying the derivative condition of the implicit function theorem. Therefore  $f(Z)$  is the generating function of the  $x_m$  sequence. The bound on  $x_m$  given by the lemma follows from the Hadamard–Cauchy root formula for the radius of convergence of  $f(Z)$  around  $Z = 0$ . If  $K_2$  is taken slightly greater than the inverse of the radius of convergence and  $K_1 > 0$ , the bound  $|X_m| < K_1 K_2^m$  holds for large enough  $m$ . So  $K_1$  can be chosen to make the bound hold for every  $m = 0, 1, 2, \dots$

An explicit lower bound for the radius of convergence in terms of  $c_0, \dots, c_7$  can be determined using the implicit function theorem proved by Lindelöf using his majorant technique [13, p. 63], [20].  $\square$

We are now ready to prove convergence of the formal psi series of Lemma 3.2.

**THEOREM 4.5.** *Consider the formal psi series of Lemma 3.2 with  $\eta = \log(b(t-t_0))$  and  $|b| = 1$ . The branch cut is the segment*

$$\{t_0 - \bar{b}p \mid p \geq 0\}.$$

*Then the psi series expansions for  $x(t)$ ,  $y(t)$ , and  $z(t)$  given by (1.2) or (3.1) converge uniformly and absolutely on the disc  $|t - t_0| \leq r$  with  $r > 0$  and with an open neighborhood of the branch cut excluded from the disc. In general,  $r$  will depend upon both  $C$  and  $D$ , which are the two undetermined constants in the psi series.*

*Proof.* We will give the proof for  $z(t)$ . The proofs for  $x(t)$  and  $y(t)$  are similar.

Excluding a neighborhood of the branch cut means that a neighborhood of  $t_0$  is excluded from the domain of convergence. Therefore  $R_{-2}(t-t_0)^{-2}$  and  $R_{-1}(t-t_0)^{-1}$  are both bounded on the domain of convergence. The other reason for excluding a neighborhood of the branch cut is to ensure that  $\eta$  is well defined.

By the definitions of  $|R_m|$  and  $|X_m|$  given at the beginning of this section,

$$\begin{aligned} |R_m(\eta)(t-t_0)^m| &\leq |R_m| \max\left(1, |\log b(t-t_0) + C|^{\lfloor (m+2)/2 \rfloor}\right) |t-t_0|^m \\ &\leq |X_m| \max\left(1, |\log b(t-t_0) + C|^{\lfloor (m+2)/2 \rfloor}\right) |t-t_0|^m \\ &< K_1 K_2^m \max\left(1, |\log b(t-t_0) + C|^{\lfloor (m+2)/2 \rfloor}\right) |t-t_0|^m, \end{aligned}$$

where  $m \geq 0$  and  $|b| = 1$ . The first inequality above uses Lemma 3.1 and the third inequality uses Lemma 4.4.

Choosing an  $r > 0$  such that

$$(4.8) \quad r < 1/K_2 \quad \text{and} \quad r(|\log r| + \pi + |C|) < 1/K_2$$

is sufficient to ensure uniform and absolute convergence. The  $\pi$  in (4.8) is explained by the inequality  $|\log b(t-t_0)| \leq |\log |t-t_0|| + \pi$ . A choice of  $r$  in accord with (4.8) suffices for the convergence of the psi series for  $x(t)$  and  $y(t)$  as well.  $\square$

A further argument is required to show that the convergent psi series actually satisfy the differential equation, a point that seems to have been overlooked on occasion. When the psi series for  $x(t)$ ,  $y(t)$ , and  $z(t)$  are substituted into the Lorenz system (1.1), the summation and multiplication of psi series on the right-hand side is justified by standard results on rearrangements of absolutely convergent series. To justify the differentiation of psi series on the left-hand side, we mention that the uniform convergence of a sequence of analytic functions on an open set implies the uniform convergence of the derivatives on any compact subset of that open set [28, Theorem 10.28]. We can now state the following theorem.

**THEOREM 4.6.** *The psi series for  $x(t)$ ,  $y(t)$ , and  $z(t)$  given by (1.2) or (3.1), whose formal validity is asserted by Lemma 3.2, satisfy the Lorenz system (1.1) in*

the disc  $|t - t_0| \leq r$ , with the branch cut excluded, for some  $r > 0$ . In general,  $r$  will depend upon both  $C$  and  $D$ , which are the two undetermined constants in the psi series.

Levine and Tabor [19] raised the possibility that the locations of the singularities of an orbit of the Lorenz system may have accumulation points in the complex  $t$ -plane. Theorem 4.6 shows that psi series singularities cannot be accumulation points.

So far, in results such as Lemma 3.2 and Theorem 4.6, we have regarded the psi series as functions of  $t$ . It is useful to consider them as functions of  $\eta$ , where  $\eta = \log b(t - t_0)$  gives a parametrization of the Riemann surface that gets rid of the branch cut in the  $t$ -plane. To be specific, we assume  $\Im(t_0) < 0$  and  $b = -i$ . In that case, we have  $(t - t_0) = i \exp(\eta)$ , and the psi series (3.1) take on the form

$$(4.9) \quad x(\eta) = \sum_{m=-1}^{\infty} i^m P_m(\eta) e^{m\eta}, \quad y(\eta) = \sum_{m=-2}^{\infty} i^m Q_m(\eta) e^{m\eta}, \quad z(\eta) = \sum_{m=-2}^{\infty} i^m R_m(\eta) e^{m\eta}$$

with  $P_m, Q_m, R_m$  being polynomials in which  $\eta$  always occurs in the group  $(\eta + C)$ . Every time  $t$  passes through the branch cut of  $\log(-i(t - t_0))$ ,  $\eta$  increases or decreases by  $2\pi i$ . Because  $\eta$  and  $C$  always occur in the group  $(\eta + C)$  in  $P_m, Q_m, R_m$ , we can allow for other branches of  $\log(-i(t - t_0))$  in the psi series of (3.1) or (1.2) by keeping the principal branch of the logarithm in the definition of  $\eta$  and incrementing  $C$  by an integer multiple of  $2\pi i$ . The change in the estimate for the radius of convergence  $r$  of Theorem 4.6 for these other branches will then be in accord with (4.8). (Note that  $K_2$  depends only on  $D$ .)

If the domain of convergence of the transformed psi series (4.9) is considered in the  $\eta$ -plane, the choice of the principal branch of  $\log(-i(t - t_0))$  implies  $-\pi < \Im(\eta) \leq \pi$ , and the  $r$  estimated by Theorem 4.6 implies  $\Re(\eta) \leq \log r$ . Thus the region of convergence of the principal branch will be a semi-infinite rectangle in the  $\eta$ -plane. To pass to other branches, we keep  $C$  fixed and allow the imaginary part of  $\eta$  to be arbitrary. For  $\eta$  corresponding to different branches, one has to use different estimates for  $r$  as explained in the previous paragraph. Therefore the estimated domain of convergence of the transformed psi series (4.9) will be a union of semi-infinite rectangles as in Figure 4.1. If we start at the principal branch of  $\log(-i(t - t_0))$  and cross its branch cut  $m$  times, then by (4.8)  $r \approx 1/(K_2 2\pi |m|)$  for large integers  $m$ . For such a branch  $-\pi + 2\pi m < \Im(\eta) \leq \pi + 2\pi m$  and  $\Re(\eta) \lesssim -\log |m|$  for convergence, which gives an approximate idea of the shape of the domain sketched in Figure 4.1.

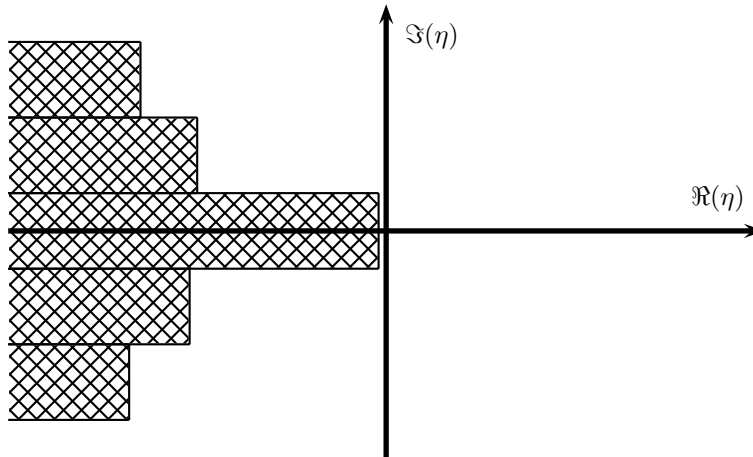
**4.2. Remarks on Theorems of Hille and Smith.** In [12], Hille proved that the plane quadratic system

$$\begin{aligned} dx/dt &= x(a_0 + a_1x + a_2y), \\ dy/dt &= y(b_0 + b_1x + b_2y) \end{aligned}$$

has a logarithmic psi series singularity if  $(a_1 - b_1)(a_2 - b_2)/(a_1b_2 - a_2b_1)$  is a positive integer. Smith [30] generalized that result to plane polynomial systems. Smith's proof is based on a reduction to results proved early in the 20th century for Briot–Bouquet systems. These results are summarized in sections 12.5 and 12.6 of Hille's book [13].

One difference between the results of Hille and Smith for plane polynomial systems and Theorem 4.6 is as follows. The singular solutions for plane polynomial systems look like simple poles near the singular point. The singularities of the Lorenz system implied by Theorem 4.6 look like double poles.





**Fig. 4.1** Schematic plot of domain of convergence of the psi series in the  $\eta$  plane, as implied by Theorem 4.5. The shape of the region is given approximately by  $\Re(\eta) \lesssim -\log|m|$  for integers  $m$  of large magnitude and  $-\pi + 2\pi m < \Im(\eta) \leq \pi + 2\pi m$ .

In [11], Hille proved the existence of logarithmic psi series solutions for the Emden–Fowler system  $d^2y/dt^2 = t^{-2/p}y^{1+2/p}$  for  $p = 2$ . At the end of the paper, Hille discussed the difficulty of extending his technique and noted remarks by a referee suggesting a proof of existence of logarithmic psi series solutions for positive integral  $2p$ . Like Smith’s proof for plane polynomial systems, the suggested proof goes through a reduction to a Briot–Bouquet system, but no complete proofs are found in the literature as far as we are aware. The result for positive integral  $2p$  was stated as Theorem 12.4.2 in Hille’s book [13]. Hille mentioned that “the various proofs are nasty” and omitted them.

The proofs using reduction to Briot–Bouquet systems are difficult to follow in their entirety, partly because they depend so crucially on results proved long ago. It appears that use of the Laplace transform and the implicit function theorem will give simpler proofs for plane polynomial systems and complete proofs that are not so “nasty” in the case of the Emden–Fowler system with  $2p$  a positive integer.

Theorem 4 of Smith’s paper [30] states that all singularities of real solutions of certain plane polynomial systems must be of the form determined in Theorem 3 of that paper. The statement occurs again as Theorem 12.6.3 of [13]. Smith’s proof begins with an ingenious change of variables. Near the end of the proof, we find the argument “in the case when  $\lambda > 0$ , the arbitrary constant  $c$  in (20) can be chosen to fit this solution  $\zeta(\xi)$  in the neighborhood of  $\xi = 0$ .” We are unable to follow that argument and believe it requires substantial explication at the very least.

**5. Complex Singularities and the Lorenz Attractor.** If  $t = t_0$  is a singularity of the Lorenz system (1.1), the solution must diverge to infinity as the singularity is approached.

**THEOREM 5.1.** *Let  $\gamma$  be a Lipschitz curve in the complex  $t$  plane that approaches  $t_0$  at one of its two endpoints. Let  $(x(t), y(t), z(t))$  be a solution of the Lorenz system (1.1) defined for  $t \in \gamma$ . If  $t_0$  is a singular point, then*

$$(5.1) \quad \liminf_{t \rightarrow t_0} |t - t_0| \left( |x(t)| + |y(t)| + |z(t)| \right) \geq \frac{1}{8},$$

as  $t$  approaches  $t_0$  along the curve  $\gamma$ .

*Proof.* Denote  $|x(t)| + |y(t)| + |z(t)|$  by  $r_t$ . Consider the set of all complex  $(x, y, z)$  in the region  $|x - x(t)| + |y - y(t)| + |z - z(t)| < b$  for some  $b > 0$ . Then the sum of the absolute values of the right-hand sides of the Lorenz system (1.1) is bounded by

$$M = 52(r_t + b) + 2(r_t + b)^2,$$

where  $10 + 10 + 28 + 1 + 8/3 < 52$  explains the first coefficient.

Theorem 8.1, Chapter 1, of [1] (also see Theorem 2.3.1 of [13]) with  $a = \infty$  and  $M$  and  $b$  as above implies that the solution admits a unique analytic continuation to all  $t'$  in the disc  $|t' - t| \leq R$  with

$$R = \frac{b}{52(r_t + b) + 2(r_t + b)^2}.$$

Taking  $b = r_t$ , we get  $R = 1/(104 + 8r_t)$ .

Being a singular point,  $t_0$  must lie outside the disc of analyticity. Therefore  $|t_0 - t|(104 + 8r_t) > 1$ . Taking the limit  $t \rightarrow t_0$  along points on  $\gamma$  completes the proof.

The curve  $\gamma$  is assumed to be Lipschitz to ensure uniqueness of the solution. □

Theorem 5.1 proves that as the singular point  $t_0$  of the Lorenz system is approached, the magnitude of the solution must diverge at a rate that is at least as great as  $0.125/|t - t_0|$ . In fact, if the answer to Question 1.1 is yes and the singularities of the Lorenz system are all given by psi series of the form (1.2), the divergence would be proportional to  $1/|t - t_0|^2$ .

Theorem 5.1 is used to prove the theorem below.

**THEOREM 5.2.** *Consider a trajectory of the Lorenz system (1.1) which is real for real values of  $t$ . In particular, assume that the state  $(x(0), y(0), z(0))$  at  $t = 0$  is real. Then there is no singularity at any finite and real value of  $t$  and the solution is defined for all real values of  $t$ .*

*Proof.* Let  $Q = x^2 + y^2 + z^2$ . From (1.1), we have

$$dQ/dt = 2(-10x^2 - y^2 - 8z^2/3 + 38xy).$$

The matrix 1-norm of the symmetric form on the right-hand side is bounded by 58 and so are the magnitudes of its eigenvalues. Therefore,  $|dQ/dt| < 58Q$  and

$$Q(t) \leq Q(0) + 58 \int_0^{|t|} Q(s) ds.$$

At this point it appears as if the proof can be completed using the Gronwall inequality (Theorem 1.6.6 of [13]) to deduce that  $Q(t) \leq Q(0) \exp(58|t|)$ . However, the bound on  $Q(t)$  holds only if we assume the existence of the solution, which is what we set out to prove.

In circumstances such as these, oscillatory singularities for which the solution does not tend to a limit as the singular point is approached must be ruled out—an important point that goes back to Painlevé [13, Chapter 3]. Theorem 5.1 forces the norm of the solution to diverge near a singular point thus making it possible to complete the proof. □

Theorem 5.2 is implied by Theorem 2.4(i) of [8]. In fact, Theorem 2.4 of [8] is a sharper result as it implies that  $Q(t) \leq C \exp(20|t|)$  for some constant  $C$  independent of  $t$ . We have given a proof that brings out the connection to the nature of the singular points.

**Table 5.1** *Imaginary parts of the singular points closest to the real line in the  $t$ -plane.*

$AB$	$\pm i0.1714501006$
$AAB$	$\pm i0.1617621257$
$AAAB$	$\pm i0.1563426260$
$AABB$	$\pm i0.1636066901$

So far we know that the Lorenz system has singularities represented by logarithmic psi series and that the solution must diverge as a singularity is approached. But do solutions such as the one shown in Figure 1.1 have complex singularities and are they represented by psi series?

Using numerical methods based on [26], we found the complex singularities closest to the real line of a few solutions listed in Table 5.1. Those solutions, of course, are all real for real  $t$ . They are assigned the labels  $AB$ ,  $AAB$ ,  $AAAB$ , and  $AABB$  following the convention explained in the caption to Figure 1.1. From Table 5.1, we see that the complex singularities are located at a distance greater than 0.037 from the real line, in agreement with Theorem 2.3 of Foias et al. [8]. In addition to computing the location of the singularities, we have verified numerically that their form matches the formal development of psi series given in section 3. This numerical work will be described in detail elsewhere.

**6. Conclusion.** Given that the Lorenz system (1.1) has resisted mathematical analysis on the real line, one may say that it is natural to think of  $t$  as a complex variable and  $x, y, z$  as analytic functions of  $t$ . When the solutions of the Lorenz system are viewed as analytic functions, it is natural to begin their investigation by looking at their singularities. We have given a complete formal development of singularities in the complex  $t$ -plane, proved convergence of the psi series representations using a new technique, and proved that the psi series indeed satisfy the Lorenz system. The development of the analytic theory appears to be a fascinating avenue for further investigations.

The geometrical theory of differential equations, in which the Lorenz system is a famous example, sprang out of problems in analytic function theory—a fact that is not well known. More specifically, the stable manifold theorem, which is undoubtedly fundamental to the geometrical theory, was first proved to understand the solution of  $dz/dw = P(z, w)/Q(z, w)$  in a neighborhood of  $z = w = 0$  when  $P$  and  $Q$  are bivariate polynomials with  $P(0, 0) = Q(0, 0) = 0$  [13, p. 97], [27]. Thus our suggestion that the mathematical analysis of the Lorenz system (1.1) could be a problem in analytic function theory is an attempt to complete the circle.

The properties of analytic functions  $x(t)$  which satisfy the nonlinear Riccati equation  $dx/dt = f_0(t) + f_1(t)x + f_2(t)x^2$ , where the  $f_i(t)$  are rational in  $t$ , is a well-studied topic. All the movable singularities of the Riccati equation are poles and the dependence of its solution on the undetermined constant is given by a fractional linear transformation. For the Lorenz system some of the movable singularities have psi series representations of the form determined in section 4. The dependence of these psi series solutions on the undetermined constants is much more complicated than for the Riccati equation.

Another well-studied topic is the classification of second order nonlinear systems all of whose movable singularities are poles. The Painlevé classification has been presented with lexicographic thoroughness by Ince [16]. There appear to be few classification results for third order systems such as the Lorenz system. Studying a

specific system will probably sidestep many difficulties of the classification problem. In any event, the movable singularities of the Lorenz system are not poles.

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