Study on the invariance of the phase of plane waves

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Abstract –In recent study, the invariance of the phase of plane waves among all inertial frames is challenged by a counter-example, that is, there is the negative frequency of plane waves in the case of"superluminal" motion of the medium, where the four-vector $(\mathbf{k}, \omega/c)$ of plane waves is exceptionally not Lorentz-covariant. A new argument that the conventional invariance of the phase of plane waves should be replaced by the invariance of the phase difference of plane waves among all inertial frames is presented. Based on our argument, the explanation to the counter-example is worked out. The result shows that the four-vector of plane waves is still Lorentz-covariant.

Introduction. – The phase of a plane wave among all inertial frames is commonly considered to be a constant, but it may be not always the case. In an EPL letter, Dr. Young-Sea Huang has given a result, that is, according to the usual argument of the invariance of the phase of plane waves, one would get a negative value for the frequency of light waves in the case that the speed of waves in a medium less than the speed of that medium moving in the direction opposite to the propagation direction of waves which give rise to a conclusion that the four-vector $(\mathbf{k}, \omega/c)$ of plane waves is in general not Lorentz-covariant. The invariance of the phase of plane waves among all inertial frames is questionable [1]. Next, the conundrum of the negative frequency of plane waves has been resolved in another EPL letter [2]. However, we think its conclusion is questionable. The reason is that in order to let the frequency changes from negative value to positive value, the direction of the wave vector and the direction of the wave surfaces transport (i.e., the phase speed) point in opposite directions in the case of "superluminal" motion of the medium. This clearly does not meet the concept of a plane wave.

In this letter, we re-investigate the reasons for the phase invariance and point out that it is not accurate. However, based on Einstein's Principle of the relativity, we can prove that the phase difference between two wave surfaces among all inertial frames is the same. We strongly suggest that the invariance of phase difference is employed in Lorentz wave vector transformation instead of the conventional phase invariance. Based on our new argument, an explanation to the meaning of negative frequency of light waves is presented. Furthermore, we conclude that the four-vector $(\mathbf{k}, \omega/c)$ of the propagation of monochromatic waves is still Lorentz-covariant, whether the phase of plane waves among inertial frames is the same or not

Proof of the invariance of phase difference and Physics meaning of negative frequency. – So far, the invariance of the phase of plane waves has never been exactly proved. The usual argument is as follows: "This is because the elapsed phase of a wave is proportional to the number of wave crests that have passed the observer. Since this is merely a counting operation, it must be independent of coordinate frame [3]." Although this is usually written in the textbook, we think it is not accurate. Because the meaning of the phase invariance implies that the instantaneous phase of one point in an inertial frame and its instantaneous phase in another inertial frame are the same, we cannot directly obtain the relationship of two instantaneous phases on one common point in two inertial frames according to the above reason. Next we will thus analyze this relationship thoroughly.

Let us consider a three-dimensional plane light wave, and suppose its wave function is sine. Assume two standard configuration inertial coordinates $\mathbf{K}$ and $\mathbf{K}'$. The frame $\mathbf{K}'$ moves with constant speed V relative to the frame $\mathbf{K}$ in the positive direction along the common $x-x'$ axis. Let their origins O and O' coincide at the initial $t = t' = 0$; a medium is at rest in the frame $\mathbf{K}$; the plane light wave in the medium propagates in the direction of wave vector $\mathbf{k}$; and its phase velocity relative to the frame $\mathbf{K}$ is $\mathbf{u^p}$. Based on the present conditions, $\mathbf{u^p}$ is also the velocity of energy (or ray) transport $\mathbf{u}$, i.e., $\mathbf{u} = \mathbf{u^p}$. According to the usual expression form of the phase of plane waves, we have

$$\Psi(\mathbf{k}\cdot\mathbf{r},t)=\mathbf{k}\cdot\mathbf{r}-\omega t+\phi. \tag{1}$$

Where it is assumed in the frame $\mathbf{K}$: $\Psi(\mathbf{k}\cdot\mathbf{r},t)$ is the phase of plane waves; $\mathbf{r}$ is the position vector; ϕ is an arbitrary constant; ω is the angular frequency. From eq. (1), when $t=0$ and $\mathbf{k}\cdot\mathbf{r}=0$, $\Psi(0,0)=\phi$ gives a wave surface, denoted with W_0. At time t the wave surface W_0 will travel to another position along the direction of $\mathbf{k}$, but the phase of W_0 remains constant, that is, its phase is still $\Psi(W_0)=\Psi(\mathbf{k}\cdot\mathbf{r}=\omega t,t)=\phi$. Next, let us consider another arbitrary wave surface W_a, its phase is $\Psi(W_a)=\Psi(\mathbf{k}\cdot\mathbf{r}=\delta,0)=\delta+\phi$ at $t=0$ and its phase is still $\Psi(W_a)=\Psi(\mathbf{k}\cdot\mathbf{r}=\delta+\omega t,t)=\delta+\phi$ at time t, where δ is an arbitrary constant. Thus, at $t=0$ the phase difference between W_0 and W_a is $\Psi(W_a)-\Psi(W_0)=\delta$ and it is still δ at time t. Therefore, we can obtain a useful result that the phase difference between two arbitrary wave surfaces in an inertial frame remains the same with respect to time.

Let us change perspectives. If the plane light wave vibration in eq. (1) is observed from $\mathbf{K}'$, should the phase of the wave remain invariant quantity? We all know that an instantaneous phase of waves is used to represent the instantaneous vibration state of one point at a moment. Einstein's Principle of the relativity tells us that an instantaneous vibration state must be independent of coordinate frame. For example, if an instantaneous state of a wave surface is wave crest, as seen from $\mathbf{K}$, then we must see that its instantaneous state is also the wave crest, as seen from $\mathbf{K}'$. However, based on the conclusion, two phases of one point as observed from $\mathbf{K}$ and $\mathbf{K}'$, respectively, are not necessarily strictly equal. In other words, based on Principle of the relativity, we can conclude that the difference between the two phases may be an integral multiple of 2π. Therefore, to meet $\sin\Psi=\sin\Psi'$, the respective phase Ψ and Ψ' of one point must satisfy eq. (2), as seen from $\mathbf{K}$ and $\mathbf{K}'$, i.e.,

$$\Psi'(\mathbf{k}'\cdot\mathbf{r}',t')=\Psi(\mathbf{k}\cdot\mathbf{r},t)+2m\pi. \tag{2}$$

Where m is an arbitrary integer, $\mathbf{r}'$ is the position vector; $\mathbf{k}'$ is the wave vector, as observed from $\mathbf{K}'$. Because the phase is not an absolute concept, its absolute quantity is dependent on the initial phase. From this perspective it is similar to the case of potential energy. Therefore, we were possibly never able to prove $\Psi=\Psi'$. Only when assuming the integer $m=0$, the usual invariance of the phase of plane waves is strictly correct.

However, if W_0 is assumed to be a wave crest and W_a is next wave crest, as seen from $\mathbf{K}$, then $\Psi(W_a)-\Psi(W_0)=2\pi$. According to Principle of the relativity, we must also see the two successive wave crests, as observed from $\mathbf{K}'$. Thus, there is still $\Psi'(W_a)-\Psi'(W_0)=2\pi$. The above result is applicable in the situation of arbitrary phase difference also. Consequently, we have proven our argument, that is, the phase difference between two arbitrary wave surfaces (or two arbitrary points) among all inertial frames is the same.

Now we will use the invariance of phase difference to study Lorentz wave vector transformation. Here, we still assume W_a is an arbitrary wave surface and $\Psi(W_0)=\phi$. According to our the invariance of phase difference, the phase difference between W_0 and W_a, as seen from $\mathbf{K}$ and $\mathbf{K}'$, respectively, is the same, i.e.,

$$\Psi(W_a)-\Psi(W_0)=\Psi'(W_a)-\Psi'(W_0). \tag{3}$$

To meet eq. (3), in eq. (2), the arbitrary integer m must be remain the same with respect to time t' and space $\mathbf{r}'$. In other words, there is the same integer m among all phase Ψ'. However, the expression of the phase Ψ' has two possible forms, as seen from $\mathbf{K}'$ [4]. Thus, we have

$$\Psi'(\mathbf{k}'\cdot\mathbf{r}',t')=\pm(\mathbf{k}'\cdot\mathbf{r}'-\omega' t')+\phi+2m\pi. \tag{4}$$

Where ω' is the angular frequency of the plane waves, as seen from $\mathbf{K}'$. Because at $t=t'=0$ W_0 is passing through the origins O and O', the two phases of W_0 are $\Psi(W_0)=\phi$ and $\Psi'(W_0)=\phi+2m\pi$ in $\mathbf{K}$ and $\mathbf{K}'$, respectively. If the wave surface W_a is passing through one coincident point of $\mathbf{K}$ and $\mathbf{K}'$ at time t and time t', then the two phases of W_a are described eqs. (1) and (4), respectively, i.e., $\Psi(W_a)=\Psi(\mathbf{k}\cdot\mathbf{r},t)$ and $\Psi'(W_a)=\Psi'(\mathbf{k}'\cdot\mathbf{r}',t')$. Therefore, on the basis of eq. (3) which describes the invariance of the phase difference, we have

$$\mathbf{k}\cdot\mathbf{r}-\omega t=\pm(\mathbf{k}'\cdot\mathbf{r}'-\omega' t'). \tag{5}$$

Where the sign $(+)$ must be taken when $\mathbf{k}\cdot\mathbf{r}>0$ and $\mathbf{k}'\cdot\mathbf{r}'>0$; or $\mathbf{k}\cdot\mathbf{r}<0$ and $\mathbf{k}'\cdot\mathbf{r}'<0$ at $t=t'=0$, otherwise the sign $(-)$ must be used. This is because according to the phase difference between two arbitrary wave surfaces is independent of time, the eq. (5) must be satisfied when $t=t'=0$. The distinction between the sign $(\pm)$ is that the two waves propagate in opposite directions. However, we usually do not know the above conditions. But still we may suppose that the sign $(+)$ is correct, i.e.,

$$\mathbf{k}\cdot\mathbf{r}-\omega t=\mathbf{k}'\cdot\mathbf{r}'-\omega' t'. \tag{6}$$

Even though eq. (6) has exactly the same form with the conventional formula of the invariance of phase of plane waves, it has the different physical meaning with the conventional one. After using Lorentz wave vector transformation, if $\omega'>0$ is obtained, then our hypothesis the sign $(+)$ is true, otherwise when $\omega'<0$, it is false. Therefore, we obtain the physics meaning of the negative frequency, that is, when a wave vector and a negative frequency are obtained from Lorentz wave vector transformation, the absolute value of the negative frequency represents the true frequency and its sign $(-)$ represents that the direction of the wave vector and its actual direction are in opposite direction. At the same time, because eq. (6) has strictly the same form with the conventional formula of the phase invariance , we conclude the four-vector $(\mathbf{k}, \omega/c)$ of plane waves is still Lorentz-covariant.

We will use the above analysis to explain the counter-example discussed in the letter [1]. In the rest frame $\mathbf{K}$ of the medium, a plane wave propagates in the direction of the positive x-axis with the wave vector $\mathbf{k}$ relative to $\mathbf{K}$. Thus, we have $\mathbf{k} = (k_x > 0, 0, 0)$, $\mathbf{u}^{\mathbf{p}} = \mathbf{u} = (u_x > 0, 0, 0)$ and $\omega = k_x u_x$. In the case of "superluminal" motion of the medium, that is, $u_x < V < c$. Where c is the velocity of light in a vacuum, V is the relative velocity between two inertial frames $\mathbf{K}$ and $\mathbf{K}'$. From Lorentz wave vector transformation, we obtain $\mathbf{k}' = [k'_x = \gamma k_x(1 - u_x V/c^2) > 0, 0, 0]$ and $\omega' = \gamma\omega(1 - V/u_x) < 0$, where $\gamma = (1 - V^2/c^2)^{-1/2}$. Based on our conclusion, we know that the actual frequency is $-\omega' = \gamma\omega(V/u_x - 1)$ and true wave vector is the $-\mathbf{k}'$, as seen from $\mathbf{K}'$. In other words, the plane wave propagates in the direction of the negative x'-axis with the wave vector $-\mathbf{k}'$ relative to $\mathbf{K}'$. In the special condition, based on the motion of the wave surface, we verify easily that the direction of the wave vector is pointing in the direction of negative x'-axis, as observed from $\mathbf{K}'$, because $u_x < V$.

The phase velocity and the ray velocity. – In eq. (1), because $\mathbf{u} = \mathbf{u}^{\mathbf{p}}$, $\omega = \mathbf{k} \cdot \mathbf{u}^{\mathbf{p}} = \mathbf{k} \cdot \mathbf{u}$. Thus, we may use the ray speed to reformulate eq. (1), i.e.,

$$\Psi(\mathbf{k} \cdot \mathbf{r}, t) = \mathbf{k} \cdot (\mathbf{r} - \mathbf{u}t) + \phi. \tag{7}$$

However, when the wave vibration in eq. (1) is observed from $\mathbf{K}'$, the direction of the velocity of energy (or ray) transport $\mathbf{u}'$ is generally not parallel to the direction of the phase velocity $\mathbf{u}^{\mathbf{p}\prime}$, because the medium is in motion with respect to the frame $\mathbf{K}'$ [3]. Nevertheless, according to the conclusion in the optics of crystals, the phase velocity $\mathbf{u}^{\mathbf{p}\prime}$ is the projection of the ray velocity $\mathbf{u}'$ onto the direction of the wave normal [4]. This conclusion is applicable in our case. To demonstrate the effect, a two-dimensional sketch of a plane wave traveling is shown in Figure 1. Because the medium is at rest with respect to $\mathbf{K}$, the ray speed $\mathbf{u}'$ is not parallel to the phase speed $\mathbf{u}^{\mathbf{p}\prime}$, as observed from $\mathbf{K}'$. The wave surface W_0 was passing the origins O' and O at time $t' = t = 0$. W_0 arrives at point P at other t', as seen from $\mathbf{K}'$. Thus, the direction of $O'P$ is the direction of energy transport and t' is the time of energy transport. But the direction of $O'A$ is the direction of the wave surface W_0 transport. Therefore, according to the geometry relations in Figure 1, we obtain easily that $\mathbf{u}^{\mathbf{p}\prime}$ is the projection of $\mathbf{u}'$ onto the direction of the $\mathbf{k}'$. In other words, we must obtain $\mathbf{k}' \cdot \mathbf{u}' \equiv \mathbf{k}' \cdot \mathbf{u}^{\mathbf{p}\prime} \equiv \omega'$ when $\mathbf{u}'$ is the ray velocity. Thus, we may also use the ray velocity reformulate the eq. (6), i.e.,

$$\mathbf{k} \cdot (\mathbf{r} - \mathbf{u}t) = \mathbf{k}' \cdot (\mathbf{r}' - \mathbf{u}'t'). \tag{8}$$

From eq. (8), we can calculate the ray velocity $\mathbf{u}'$ if $\mathbf{k}$ is given. However, this is inconsistent with Gjurchinovski's conclusion on the conundrum of the negative frequency of plane waves [2]. But, it is evident that his conclusions are not correct because it is based on a false assumption, namely, $\mathbf{k}' \cdot \mathbf{u}' < 0$.

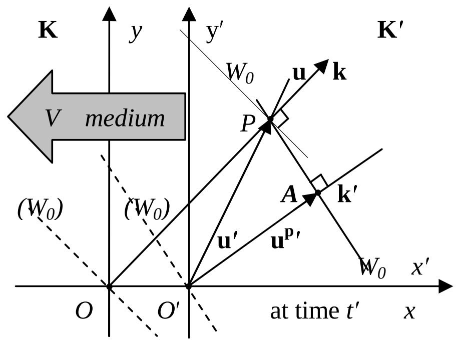

Fig. 1: The wave surface W_0 of a plane light waves propagates with the wave vector $\mathbf{k}$ relative to $\mathbf{K}$ from O to P and the medium is at rest with respect to $\mathbf{K}$, as seen from $\mathbf{K}$. W_0 propagates with the wave vector $\mathbf{k}'$ relative to $\mathbf{K}'$ from O' to P and the medium is in motion with respect to $\mathbf{K}'$, as seen from $\mathbf{K}'$. The ray speed $\mathbf{u}'$ is not parallel to the phase speed $\mathbf{u}^{\mathbf{p}\prime}$.

Conclusion. – Although we were possibly never able to prove the invariance of the phase of plane waves, we have proven that the phase difference between two wave surfaces among all inertial frames is the same. Based on the invariance of the phase difference, we proved that the phases of time and space (i.e. $\mathbf{k} \cdot \mathbf{r} - \omega t$ or $\omega t - \mathbf{k} \cdot \mathbf{r}$) among all frames are the same in spite of possible different initial phases. According to the invariance of the phase difference, the expressions of the phase difference are not necessary to use strictly the same form. In other words, when one of them is $\mathbf{k} \cdot \mathbf{r} - \omega t$, another may be $\omega' t' - \mathbf{k}' \cdot \mathbf{r}'$. If strictly the same form is employed, one might get the negative frequency ω' (or ω). However, it should be noted that when the negative frequency appears, its absolute value is the true frequency of plane waves and $-\mathbf{k}'$ (or $-\mathbf{k}$) is actual wave vector. On the basis of the invariance of the phase difference, we proved that the four-vector $(\mathbf{k}, \omega/c)$ of plane waves is still Lorentz-covariant. Based on eq. (8), we can calculate the velocity of energy (or ray) transport.

Finally, it should be stressed here that the eqs. (5) and (6) are applicable not only to relativistic plane waves transformation, but also to classical plane waves transformation. This is because our formula of the invariance of phase difference of plane waves has strictly the same form with the conventional formula of the invariance of phase of plane waves.

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