The Fractal Dimensions of Complex Networks *

GUO Long(郭龙)**, CAI Xu(蔡勖)

Complexity Science Center and Institute of Particle Physics, Huazhong Normal University, Wuhan 430079

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It is shown that many real complex networks share the distinctive features, such as the small-world effect and the heterogeneous property of connectivity of vertices, which are different from the random networks and the regular lattice. Although these feathers capture the important characteristic of complex networks, their applicability depends on the style of networks. To unravel the universal characteristic many complex networks have in common, we study the fractal dimensions of complex networks using the method introduced by Shanker. We find that the average 'density' $\langle\rho(r)\rangle$ of complex network follows a better power-law function as a function of distance r with the exponent d_f, which is defined as the fractal dimension, in some real complex networks. Furthermore, we study the relation between d_f and the shortcuts $N_{\rm add}$ in small-world network and the size N in regular lattices. Our present work provides a new perspective to understand the dependence of the fractal dimension d_f on the complex network structure.

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Recently, complex networks have been studied extensively in interdisciplinary fields including mathematic, statistical physics, computer science, sociology, economics, biology, etc. Complex networks are ubiquitous in the real world, e.g., there are technological networks such as the power grid,[1] biological networks such as the protein interaction networks,[2] and social networks such as scientific collaboration networks,[3,4] and human communication networks,[5] to name a few.

It has been shown that many real complex networks share distinctive characteristic properties that differ in many ways from the random and regular networks. One such property is the "small-world effect",[1] which means that the average shortest path length between vertices in network is short, usually scaling logarithmically with the size N of network, while maintaining high clustering coefficient. A famous example is the so-called "six degrees of separation" in social networks.[6] Another is the scale-free property that many networks possess. The probability distribution of the number of links per node, $P(k)$ (also known as the degree distribution) satisfies a power-law $P(k) \sim k^{-\gamma}$ with the degree exponent γ in the range of $2 < \gamma < 3$.[7] Although these properties capture the important characteristic of complex networks, their applicability depends on the style of networks. With the aim of providing a deeper understanding of the underlying mechanism of these common properties and unravelling the universal characteristics that many complex networks possess, many researchers have studied the self-similarity property and the dimension of complex networks. Song *et al.* discussed the mechanism that generates fractality, i.e., the repulsion between hubs, using the concept of renormalization.[8] In order to unfold the self-similar properties of complex networks, Song *et al.* calculated the fractal dimension using a 'box-counting' method and a 'cluster-growing' method, and found that the box-counting method is a powerful tool for further investigations of network properties.[9] The degree exponent γ can be related to a more fundamental length-scale invariant property, characterized by the box dimension d_B and the renormalized index d_k, as a function of $\gamma = 1 + d_B/d_k$.[9] Kim *et al.*[10,11] studied the skeleton and fractal scaling in complex networks using a new box-covering algorithm that is a modified version of the original algorithm introduced by Song *et al.* What is more, Kim *et al.* discussed the difference of fractality and self-similarity in scale-free networks, which has been helpful for us to understand the complex networks better.[12] Zhou *et al.* proposed an alternative algorithm, based on the edge-covering box counting, to explore self-similarity of complex cellular networks.[13] Furthermore, Lee and Jung studied the statistical self-similar properties of complex networks adopting the clustering coefficient as the probability measure and found that the probability distribution of the clustering coefficient is best characterized by the multifractal.[14] On the other hand, several algorithms have been proposed to calculate the fractal dimension of complex network, such as the box-covering algorithm[15] and the ball-covering approach.[16] Shanker defined the dimension of complex network in terms of the scaling property of the volume, which can be extended from regular lattices to complex networks.[17,18] Nevertheless, understand-

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**Email: longkuo0314@gmail.com;longkuo0314@hotmail.com

ing the self-similar properties of complex networks remains a challenge.

In order to unfold the universal scaling properties of complex networks, we study the fractal dimension of some real complex networks using the dimension measurement algorithm based on the scaling property of the volume in Refs. [17,18]. We find that there exists a universal scaling relation between the average density $\langle\rho(r)\rangle$ and the box linear size r with the exponent d_f. Furthermore, we study the fractal dimension d_f in small-world networks and in the regular lattices. We find that the dependence of the fractal dimension d_f on the average adding shortcuts $N_{\rm add} = Np$ in the NW small-world networks and the size N in the regular lattices.

Generally, we adopt the abstract space, which is different from one-dimensional linear space and two-dimensional flat space, to analyze the characteristics of complex networks, such as the structure of complex networks and the dynamics behavior of and on complex networks. In order to analyze the dimension property of complex networks, we define the distance d_{ij} between two vertices, say i and j, is the shortest path length from vertex i to vertex j. We set all the nodes as the seeds in turn and a cluster of nodes centrad at each seed within the box of the linear size r. Then, the average density $\langle\rho(r)\rangle$, defined as the ratio of the number of nodes in all the boxes with the size r and the complex network size N, is calculated as a function of r to obtain the following scaling:

$$\langle\rho(r)\rangle \simeq kr^{d_f}, \tag{1}$$

where d_f is defined as the fractal dimension of complex network and k is a geometric constant which depends on the complex network. The most important is that the definition of the fractal dimension reduces the fluctuation of the heterogeneous property of connectivity degree of vertices in complex networks, since all the nodes as the seeds in turn during covering complex network. The definition here is different from the box-covering algorithm, where the fractal dimension relation $N(l) \sim l^{-d_B}$ and $N(l)$ is the minimum number of boxes needed to tile a given network. However, to identify the minimum $N(l)$ value for any give l belongs to a family of NP-hard problems.[16]

Table 1. General characteristics of several real networks. For each network we have indicated the type (undirected network or directed network) of complex network, the number of nodes, the average degree $\langle k\rangle$, the average path length l, the clustering coefficient C and the degree distribution $P(k)$. Here empty shows that there is no obvious degree distribution since the size is too small. The various types of networks datasets are obtained from the Pajek datasets (http://vlado.fmf.uni-lj.si/pub/networks/data/).

Network	Type	Size	$\langle k\rangle$	l	C	$P(k)$
Power grid	undirected	4941	2.67	18.7	0.08	$e^{-0.59k}$
C.Elegans	directed	306	7.66	3.97	0.147	
Yeast	directed	2361	2.82	4.62	0.04	$k^{-2.11}$
CNCG	undirected	7343	1.62	3.92	0.103	$k^{-2.17}$
E-mail	directed	1133	9.62	3.606	0.166	$e^{-0.11k}$

We apply the definition of the fractal dimension above mentioned to some real complex networks, e.g., the chemical biology networks such as the protein-protein interaction network (PIN) in budding yeast,[19] the neural network of the nematode worm C.elegans,[1] the social networks such as the email network of University at Rovira i Virgili (URV)[5] and the collaboration network in computational geometry (CNCG), the technological network such as the electrical power grid of the western United States.[1] All those real complex networks are of scientific interest. The PIN in budding yeast plays a key role in predicting the function of uncharacteristic proteins based on the classification of known proteins within topological structures. The C.elegans is an important example of a completely mapped neural networks. The graph of the email network at URV and the graph of CNCG are the surrogates for social networks where the agents interact with others by the means of collaboration and information transition. The graph of the power grid is related to the efficiency and robustness of power networks.[1] Table 1 shows that those real complex networks are sparse ones with the small-world effect and the heterogeneous property of connectivity degree of vertices.

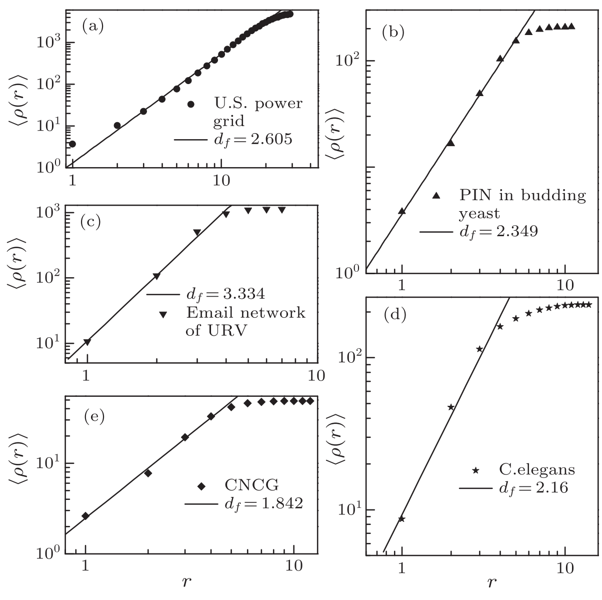

Fig. 1. (color online) The fractal dimensions in some real complex networks. (a) The U.S. power grid with $d_f = 2.286$. (b) The PIN in budding yeast with $d_f = 2.349$. (c) The email network of URV with $d_f = 3.334$. (d) The neural network of C. elegans with $d_f = 2.16$. (e) The CNCG with $d_f = 1.842$. The red solid lines represent the power-law fit for those real complex networks.

Figure 1 displays the evolution of $\langle\rho(r)\rangle$ as a function of r for various real complex networks. We find that $\langle\rho(r)\rangle$ evolves as a scaling function of r with the exponent d_f in all those complex networks. Interestingly, the scaling function is independent of the style of complex networks, which may show the universal scaling property in complex networks. However, the fractal dimensions d_f values are different in those real complex networks, such as the U.S. power grid with $d_f = 2.286$, The PIN in budding yeast with $d_f = 2.349$, the email network of URV with $d_f = 3.334$, the neural network of C. elegans with $d_f = 2.16$ and the CNCG with $d_f = 1.842$, see Fig. 1. The fractal dimension d_f maybe is related to the complex network structure, such as the shortcuts and the size N. Here, the d_f value is different from the d_B value obtained from the box-covering algorithm[12] and the d_{ball} value from the ball-covering approach,[16] because of the different physical quantities in those fractal definition. The average density $\langle\rho(r)\rangle$ of the vertices in the boxes with size r is an exact solution in our present work, and the minimum number $N(l)$ of boxes needed to tile a give network is an approximate solution in the box-covering algorithm. For example, in the C.elegans, $d_f = 2.16$ is smaller than $d_B = 3.5$ and $d_{\rm ball} = 3.7$,[16] respectively.

Further light can be shed on the dependence of the fractal dimension d_f on the complex network structure, such as the shortcuts in small-world network and the complex network size. In order to do this, we study the dimensions of the small-world network and the regular lattice with open boundary condition using the finite-size effect method.

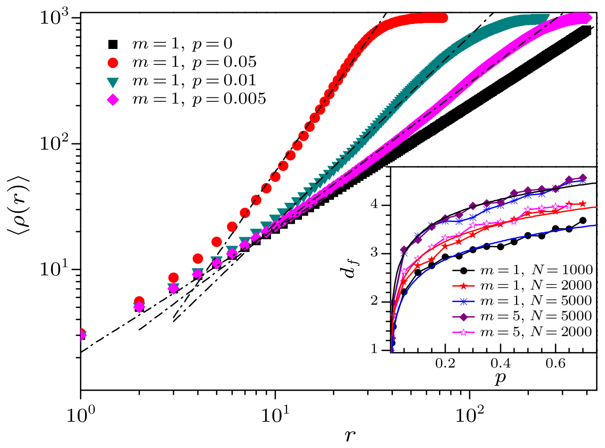

Fig. 2. (color online) The evolution of the density $\langle\rho(r)\rangle$ as a function of the boxes linear size r in NW small-world network with various shortcuts density p. The dot-dashed lines are the fit lines related to various p, respectively. The size of the network is $N = 1000$. Inset: the fractal dimension as a function of p in NW small-world network. The curves satisfy the function of $d_f = 1.25\log(1 + Np)$ for $p > 0$, where $N_{add} = Np$ is the average number of shortcuts in the NW small-world network.

Here the small-world network is built as the algorithm of the Newman-Watts (NW) small-world network.[21] The NW small-world network is defined on a lattice consisting of N nodes arranged in a ring. Initially each node is connected to all of its neighbors up to some fixed range m to make the network with average coordination number $z = 2m$. Randomness is then introduced by taking each node in turn and, with probability p, adding an edge to a randomly chosen node, so that there are again (Np) shortcuts average. For convenience, we call m the first neighbor parameter (FNP) and p the shortcuts density. Tuning m and p, we can obtain a series of complex networks with different structural properties. This model is equivalent to the Watts-Strogatz model[1] for small p, whilst being better behaved when p becomes comparable to 1.[21]

In Fig. 2, we represent the evolution of the average density $\langle\rho(r)\rangle$ as a function of the box size r with the same FNP $m = 1$ and different shortcuts density p. We find that the relation between $\langle\rho(r)\rangle$ and r satisfies the scaling function as Eq. (1) with the fractal dimension d_f better. Furthermore, we find that $d_f = 0.998 \simeq 1$ for $p = 0$ and $d_f > 1$ for $p > 0$. Here d_f increases as the shortcut density p increases. Namely, the larger the shortcuts density p is, the larger the fractal dimension d_f of NW small-world network is. Hence, the dimension d_f can reflect the disorder degree of complex systems. On the other hand, we study the evolution of $\langle\rho(r)\rangle$ as a function of p using the finite-size effect, see the inset of Fig. 2. We find that the fractal dimension d_f, which is independent of the FNP m, increases as the size N and the shortcut density p of NW small-world network increases. We fit the evolution of d_f as a function of the shortcuts density p and the network size N for $p > 0$ using the nonlinear fitting method, and find that $d_f(N, p)$ satisfies the relation

$$d_f(N, p) = 1.25\log(1 + Np), \tag{2}$$

where $N_{\rm add} = Np$ is the average number of shortcuts in the NW small-world network.

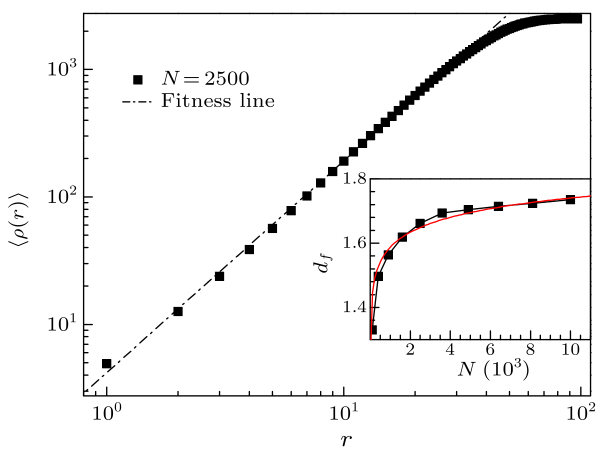

Fig. 3. (color online) The evolution of the density $\langle\rho(r)\rangle$ as a function of the boxes linear size r in the regular lattice with the size $N = 2500$. The dash dot line is the fitness line with the slope $\gamma = 1.649$. Inset: the fractal dimension as a function of the size N in regular network. The red curve satisfies the function of $d_f = 2 - exp(-N^{0.183}/4)$.

What is more, we also study the dimension of the regular lattice with open boundary condition using the above mentioned method. In Fig. 3, we represent the evolution of $\langle\rho(r)\rangle$ as a function of r with the size $N = 2500$. We find that the relation between $\langle\rho(r)\rangle$ and r satisfies the strictly scaling function as Eq. (1) with the exponent $d_f = 1.649$. Surprisingly, the dimension calculated by the above mentioned is not equal to the integer 2. We analyze the dependence of d_f on the size N of the regular lattice using the finite-size effect, since the regular lattice with finite size is embedded in the flat space. We find that the fractal dimension d_f increases as the size N increases, see the inset of Fig. 3. Interestingly, we also fit the function of $d_f(N)$ using the nonlinear fitting method, and find that $d_f(N)$ satisfies the relation

$$d_f(N) = 2 - \exp(-N^{0.183}/4), \qquad (3)$$

where 4 is the connectivity degree that most vertices are in the regular lattice. From Eq. (3), we find that $d_f \to 2$ for $N \to \infty$. Combining $d_f \simeq 1$ for $p = 0$ in the NW small-world network and $d_f \to 2$ for $N \to \infty$ in the regular lattice, we find that the definition of the fractal dimension here can be applied to the regular lattices. Hence, the finite size plays a crucial role in the complex network structure and the dynamics of and on complex networks.[22,23]

In summary, we have studied the fractal dimensions of complex networks using the method introduced by Shanker. We find that the evolution of the average density $\langle\rho(r)\rangle$ as a scaling function of the boxes linear size r in some real complex networks. The scaling property is independent of the style of complex networks and universal, since the calculation of the $\langle\rho(r)\rangle$ is averaged over all the vertices in complex networks in the definition of the fractal dimension. The average density reduces the fluctuation in complex networks. Furthermore, we study the dependence of d_f on the shortcuts (including the size N and the shortcuts density p) in small-world networks and the size N in the regular lattices. Our present work shows the important role of complex network structure in the fractal dimension d_f and provides a new perspective to understand the fractal dimension of complex networks.

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