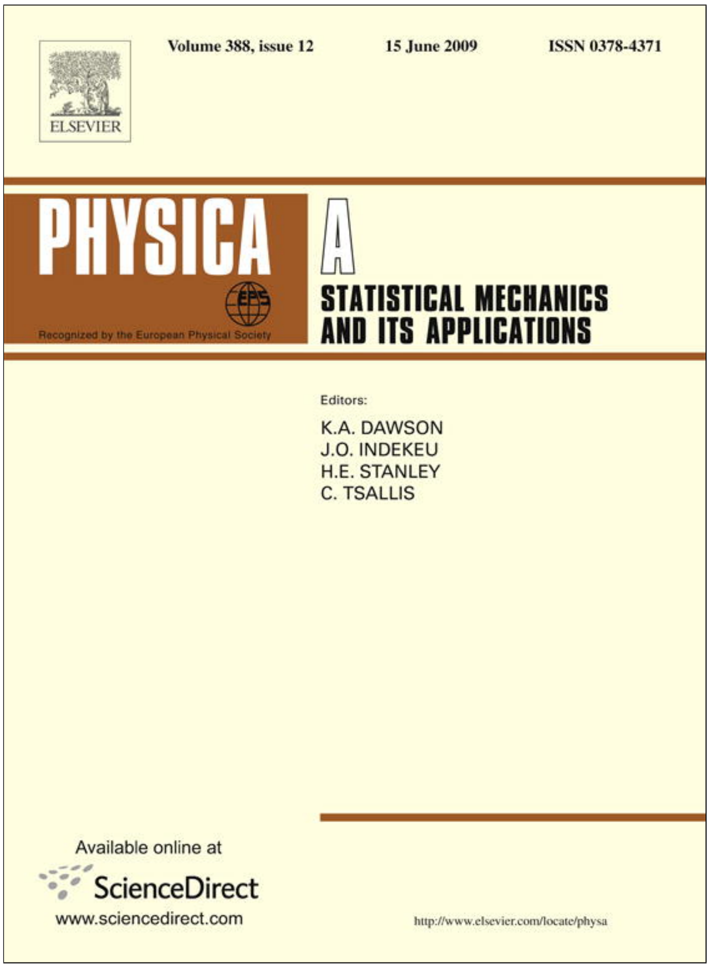
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Bose–Einstein condensation of bouncing balls

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ABSTRACT

Microscopic bouncing balls, i.e., particles confined within a positive one-half-dimensional gravitational potential, display Bose–Einstein condensation (BEC) not only in the thermodynamic limit but also in the case of a finite number of particles, and the critical temperature with a finite number of particles is higher than that in the thermodynamic limit. This system is different from the one-dimensional harmonic potential one, for which the standard result indicates that the BEC is not possible unless the number of particles is finite.

1. Introduction

Along with creating the Bose–Einstein statistics for the ideal boson gas, Einstein in 1925 showed that from a certain temperature on, the molecules condense without attractive forces [1], and it was named later as Bose–Einstein condensation (BEC). Seventy years after the prediction of Einstein, Bose–Einstein condensates formed by atomic gases confined in harmonic magnetic traps were observed at very low temperatures [2]. Inspired by either the theoretical modeling or the real experimental setup, theoreticians search in wider domains for the relation between BEC and the form of the potentials trapping the bosons. The well-studied situation is closely related to that of three-dimensional harmonic magnetic traps [3–5], where the mark of the transition temperature with a finite number of particles appears lower than that in the thermodynamic limit [6], and lowering the dimension increases the transition temperature, and therefore is favorable for BEC [3].

It is well-known that without any external field, the free bosons confined in a box in dimensions fewer than 3 will not condense. Starting from the one-dimensional gas of particles confined by a power-law potential $U(x) \sim |x|^{\eta}$, studies show that in the thermodynamic limit it will display BEC only if the potential power η satisfies $0 \prec \eta < 2$ [7]. In other words, the standard treatment indicates that the BEC is not possible for a one-dimensional harmonic potential $\eta = 2$, even it can be defined with a finite number of particles [3,7].

Once physical systems are studied on the Earth, the influence of the gravitational field cannot be overlooked. However, even with the lack of quantitative results, a qualitative analysis was made forty-one years ago [8], showing that in the presence of the gravitational field, the inhomogeneous boson gas can condense in one and two dimensions. It is therefore understandable that the gravitational field added into the independent free bosons in a three-dimensional box can quantitatively affect the critical temperature of BEC [9]. In the present note, we give quantitative results for BEC in a

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one-dimensional half-space decorated with a gravitational field, with emphasis on possible effects resulting from the finite number of particles.

In Section 2, a semiclassical treatment of the one-dimensional noninteracting bosons in the presence of a gravitational field is given, which is applicable in the thermodynamic limit. In Section 3, the effect of the finite number of particles is discussed. This article is closed with a brief conclusion in Section 4.

2. A semiclassical treatment of one-dimensional BEC in a gravitational field

In Bose–Einstein statistics, the average number of particles in an energy eigenstates E_n is given by the Bose–Einstein distribution,

$$N_n = \frac{1}{\exp((E_n - \mu)/k_B T) - 1}, \tag{1}$$

where μ is the chemical potential and k_B is Boltzmann's constant. The chemical potential μ is determined by the constraint that the total number of particles in the system is N,

$$N = \sum_{n=0}^{\infty} N_n. \tag{2}$$

The particles of mass M are confined by a gravitational potential that is given by

$$U(x) = \begin{cases} Mgx, & (x > 0) \\ \infty, & (x \leq 0). \end{cases} \tag{3}$$

So, the particle is actually a classical bouncing ball that moves in a positive half-space in one dimension. For the case where the temperature $k_B T$ is much higher than that for any two adjoining quantum levels, i.e., $k_B T \gg \varepsilon_2 - \varepsilon_1$, the system can be well described as a continuum of energy levels plus a separate ground state. The density of states is then given by

$$\rho(\varepsilon) = \frac{1}{h} \int \mathrm{d}x\mathrm{d}p\,\delta(H - \varepsilon), \tag{4}$$

where $H = p^2/(2M) + U(x)$ is the Hamiltonian for the particle. Then Eq. (4) becomes

$$\rho(\varepsilon) = \frac{\sqrt{2M}}{h} \int_0^{l(\varepsilon)} \frac{\mathrm{d}x}{\sqrt{\varepsilon - U(x)}} = \frac{2}{hg}\sqrt{\frac{2}{M}}\sqrt{\varepsilon}, \tag{5}$$

where $l(\varepsilon) = \varepsilon/(Mg)$ is the maximum height for classical particles with energy ε. The total number of particles satisfies, from Eq. (2),

$$N = N_0 + \int_0^{\infty} \frac{\rho(\varepsilon)\mathrm{d}\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1}, \tag{6}$$

where N_0 is the number of particles in the ground state $\varepsilon = 0$. For a given temperature, the maximum number of particles accommodated in excited states is reached when $\mu = 0$. Then the critical temperature T_C can be determined using the following equation:

$$N = \int_0^{\infty} \frac{\rho(\varepsilon)\mathrm{d}\varepsilon}{\exp(\varepsilon/k_B T_C) - 1}, \tag{7}$$

which can be rewritten as

$$T_C = \left(\frac{MghN}{2.612 k_B^{3/2}\sqrt{2\pi M}}\right)^{2/3} = 1.226 N^{2/3} \left(\frac{Mg^2\hbar^2}{2}\right)^{1/3} \frac{1}{k_B}. \tag{8}$$

To see quantitatively at what number N the molecules start to condense, we obtain some numerical results. For an air molecule of average mass $M = 4.82 * 10^{-26}$ kg, with the requirement that $T_C = 300$ K, BEC starts once $N \geq 1.28 * 10^{15}$, while for a hydrogen gas molecule of mass $M = 3.35 * 10^{-27}$kg, $N \geq 4.84 * 10^{15}$. At the extreme, $N(T_C) = 1$ gives from Eq. (7) $T_C = 25.5$ nK and 10.5 nK for air and hydrogen gas respectively, and no molecule can be excited as long as $T \prec T_C$. These numerical results show that the continuum treatment is applicable once the critical temperature is much higher than 10 nK, the temperature corresponding to the energy difference between the ground state and the first excited state. In other words, at temperature lower than 10 nK, a few particles are sufficient for condensation. So, the finite number effects can never be overlooked at low temperature. Moreover, we will see in the next section an interesting behaviour: the critical temperature is higher than that in the thermodynamic limit.

By combining Eqs. (6) and (8), the ground-state population fraction for $T < T_C$ can be obtained. The result is

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^{3/2}. \tag{9}$$

The total energy of the system is given by

$$E(T) = \int_0^\infty \frac{\varepsilon\rho(\varepsilon)\mathrm{d}\varepsilon}{\exp((\varepsilon-\mu)/k_BT)-1} \tag{10}$$

$$= \begin{cases} \frac{3}{2}Nk_BT\frac{g_{5/2}(z)}{g_{3/2}(z)}, & (T > T_C) \\ \frac{3}{2}Nk_BT\frac{\zeta(5/2)}{\zeta(3/2)}\left(\frac{T}{T_C}\right)^{3/2}, & (T \le T_C), \end{cases} \tag{11}$$

where $z = \exp(\mu/(k_BT))$ is the fugacity, and $g_n(z) = \sum_{j=1}^{\infty}(z^j/j^n)$ is the Bose functions and $g_n(1) = \zeta(n)$.

From Eq. (11), taking $\mu = 0$ $(T \le T_C)$, the heat capacity $C(T) = \partial E(T)/\partial T$ is

$$\frac{C(T)}{Nk_B} = \begin{cases} \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}, & (T > T_C) \\ \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)}\left(\frac{T}{T_C}\right)^{3/2}, & (T \le T_C), \end{cases} \tag{12}$$

which is continuous at critical temperature T_C. Moreover, one can easily find that $\partial C(T)/\partial T$ is discontinuous at T_C, so this is a third-order phase transition.

3. Finite number effects near nanokelvins

For the bouncing ball in the gravitational potential (3), quantum mechanics gives the stationary states $\psi_n(x)$, [10] $(n = 1, 2, 3, \ldots)$

$$\psi_n = \begin{cases} C_nA(-\gamma_n + x/l), & (x > 0) \\ 0, & (x = 0), \end{cases} \tag{13}$$

where $A(x)$ is the Airy function with zeros $-\gamma_n$, and $C_n = (2M^2g/\hbar^2)^{1/6}\mathrm{d}A(x)/\mathrm{d}x\,|_{x\to-\gamma_n}$ are the normalization constants, $l = (\hbar^2/2M^2g)^{1/3}$ is the characteristic length, and $n = 1, 2, \ldots$ are the quantum numbers. The corresponding energy eigenvalues E_n are

$$E_n = \left(\frac{Mg^2\hbar^2}{2}\right)^{1/3}\gamma_n. \tag{14}$$

The mean value of the potential of particle in a stationary state $\psi_n(x)$ is simply

$$U_n = \langle\psi_n|Mgx|\psi_n\rangle = \frac{2}{3}\gamma_nMgl. \tag{15}$$

For a finite number N of particles, the total number of particles is given by

$$N = N_0 + \sum_{n=2}^{\infty}\frac{1}{\exp((E_n-\mu)/k_BT)-1}, \tag{16}$$

where N_0 is the number in the ground state,

$$N_0 = \frac{1}{\exp((E_1-\mu)/k_BT)-1}. \tag{17}$$

With the introduction of a dimensionless temperature t via $T = t(Mg^2\hbar^2/2)^{1/3}/k_B$, Eq. (16) becomes

$$N = \frac{1}{\exp((\gamma_1-\mu)/t)-1} + \sum_{n=2}^{\infty}\frac{1}{\exp(-\mu/t)\exp(\gamma_n/t)-1}. \tag{18}$$

Similarly, the total energy of the system is

$$E = N_0E_1 + \sum_{n=2}^{\infty}\frac{E_n}{\exp(-\mu/t)\exp(\gamma_n/t)-1}. \tag{19}$$

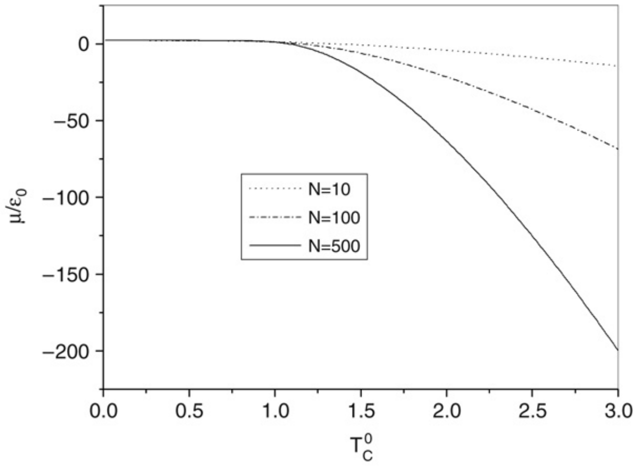

Fig. 1. For different numbers N of particles, the chemical potential μ changes with temperature T. When the temperature approaches the critical one T_C^0 (8) and below it, we have $\mu \approx E_1$. In this figure, $E_1 = \varepsilon_0\gamma_1$ where $\varepsilon_0 \equiv (Mg^2\hbar^2/2)^{1/3}$ has the dimension of energy, so μ/ε_0 is a dimensionless quantity.

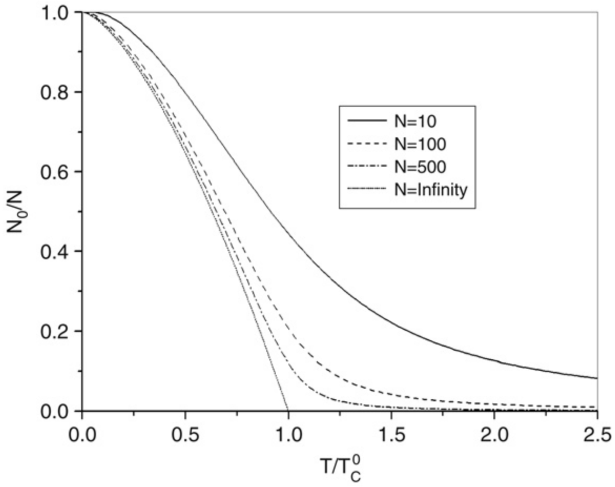

Fig. 2. Ratio N_0/N versus T/T_C^0 for different values of N. Even with a finite number of particles, there is clearly an abrupt increase of population in the ground state at temperature higher than T_C^0, the critical temperature given by the thermodynamical limit.

The total potential energy of the system is

$$U = \sum_{n=1}^{\infty} \frac{U_n}{\exp((E_n - \mu)/k_B T) - 1} = \frac{2}{3} Mgl \sum_{n=1}^{\infty} \frac{\gamma_n}{\exp(-\mu/t)\exp(\gamma_n/t) - 1}. \tag{20}$$

The mean height of these N particles is

$$h = \frac{U}{NMg} = \frac{2l}{3N} \sum_{n=1}^{\infty} \frac{\gamma_n}{\exp(-\mu/t)\exp(\gamma_n/t) - 1}. \tag{21}$$

The dimensionless heat capacity per particle is

$$\frac{C(T)}{Nk_B} = \frac{1}{N} \sum_{n=2}^{\infty} \frac{(\gamma_n^2 + \gamma_n t\mathrm{d}\mu/\mathrm{d}t - \gamma_n\mu)\exp(\gamma_n/t)\exp(-\mu/t)}{t^2(\exp(\gamma_n/t)\exp(-\mu/t) - 1)^2}. \tag{22}$$

These quantities cannot be expressed in closed form. Numerical calculations are needed and the details are presented in Figs. 1–4.

First of all, for a given N we need the value of the chemical potential μ determined by the constraint equation (18), and it is a function of temperature $\mu = \mu_N(T)$. The results with different N are plotted in Fig. 1. It clearly shows that when temperature reaches the critical value T_C^0 (8) and below it, we have $\mu \approx E_1$. Once the relation $\mu = \mu_N(T)$ is known, the ground-state population fraction, the heat capacity and the mean height of particles are all determined. The ratios N_0/N versus T/T_C^0 for different values of N are plotted in Fig. 2. Curves in Fig. 2 show that the ratio N_0/N changes gradually, exhibiting a common feature of BEC with a finite number of particles [3,6–9]. Fig. 3 depicts the dimensionless mean height

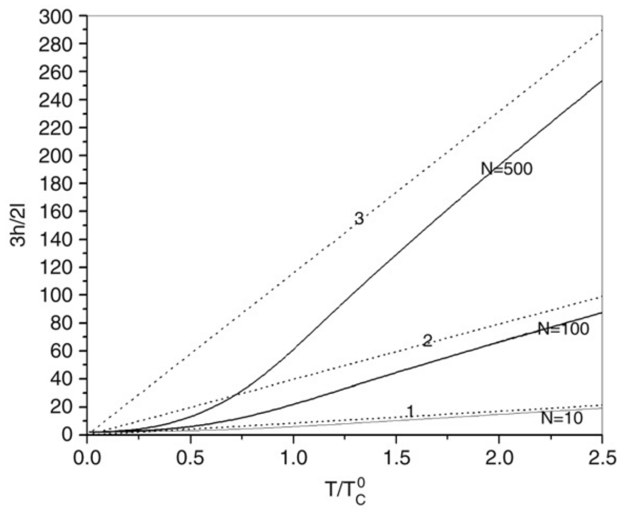

Fig. 3. The mean height of particles $3h/2l$ versus T/T_C^0 for different values of N. The dashed lines 1, 2 and 3 correspond to the Boltzmann statistics for $N = 10$, 100 and 500. In the Boltzmann statistics, the total potential energy of the system is $U = NkT$, so the mean height $h = 1.226N^{2/3}(T/T_C^0)l$. The mean height of particles gradually approaches its classical value from below.

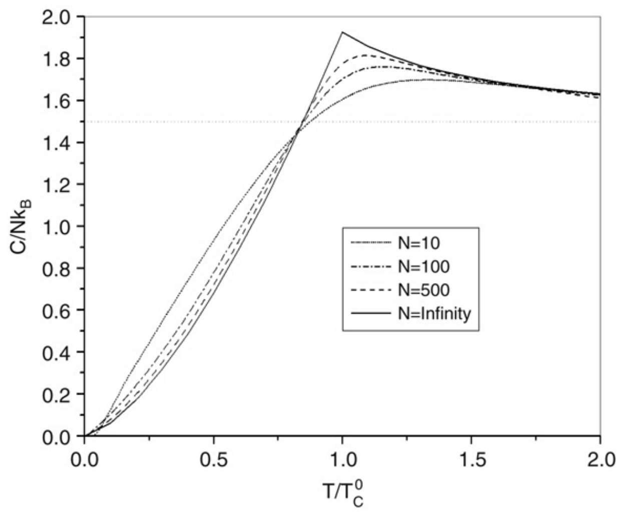

Fig. 4. Dimensionless heat capacity per particle C/Nk_B versus T/T_C^0 for different numbers of particles N. We see clearly a maximum of C occurring at a temperature that is higher than T_C^0, the phase transition temperature given by the thermodynamical limit.

of particles $3h/2l$ versus temperature T/T_C^0 for various values of N. From the figure, we see that when $T \lessapprox T_C$, the mean height greatly decreases and the particles "condense to the floor". The change of C/Nk_B as a function of T/T_C^0 for different values of N is plotted in Fig. 4. Note that the phase transition defined by the discontinuity of the thermodynamic quantity occurs in the thermodynamic limit only. As long as an ensemble is composed of a finite number N of ideal bosons, the transition temperature can be quantitatively marked by the maximum point of the heat capacity at $T = T_C$ that is mathematically defined as the maximum of the curve of $C(T)$ for finite N, i.e., $(\partial C(T)/\partial T)_{T=T_C} = 0$. The critical temperature T_C for finite N is higher than that in the continuum limit and we have explicitly $T_C/T_C^0 = 1.335$, 1.155 and 1.095 for $N = 10$, 100 and 500 respectively. Fig. 4 shows also that as $T \to \infty$, the heat capacity approaches $C \to 1.5Nk$, as it must.

4. Conclusion

We have discussed BEC of noninteracting bosons in the presence of a one-half-dimensional gravitational field. The semiclassical treatment is applicable in the thermodynamic limit, which shows that the larger the number of particles, the higher the phase transition temperature. However, with a finite number of particles, the critical temperature is higher than that in the thermodynamic limit. We take the one-dimensional harmonic potential for a comparison. When the number of particles is finite, the BEC for the potential can also be defined from the fraction of ground-state bosons, but the standard result indicates that BEC is not possible in the thermodynamic limit.

Acknowledgments

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