## Response to 'Comment on 'Conjectures on exact solution of three-dimensional (3D) simple orthorhombic Ising lattices ' ' by Perk

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The error of eq. (15b) in my article [Z.D. Zhang, Phil. Mag. **87**, 5309 (2007) and also see arXiv: 0705.1045] in the application of the Jordan-Wigner transformation does not affect the validity of the putative exact solution, since the solution is not derived directly from it. Other objections of Perk's Comment [J.H.H. Perk, Phil. Mag. **88**, (2008) in press, also see arXiv:0811.1802v2] are the same as those in Wu et al.'s Comments [F.Y. Wu et al., Phil. Mag. **88**, (2008) 3093; 3103], which do not stand on solid ground and have been rejected in my previous Response [Z.D. Zhang, Phil. Mag. **88**, (2008) 3097]. The conjectured solution can be utilized to understand critical phenomena in various systems, while the conjectures are open to prove rigorously.

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This is a Response to Perk's Comment [1] on the conjectured solution of the three-dimensional (3D) Ising model [2]. At first, I would like to appreciate Dr. Perk for pointing out the error of eq. (15b) in [2] in the application of the Jordan-Wigner transformation, which should be corrected as eq. (3) of [1].<sup>1</sup> However, although this error is the same as Maddox's [4], the essential difference is that my putative solution in [2] is obtained by introducing two conjectures dealing with the topologic problem in the 3D Ising model, not derived directly from the error. Thus it does not affect the validity of the putative exact solution. When I wrote [2], I thought that the topologic troubles are due to the U factors in eq. (15). According to Lou and Wu [3] and discussion with Perk, the appearance of the high-order terms in eq. (3) of [1] and the corresponding exponential factors of the transfer matrix (not the U factors in eq. (15) of [2]) is the root of difficulties of the 3D Ising model. The U factors come from the periodic boundary conditions, but disappear for open boundary condition. It is clear now that there is no need to get rid of the U factors at the boundary by some topologic trick, but the high-order 'internal' factors might need. Since there is an 'internal' factor for each j (j runs from 1 to *nl* in [1], corresponding to (r, s) running from (1, 1) to (n, l) in [2]), the number of the 'internal' factors is in the order of ln more than that of the U factors. These 'internal' factors raise more difficulties since they do not commute with the rest in the transfer matrix (e.g., the product of the factors like  $e^{\frac{1}{2}\theta\Gamma\Gamma}$  [3]) and one does not have representations of the rotation group so that one

<sup>&</sup>lt;sup>1</sup> Lou and Wu [3] also apply correctly an equivalent equation. An error in eq. (16) of [2] should be corrected as eq. (1) of [1]. A factor of 2 should be added in all the exponentials on the right hand of eq. (28) and also the **Conjecture 2** on page 5321 of [2] (e.g.  $e^{i\frac{2\pi}{n}}, \dots, e^{i\frac{2t_x\pi}{n}}$ , etc.).

cannot continue as Kaufman did [5]. Although the situation becomes more complicating, we can still make the same conjectures, with the motivation for the conjecture only slightly different with what I had in [2]. Suppose that we are given a 3D manifold bounding a 4D manifold [6,7]. It might be possible to attach an "internal" space on every point of the 3D lattice to provide with some operators to allow these "internal" factors to commute with the transfer matrix. In this sense, we add an extra dimension with an additional rotation as a kind of boundary condition as what I conjectured in my original paper [2] and then we might solve simultaneously the topologic problem in eq. (15) as a whole regarding to its non-local behavior, no matter how complicating it is.

Istrail showed that the essential ingredient in the NP-completeness of the Ising model is nonplanarity [8], which indicates also that the origin of difficulties is topologic. As discussed on page 5393 of [2], the NP-completeness only prevents algorithms from solving all instances of the problem in polynomial time [9,10]. Such NP-completeness from the point view of computer sciences cannot be fully used to judge the advances in mathematics which are needed to uncover the exact solution. Furthermore, as Istrail and Cipra claimed [8-10], there exists the possibility for exact answers in the ferromagnetic 3D Ising model dealt with in [2]. As discussed above, the main difficulties caused by these high-order terms are topologic [1,8-10]. Therefore, the conjectures of introducing the fourth dimension [2], which serve for dealing with the topologic problem in the 3D Ising model, are still meaningful, and open to be proved rigorously (however, with the new form of eq. (15b) for the matrix

V [2], thanks to Dr. Perk).

Other objections in [1], concentrating on the low- and high-temperature expansions and the different choices of the weight functions, are all the same as those in recent Comments by Wu et al. [11,12], which have been rejected in my previous Response [13]. The only exception is that literatures referred in [1] for rigorously proving the convergence of the high-temperature series [14-18] are different with those [19-22] in [11]. As remarked in [1], the proof of [14-18] is based on the proof of Gallavotti and Miracle-Solé [14]. However, just below eq. (5) of [14], the authors put for convenience  $\beta = 1/(k_B T) = 1$ , which is inconsistent with  $\beta = 0$  for infinite temperature. Some important conditions for Theorems in [14] are not valid for  $\beta = 0$ . For instance, the condition for ii), iii) and iv) of Theorem 1, Theorems 2 and 3, will be invalid if  $\beta = 0$  is put into eq. (24) of [14]. One may argue that infinite temperature just requires that all interaction energies equal to zero. But in this case, this condition is invalid still and, moreover, one should face a change of all the interaction energies from zero to non-zero at/near  $\beta = 0$ . Such change results in an intrinsic change of the geometrical (topologic) structure in the 3D Ising interaction system as revealed in [13]. As has been already pointed out in [13], Lebowitz and Penrose [15] and Griffiths [17] distinguished  $\beta > 0$  and  $\beta = 0$ , and started with the condition  $\beta > 0$  to prove their theorems. The basic difficulty of these well-known theorems originates from a fact that a phase transition may occur at  $\beta = 0$  according to the Yang-Lee theorems [23,24].

Everyone has been brought in a situation in which it is impossible to satisfy the

opposite wishes: being convergent as an exact solution is, while it must agree exactly with a divergent series. As remarked in [1], the low-temperature series of my putative exact solution has a finite radius of convergence up to its critical point. So it is normal that it does not reproduce term by term the well-known low-temperature series that is divergent. The lack of information of the global behaviours of the 3D Ising system is the root of such divergence in the well-known low-temperature series. The troubles in it may originate from some difficulties in the foundation of statistical mechanics [25-29].

In [13], I indicated the necessity of introducing a (3+1)–dimensional framework for dealing with the 3D Ising model and discussed briefly the physics beyond the extra dimension. According to [12],<sup>2</sup> it is very significant to inspect further the mathematical basis of statistical mechanics, i.e., *ergodic hypothesis* and *mixing hypothesis* [25-31]. The *ergodic hypothesis* has been proved to be one of the most difficult problems and a proof of the *ergodic hypothesis* under fairly general conditions has been lacking [25-31]. For the *mixing hypothesis* that is stronger than *the ergodic hypothesis*, talking about a distribution of points on the surface  $\Gamma$  (E), one is no longer discussing a single system, and mixing is irrelevant for a truly isolated system [25]. In statistical mechanics, one simply assumes that the time average can be replaced by the ensemble average [25-31]. Actually, most systems studied in statistical mechanics are not ergodic [29-31]. It is my understanding that the lack of ergodicity of the 3D Ising model would lead to that the time average could be

<sup>&</sup>lt;sup>2</sup> I take this opportunity to reply briefly the last sentence in Wu et al.'s Rejoinder [12] to my Response [13].

different from the ensemble average that may not contain complete information of the system. Neglecting the difference between the two averages may work well in other models with dimensions  $D \neq 3$ , but cause serious troubles in the 3D Ising system because of its global topologic behaviour and geometrical structure [2,13]. Because the well-known low- and high-temperature series of the 3D Ising model might not account properly the time average of the system, they might be invalid at finite temperatures. In my view, it is unjustified to reply upon successes of statistical mechanics to dismiss questions regarding its foundation.

In summary, the error in [2] should be corrected as Perk suggested in [1], but it does not affect the validity of the putative exact solution that is not derived directly from the error. All these well-known theorems in [14-22] are proved only for  $\beta > 0$ , not for infinite temperature and, other objections in [1] and also those in [11,12] do not stand on solid ground. The conjectured solution can be utilized to understand critical phenomena in various systems [32,33], while the conjectures are open to be rigorously proved.

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