

On principle of inertia in closed universe

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Abstract

If our universe is asymptotic to a de Sitter space, it should be closed with curvature in $O(\Lambda)$ in view of dS special relativity. Conversely, its evolution can fix on Beltrami systems of inertia in the ds -space without Einstein's 'argument in a circle'. Gravity should be local ds -invariant based on localization of the principle of inertia.

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1. Introduction

In classical physics, it is well known that for both Newton theory and Einstein's special relativity the principle of inertia (*PoI*) with Galilean symmetry and Poincaré symmetry, respectively, plays an extremely important role as the benchmark of physics for defining physical quantities and introducing physical laws. But, in Einstein's point of view, there is an 'argument in a circle' for the *PoI* as the benchmark.

Some eighty five years ago, Einstein claimed:

The weakness of the *PoI* lies in this, that it involves an argument in a circle: a mass moves without acceleration if it is sufficiently far from other bodies; we know that it is sufficiently far from other bodies only by the fact that it moves without acceleration. Are there at all any inertial systems for very extended portions of the space–time continuum, or, indeed, for the whole universe? We may look upon the principle of inertia as established, to a high degree of approximation, for the space of our planetary system, provided that we neglect the perturbations due to the sun and planets. Stated

more exactly, there are finite regions, where, with respect to a suitably chosen space of reference, material particles move freely without acceleration, and in which the laws of the special theory of relativity, . . . , hold with remarkable accuracy. Such regions we shall call "Galilean regions" [1].

In fact, to avoid this 'weakness' is one of the main motivations for Einstein from special relativity to general relativity based on his equivalence principle and general principle of relativity as an extension of the special principle of relativity.

In general relativity, however, what is realized for the general principle of relativity is the principle of general covariance. Although it is always possible to analyze physics in terms of arbitrary (differentiable) coordinate systems at classical level, 'the principle of covariance has no forcible content' [2]. For the equivalence principle, it requires that physical quantities and laws are in 'their familiar special-relativistic forms' in local Lorentz frames [2]. The symmetry for physical quantities and laws, however, is local $GL(4, R)$ or its subgroup $SO(1, 3)$ without local translation in general. Thus, in 'Galilean regions', Poincaré symmetry of *PoI* as the benchmark in special relativity is partially lost. These seem away from Einstein's original intention and lead to the benchmark of physics with gravity is not completely in consistency with that in special relativity without gravity.

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Recent observations show that our universe is accelerated expanding [3,4]. It is certainly not asymptotic to a Minkowski (*Mink*)-space, rather quite possibly asymptotic to a de Sitter (*dS*)-space with a tiny cosmological constant Λ . These present great challenges to the foundation of physics on the cosmic scale (see, e.g., [5]). In fact, it is the core of challenges: What are the benchmarks of physics on the cosmic scale? Are they consistent?

In view of the *dS*-invariant special relativity [6–14], however, there is a *PoI* of *dS*-invariance on *dS*-space with Beltrami systems of inertia (denoted *BdS*-space). Here we show that if the universe is asymptotic to a *dS*-space, it should be closed with a tiny curvature in the order of Λ , $O(\Lambda)$. Conversely, the evolution of the universe can fix on the Beltrami systems. Thus, the universe acts as the origin of the *PoI* of *dS*-invariance without Einstein's 'argument in a circle' so that the benchmark of physics on the cosmic scale should still be the *PoI* of *dS*-invariance. Then, we explain that the benchmark of physics with gravity should be the localization of the *PoI* of the *dS* special relativity. Thus, the *PoI* of *dS*-invariance and its localization should play the role of the consistent benchmarks of physics on the cosmic scale in the universe.

Actually, based on the principle of relativity [6,7] and the postulate on invariant universal constants, the speed of light c and the curvature radius R [8,9], the *dS* special relativity can be set up on the *BdS*-space. While Einstein's special relativity is the limiting case of $R \rightarrow \infty$.

In the *dS* special relativity, Beltrami coordinate systems [15] with Beltrami time simultaneity are very similar to *Mink*-systems in Einstein's special relativity. Namely, in the *BdS*-space geodesics are all *straight world lines* so that there is a *PoI* with a *law of inertia* for free particles and light signals. All these issues are transformed symmetrically under the fractional linear transformations with a common denominator (*FLT*s) of *dS*-group $SO(1, 4)$ in the Beltrami atlas chart by chart. It is significant that the Beltrami systems and their Robertson–Walker-like *dS*-counterpart with respect to proper-time simultaneity provide an important model. In this model, the *dS*-group as a maximum symmetry ensures that there are both the *PoI* and the cosmological principle on *dS*-space as two sides of a coin. On one side, there is the *BdS*-space with the *PoI*, while on the other there is a Robertson–Walker-like *dS*-space with the cosmological principle having an accelerated expanding closed 3d cosmos S^3 of curvature in the order of $O(R^{-2})$. Since the both can be transformed each other explicitly by changing the simultaneity just like flip a coin, the Robertson–Walker-like *dS*-space displays as an origin of the *PoI*, while the *PoI* provides a benchmark of physics on the *dS*-space.

If the universe is asymptotic to a *dS*-space with $R \simeq (3/\Lambda)^{1/2}$. In view of the *dS* special relativity, the universe should be asymptotic to the Robertson–Walker-like *dS*-space in the model so that it should be closed and the deviation from flatness is in the order of Λ , $O(\Lambda)$. This is an important prediction more or less consistent with recent data from WMAP [4] and can be further checked.

Conversely, the asymptotic behavior of the universe should naturally pick up a kind of the Robertson–Walker-like *dS*-

systems with such a 'cosmic' time τ that its axis coincides with the revolution time arrow of the real cosmic time τ_u in the universe. Since the 'cosmic' time τ in the Robertson–Walker-like *dS*-space is explicitly related to the Beltrami time x^0 , the universe should also fix on a kind of Beltrami systems with x^0 transformed from the 'cosmic' time τ . Therefore, via its evolution time arrow of τ_u picking up a 'cosmic' time τ on the Robertson–Walker-like *dS*-space, the universe should just act as an origin of such kind of Beltrami systems in which the *PoI* holds. Thus, there do exist the inertial systems in the universe and there is no Einstein's 'argument in a circle' for the *PoI*.

In general relativity, there is no special relativity in *dS*-space. In the *dS* special relativity, there is no gravity in *dS*-space. How to describe gravity?

In the light of Einstein's 'Galilean regions' [1], where his special relativity with full Poincaré symmetry should hold locally, the *PoI* should be localized. Therefore, in view of the *dS* special relativity, on spacetimes with gravity there should be local *dS*-frame anywhere and anytime so that the *PoI* of the *dS* special relativity should hold locally. If so, the localized *PoI* of the *dS* special relativity should be the benchmark of physics with gravity. This is in consistency with the role played by the *PoI* of the *dS* special relativity. We may further require that gravity have a gauge-like dynamics characterized by a dimensionless constant $g \simeq (\Lambda G \hbar / c^3)^{1/2} \sim 10^{-61}$ from the cosmological constant Λ and the Planck length. A simple model has implied this should be the case [23–25].

This Letter is arranged as follows. In Sections 2, we argue why there is a *PoI* on *dS*-space and very briefly introduce the *dS* special relativity. In Section 3, we introduce the relation between the *PoI* and the cosmological principle on *dS*-space as well as the cosmological meaning of *dS* special relativity. In Section 4, we explain why the universe can fix on the Beltrami systems of inertia without Einstein's 'argument in a circle'. In Section 5, we very briefly discuss that gravity should be based on localization of the *dS* special relativity with *PoI* and introduce the simple model for *dS*-gravity. Finally, we end with a few remarks.

2. On de Sitter special relativity

Is there special relativity with a *PoI* on *dS*-space?

Yes! Absolutely. This can be enlightened from two deferent but related angles [6–12].

Firstly, as is well known, weakening the Euclid fifth axiom leads to Riemann and Lobachevsky geometries on an almost equal footing with Euclid geometry. There is a physical analog via an inverse Wick rotation of 4d Euclid space, Riemann sphere and Lobachevsky hyperboloid $E^4/S^4/L^4$, respectively. Namely, there should be two other kinds of the *dS*/*AdS*-invariant special relativity on an almost equal footing with Einstein's special relativity [12]. In fact, there is a one-to-one correspondence between three kinds of geometries and their physical counterparts. We list the correspondence as follows:

Geometry	Space–time physics
$E^4/S^4/L^4$	$M^{1,3}/dS^{1,3}/AdS^{1,3}$
$ISO(4)/SO(5)/SO(1, 4)$	$ISO(1, 3)/SO(1, 4)/SO(2, 3)$
Descartes, Beltrami atlas	Minkowski, Beltrami atlas
Points	Events
Straight line	Straight world line
Principle of invariance	Principle of relativity
Klein’s Erlangen program	Theory of special relativity

Secondly, owing to Umov, Weyl and Fock [16], it can be proved that the most general form of the transformations among inertial coordinate systems

$$x^i = f^i(x^i), \quad x^0 = ct, \quad i = 0, \dots, 3, \quad (2.1)$$

which transform a uniform straight line motion, i.e. the inertial motion, in $F(x)$

$$\begin{aligned} x^a &= v^a(t - t_0) + x_0^a, \\ v^a &= \frac{dx^a}{dt} = \text{const}, \quad a = 1, 2, 3, \end{aligned} \quad (2.2)$$

to a motion of the same nature in $F'(x')$, are of *FLT*-type.

As in Einstein’s special relativity, the principle of relativity implicates that there is a metric in inertial systems on 4d space–time with signature ± 2 and it is invariant under a transformation group with ten parameters including space–time ‘translations’, boosts and space rotations. Thus, these 4d spaces are maximally symmetric, i.e. *Mink/dS/AdS* of zero, positive or negative constant curvature, invariant under group $ISO(1, 3)/SO(1, 4)/SO(2, 3)$, respectively. As for invariant universal constants, in addition to the speed of light c there is another invariant constant R , the radius of *dS/AdS*-spaces. Therefore, the *dS/AdS* special relativity can be set up based on the principle of relativity and the postulate on invariant universal constants [8,9].

The *dS*-space as a 4d hyperboloid \mathcal{H}_R can be embedded in a $(1 + 4)$ -dimensional *Mink*-space, $\mathcal{H}_R \subset M^{1,4}$:

$$\mathcal{H}_R: \quad \eta_{AB}\xi^A\xi^B = -R^2, \quad (2.3)$$

$$ds^2 = \eta_{AB}d\xi^A d\xi^B, \quad (2.4)$$

where $\eta_{AB} = \text{diag}(1, -1, -1, -1, -1)$, $A, B = 0, \dots, 4$.

On the hyperboloid, a kind of uniform great ‘circular’ motions of a particle with mass m_R can be defined by a conserved 5d angular momentum:

$$\frac{d\mathcal{L}^{AB}}{ds} = 0, \quad \mathcal{L}^{AB} := m_R \left(\xi^A \frac{d\xi^B}{ds} - \xi^B \frac{d\xi^A}{ds} \right). \quad (2.5)$$

For the particle, there is an Einstein-like formula:

$$-\frac{1}{2R^2}\mathcal{L}^{AB}\mathcal{L}_{AB} = m_R^2, \quad \mathcal{L}_{AB} := \eta_{AC}\eta_{BD}\mathcal{L}^{CD}. \quad (2.6)$$

Obviously, Eqs. (2.3), (2.4), (2.5) and (2.6) are invariant under linear transformations of *dS*-group $SO(1, 4)$. For a massless particle or a light signal with $m_R = 0$, similar motion can be defined as long as the proper time s is replaced by an affine parameter λ .

Via a ‘gnomonic’ projection without antipodal identification, \mathcal{H}_R becomes the *BdS*-space with a Beltrami atlas [8,9] chart by

chart. In the charts $U_{\pm 4}$, for instance,

$$x^i|_{U_{\pm 4}} = \frac{R\xi^i}{\xi^4}, \quad i = 0, \dots, 3; \quad (2.7)$$

$$\xi^4|_{U_{\pm 4}} = \left(\xi^{0^2} - \sum_{a=1}^3 \xi^{a^2} + R^2 \right)^{1/2} \geq 0, \quad (2.8)$$

there are condition from (2.3) and *BdS*-metric from (2.4)

$$\sigma(x) = \sigma(x, x) := 1 - R^{-2}\eta_{ij}x^i x^j > 0, \quad (2.9)$$

$$ds^2 = [\eta_{ij}\sigma(x)^{-1} + R^{-2}\eta_{ik}\eta_{jl}x^k x^l \sigma(x)^{-2}] dx^i dx^j, \quad (2.10)$$

where $\eta_{ij} = \text{diag}(1, -1, -1, -1)$. Under *FLT*s of $SO(1, 4)$ sending an event $A(a^i)$ to the origin

$$\begin{aligned} x^i &\rightarrow \tilde{x}^i = \pm \sigma(a)^{1/2} \sigma(a, x)^{-1} (x^i - a^i) D_j^i, \\ D_j^i &= L_j^i + R^{-2} \eta_{jk} a^k a^l (\sigma(a) + \sigma(a)^{1/2})^{-1} L_l^i, \\ L &:= (L_j^i)_{i,j=0,\dots,3} \in SO(1, 3), \end{aligned} \quad (2.11)$$

(2.9) and (2.10) are invariant. Thus, inertial systems and inertial motions transform among themselves, respectively.

For a pair of events $(A(a^i), X(x^i))$,

$$\Delta_R^2(a, x) = R^2 [\sigma^2(a, x) - \sigma(a)\sigma(x)] \geq 0 \quad (2.12)$$

is invariant under (2.11). Thus, the pair is time-like, null, or space-like, respectively.

The Beltrami light-cone at an event A with running points X is

$$\mathcal{F}_R := R \{ \sigma(a, x) - [\sigma(a)\sigma(x)]^{1/2} \} = 0. \quad (2.13)$$

At the origin $a^i = 0$, it is just Minkowskian $\eta_{ij}x^i x^j = 0$.

Under the ‘gnomonic’ projection, the uniform great ‘circular’ motions are projected as a kind of inertial motions along geodesics. In fact, the geodesics are Lobachevsky-like straight world lines and vice versa. A time-like geodesic, along which a particle with mass m_R moves, is equivalent to

$$\frac{dp^i}{ds} = 0, \quad p^i := m_R \sigma(x)^{-1} \frac{dx^i}{ds} = C^i = \text{const}. \quad (2.14)$$

Under certain initial condition it is just a straight world line with respect to $w = w(s)$

$$x^i(w) = c^i w + b^i. \quad (2.15)$$

A light signal moves along a null geodesic with an affine parameter λ can be written as

$$\frac{dk^i}{d\lambda} = 0, \quad k^i := \sigma^{-1}(x) \frac{dx^i}{d\lambda} = \text{const}. \quad (2.16)$$

It can also be expressed as a straight line [8,9].

From both (2.14) and (2.16), it follows that the coordinate velocity components are constants

$$\frac{dx^a}{dt} = v^a = \text{const}, \quad a = 1, 2, 3. \quad (2.17)$$

Thus, the both motions of free particles and light signals are indeed of inertia as in (2.2), chart by chart. Namely, *the law of*

inertia holds on the BdS -space. This together with the principle of relativity is just the *PoI* in the BdS -space.

For such a free massive particle a set of conserved quantities p^i in (2.14) and L^{ij} can be defined as a pseudo 4-momentum vector and a pseudo 4-angular-momentum, respectively

$$L^{ij} = x^i p^j - x^j p^i, \quad \frac{dL^{ij}}{ds} = 0. \quad (2.18)$$

In fact, p^i and L^{ij} constitute the conserved 5d angular momentum in (2.5). And the Einstein-like formula (2.6) becomes a *generalized Einstein's formula*

$$E^2 = m_R^2 c^4 + p^2 c^2 + \frac{c^2}{R^2} j^2 - \frac{c^4}{R^2} k^2, \quad (2.19)$$

with energy $E = p^0$, momentum p^a , $p_a = \delta_{ab} p^b$, ‘boost’ k^a , $k_a = \delta_{ab} k^b$ and 3-angular momentum j^a , $j_a = \delta_{ab} j^b$. For a massless particle or a light signal with $m_R = 0$, similar issues hold so long as the proper time is replaced by an affine parameter.

If we introduce the Newton–Hooke constant ν [13] and take R as $R \simeq (3/\Lambda)^{1/2}$,

$$\nu := \frac{c}{R} \simeq c(3/\Lambda)^{-1/2}, \quad \nu^2 \sim 10^{-35} \text{ s}^{-2}. \quad (2.20)$$

It is so tiny that all experiments at ordinary scales cannot distinguish the dS special relativity from Einstein's one.

In order to make sense of inertial motions and these observables for an inertial observer \mathcal{O}_I rested at the spacial origin of the Beltrami system, the simultaneity should be defined. Similar to Einstein's special relativity, two events A and B are simultaneous if and only if their Beltrami temporal coordinate values $x^0(A)$ and $x^0(B)$ are equal:

$$x^0 := x^0(A) = x^0(B) =: x^0. \quad (2.21)$$

It is called the *Beltrami simultaneity* and defines a 1 + 3 decomposition of the BdS -metric (2.10)

$$ds^2 = N^2(dx^0)^2 - h_{ab}(dx^a + N^a dx^0)(dx^b + N^b dx^0) \quad (2.22)$$

with lapse function, shift vector, and induced 3-geometry on Σ_c in the chart, respectively,

$$\begin{aligned} N &= \{\sigma_{\Sigma_c}(x)[1 - (x^0/R)^2]\}^{-1/2}, \\ N^a &= x^0 x^a [R^2 - (x^0)^2]^{-1}, \\ h_{ab} &= \delta_{ab} \sigma_{\Sigma_c}^{-1}(x) - [R\sigma_{\Sigma_c}(x)]^{-2} \delta_{ac} \delta_{bd} x^c x^d, \end{aligned} \quad (2.23)$$

where $\sigma_{\Sigma_c}(x) = 1 - (x^0/R)^2 + \delta_{ab} x^a x^b / R^2$. In particular, at $x^0 = 0$, $\sigma_{\Sigma_c}(x) = 1 + \delta_{ab} x^a x^b / R^2$, 3-hypersurface Σ_c is isomorphic to an S^3 in all Beltrami coordinate charts.

3. Principle of inertia and cosmological principle as two sides of a coin

On the dS -space, there is an important relation between the *PoI* and the cosmological principle. It is just like two sides of a coin.

In fact, for an observer rest at spacial origin $x^a = 0$ of Beltrami system, there is another simultaneity: the *proper-time simultaneity* with respect to an ideal clock's proper-time τ . It is easy to see that the proper-time τ is explicitly related to the Beltrami time x^0 :

$$\tau := \tau_R = R \sinh^{-1}(R^{-1} \sigma^{-1/2}(x) x^0). \quad (3.1)$$

Thus, the *proper-time simultaneity* can be defined as: all events $X(x^i)$ are simultaneous with respect to the observer if and only if their proper time are equal. Namely,

$$x^0 \sigma^{-1/2}(x, x) =: \xi^0 = R \sinh(R^{-1} \tau) = \text{const}. \quad (3.2)$$

In fact, these events are comoving with the observer, who now becomes a comoving one \mathcal{O}_C with respect to all these events. The line-element on a simultaneous hypersurface Σ_τ now is

$$dl^2 = -ds_{\Sigma_\tau}^2, \quad (3.3)$$

where

$$\begin{aligned} ds_{\Sigma_\tau}^2 &= R_{\Sigma_\tau}^2 dl_{\Sigma_\tau 0}^2, \\ R_{\Sigma_\tau}^2 &:= \sigma^{-1}(x, x) \sigma_{\Sigma_\tau}(x, x) = 1 + (\xi^0/R)^2, \\ \sigma_{\Sigma_\tau}(x, x) &:= 1 + R^{-2} \delta_{ab} x^a x^b > 0, \\ dl_{\Sigma_\tau 0}^2 &:= \{\delta_{ab} \sigma_{\Sigma_\tau}^{-1}(x) - [R\sigma_{\Sigma_\tau}(x)]^{-2} \delta_{ac} \delta_{bd} x^c x^d\} dx^a dx^b. \end{aligned} \quad (3.4)$$

It is clear that this simultaneity is directly related to the cosmological principle on the dS -space. In fact, if the proper time τ is taken as a temporal coordinate for the observer \mathcal{O}_C , the BdS -metric (2.10) becomes as a Robertson–Walker-like dS -metric with τ being a ‘cosmic’ time and an accelerated expanding 3d cosmos isomorphic to S^3 :

$$ds^2 = d\tau^2 - dl^2 = d\tau^2 - \cosh^2(R^{-1} \tau) dl_{\Sigma_\tau 0}^2. \quad (3.5)$$

It is important that two kinds of simultaneity relate the BdS -metric (2.10) with the *PoI* and the Robertson–Walker-like dS -metric (3.5) with the cosmological principle. They do make sense in two types of measurements: the Beltrami simultaneity is for those of the inertial observer \mathcal{O}_I relevant to the *PoI* and the proper time simultaneity for those of the comoving observer \mathcal{O}_C concern ‘cosmic’ effects of all distant stars and cosmic objects except the cosmological constant as test stuffs. Thus, on the dS -space there is a kind of inertial-comoving observers \mathcal{O}_{I-C} who play two roles with apparatus having two different types of time scales and relevant rulers. What should be done for them from their comoving observations to another type of measurements is to switch off the ‘cosmic’ time τ with the ‘cosmic’ rule and on the Beltrami time $x^0 = ct$ with the Beltrami rule, respectively, and vice versa. Namely, if the observers as comoving ones, \mathcal{O}_C , on (3.5) would change their measurements from the proper-time simultaneity to the Beltrami time one according to the relation (3.1), they become inertial ones \mathcal{O}_I , for whom the *PoI* makes sense, and vice versa.

Actually, for the dS -space this provides a very meaningful model like a coin with two sides. On one side, there is the *PoI* on the BdS -space (2.10) together with the law of inertia on inertial

systems with respect to a set of inertial observers \mathcal{O}_I . On another side, the Robertson–Walker-like dS -space (3.5) displays the cosmological principle with respect to a set of comoving observers \mathcal{O}_C . In other words, the ‘cosmic’ background of the Robertson–Walker-like dS -space (3.5) supports the PoI on the BdS -space (2.10). And conversely, the PoI provides a benchmark of physics related to ‘cosmic’ observations.

4. Are there any inertial systems for the whole universe?

‘Are there at all any inertial systems for very extended portions of the space–time continuum, or, indeed, for the whole universe?’ [1]. For Einstein, the answer seems to be negative unless for the ‘Galilean regions’. However, in view of the dS special relativity, the answer is positive!

Actually, the universe does fix on a kind of inertial systems in the following manner. Firstly, if the universe is accelerated expanding and asymptotic to a dS , its fate should be the Robertson–Walker-like dS -space (3.5). This is very natural in view of the dS special relativity. Secondly, the time direction and the homogeneous space of the universe tend to the ‘cosmic’ time and the 3d cosmos as an accelerated expanding S^3 of the Robertson–Walker-like dS -space, respectively. These set up the directions of the ‘cosmic’ time axis and the spacial axes for the Robertson–Walker-like dS up to some spacial rotations in all them transformed each other by dS -group. Thirdly, by means of the important relation between the BdS -metric (2.10) and the Robertson–Walker-like dS -metric (3.5) by changing the simultaneity, or just simply via the relation (3.1) between the Beltrami time x^0 and the ‘cosmic’ time τ , the directions of the axes of the Beltrami systems can be given. In fact, the Beltrami time axis is related to the ‘cosmic’ time axis in the Robertson–Walker-like dS -space, while the spacial axes of the Robertson–Walker-like dS -metric (3.5) are just the Beltrami spacial ones in the BdS -metric (2.10). Thus, the evolution of the universe does fix on the Beltrami inertial systems.

It is important that such a way of determining the Beltrami systems of inertia is completely different from the way of Einstein [1]. Actually, the gravitation in the universe does not explicitly play any roles here and there is nothing related to Einstein’s ‘argument in a circle’.

In the Beltrami systems, there are two universal constants, c and R . In order to set up the Beltrami systems, it is also needed to determine their values concretely. However, it is clear that as inertial-frames the Beltrami systems do not depend on their concrete values unless they are related to observations in the universe. In this case, their values should be given by two independent experiments or observations. Note that these constants are supposed to be invariant and universal approximately. So, the speed of light c may still be taken as that in Einstein’s special relativity, which is just a limiting case $R \rightarrow \infty$ of the dS special relativity. Thus, this also fixes on the origin of the Beltrami systems since the Beltrami light cone (2.13) at the origin is just Minkowskian. As for the value of R , it may also be given by $R \simeq (3/\Lambda)^{1/2}$ with the Λ being taken in the precise cosmology nowadays. Furthermore, the re-scaling of the curvature radius R may lead to the conformal extension and compactifi-

cation of the dS -space together with that of the $Mink$ -space and the AdS -space [14].

It is also clear and important that although the temporal axis of such kind of Beltrami systems can be fixed on by the evolution of the universe in the above manner, the symmetry among all Beltrami systems is still of the dS -group so long as the cosmological effects are not be taken into account. Otherwise, the symmetry should be reduced to the group $SO(4)$ for the comoving observations in the universe. This may shed light on the inconsistency between the principle of relativity and the cosmological environment (see, e.g. [17]).

Further, different kinds of PoI together with relevant inertial-frames in all possible kinematics, such as Einstein’s special relativity, Newton mechanics, Newton–Hooke mechanics [13] and so on can be viewed as certain contractions in different limits of c and R , respectively. Therefore, the origin of all these PoI should be inherited from the PoI in the BdS -space and in this sense they can also be set up by the evolution of the universe.

In conclusion, the Beltrami systems of inertia and their contractions does exist in the universe. Their coordinate axes can be fixed on by the cosmic time’s arrow of the universe via the Robertson–Walker-like dS -space, to which the universe is asymptotic. This is independent of the gravitational effects. In this sense, for the PoI in the dS special relativity and all other kinds of PoI as its contractions, there is no longer Einstein’s ‘argument in a circle’ [1].

Of course, in the universe except at its fate as a dS -space, there is gravity anywhere and anytime. How to take into account the gravitational effects and what should be done for the PoI ? What is the benchmark of physics with gravity?

5. Gravity and localized principle of inertia

In view of the dS special relativity, there is no gravity in the dS -space. The ‘gravitational effects’ in the dS -space with coordinate atlas other than the Beltrami one should be a kind of non-inertial effects. Temperature and entropy in the static dS -system are just this case in analogy with the Rindler space in view of Einstein’s special relativity in the $Mink$ -space [11]. Thus, the dS -space does not like a black hole.

In order to describe gravity, we would like to recall Einstein’s description on ‘Galilean regions’ first. In these finite regions, ‘the laws of the special theory of relativity, . . . , hold with remarkable accuracy’ [1]. Namely, all gravitational effects can be ignored on Einstein’s ‘Galilean regions’ in such a way that his special relativity with full Poincaré symmetry should hold *locally*. This is because all these regions are *finite*. Although in practice, it may still be regarded as *global symmetry* approximately *with remarkable accuracy*.

If there are two such kind of ‘finite regions’ of full local Poincaré invariance at different but nearby positions, how to pass from one to another?

According to Einstein, there should be gravity in-between these ‘regions’. Therefore, in order to transit from one to another, some curved space–time with gravity in-between should be passed. In other words, in order to connect these ‘regions’ together, some gravitational field as interaction should be taken

into account. Since there is local Poincaré symmetry in these ‘regions’, in order to transit in-between, the space–time with gravity should also be of local Poincaré symmetry! Otherwise, it cannot be consistently transited from one ‘region’ to another if Poincaré symmetry cannot be maintained locally in the course of transition. For any number of such ‘finite regions’, it is the same.

This may also be seen from another angle in terminology of differential geometry. Each of *finite* ‘Galilean regions’ is essentially a portion of a *Mink*-space with Poincaré symmetry isomorphic to an R^4 , so that there are intersections among these *Mink*-spaces with different ‘finite regions’ at different positions. And the transition functions on these intersections should also be valued in Poincaré symmetry. Further, these *Mink*-spaces with ‘finite regions’ may be viewed as tangent spaces at different positions of a curved manifold as the space–time with gravity and the transition functions are valued in *local* Poincaré symmetry.

Thus, it seems to be the core of Einstein’s idea on gravity that the theory of gravity should be based on the localization of his special relativity with Poincaré group as full symmetry anywhere and anytime on some curved space–times. For the sake of definiteness, we name this principle as the localized *PoI* with full local symmetry or *the principle of localization*. Mathematically, this indicates that space–times with gravity might be such a kind of manifolds that on them the *Mink*-space with (local) full Poincaré symmetry should be as a kind of tangent spaces anywhere and anytime in the universe. If so, the *PoI* as a benchmark should be localized on the space–times with gravity and this should be in consistency with the case of the *Mink*-space as a free space–time where gravity might be ignored.

But, in general relativity, it is not really the case as was mentioned at beginning.

Due to the asymptotic behavior of the universe and in the light of Einstein’s ‘Galilean regions’ as well as in view of the *dS* special relativity, we may require that gravity in the universe should be based on the localization of the *dS* special relativity with localized *PoI* in local *dS*-frame anywhere and anytime in the universe. Further, its dynamics should also be properly of local *dS*-invariance characterized by a dimensionless constant $g \simeq (\Lambda G \hbar / c^3)^{1/2} \sim 10^{-61}$ from the cosmological constant Λ and the Planck length (see, e.g. [18,19]). If so, the benchmark of physics is either the *PoI* on the *dS*-space as a free space on the cosmic scale or its localization with local *dS*-invariance anywhere and anytime in the universe. In addition, the evolution of the universe can also fix on the local inertial frames of *dS*-invariance in the same manner as the case without gravity or where gravitational effects can be ignored at very high accuracy.

A simple model for the *dS*-gravity has implied that these points should work.

In fact, from Cartan connection 1-form $\theta^{ab} = B_j^{ab} dx^j \in \mathfrak{so}(1, 3)$ and Lorentz frame 1-form $\theta^a = e_j^a dx^j$ on Riemann–Cartan manifold of Einstein–Cartan theory [20–22], it follows a kind of connections valued at *dS*-algebra [23–25]

$$\mathcal{B} := \mathcal{B}_j dx^j,$$

$$\mathcal{B}_j := (\mathcal{B}^{AB}{}_j)_{A,B=0,\dots,4} = \begin{pmatrix} B^{ab}{}_j & R^{-1} e_j^a \\ -R^{-1} e_j^b & 0 \end{pmatrix} \in \mathfrak{so}(1, 4). \quad (5.1)$$

The curvature valued at *dS*-algebra reads:

$$\mathcal{F}_{jk} = (\mathcal{F}^{AB}{}_{jk}) = \begin{pmatrix} F^{ab}{}_{jk} + 2R^{-2} e^{ab}{}_{jk} & R^{-1} T^a{}_{jk} \\ -R^{-1} T^b{}_{jk} & 0 \end{pmatrix} \in \mathfrak{so}(1, 4), \quad (5.2)$$

where $e^a{}_{bjk} = \frac{1}{2}(e_j^a e_{bk} - e_k^a e_{bj})$, $e_{bj} = \eta_{ab} e_j^a$, $F^{ab}{}_{jk}$ and $T^a{}_{jk}$ are curvature and torsion of Cartan connection.

The total action of the model with source may be taken as

$$S_T = S_{\text{GYM}} + S_m, \quad (5.3)$$

where S_m is the action of source and S_{GYM} the Yang–Mills-like action of gravity:

$$\begin{aligned} S_{\text{GYM}} &= \frac{1}{4g^2} \int_M d^4x e \text{Tr}_{dS}(\mathcal{F}_{jk} \mathcal{F}^{jk}) \\ &= \int_M d^4x e \left[\chi (F + 2\Lambda) - \frac{1}{4g^2} F^{ab}{}_{jk} F_{ab}{}^{jk} \right. \\ &\quad \left. + \frac{\chi}{2} T^a{}_{jk} T_a{}^{jk} \right]. \end{aligned} \quad (5.4)$$

Here $e = \det(e_j^a)$, a dimensionless constant g should be introduced as usual in gauge theory to describe the self-interaction of the gravitational field, χ a dimensional coupling constant related to g and R , and $F = \frac{1}{2} F^{ab}{}_{jk} e_{ab}{}^{jk}$ the scalar curvature of Cartan connection, the same as the action in Einstein–Cartan theory. In order to make sense in comparison with Einstein–Cartan theory, we take $\chi = 1/(8\pi G)$ and $g^{-2} \simeq 3\chi \Lambda^{-1}$ with $\hbar = c = 1$. In fact, $g^2 \simeq G \hbar c^{-3} \Lambda$.

Although the gravitational field equation now should be of Yang–Mills type, this model does pass the observation tests in solar-scale and there are simple cosmic models having ‘Big Bang’. But, different from general relativity, there are ‘energy–momentum-like tensors’ for gravity from the F^2 and T^2 terms as a kind of the ‘dark stuffs’ in the action (5.4). In fact, by means of the relation between Cartan connection $B^{ab}{}_j$ and Ricci rotational coefficients $\gamma^{ab}{}_j$, we may pick up Einstein’s action from Einstein–Cartan’s action F , and the rest terms in (5.4) are all ‘dark stuffs’ in view of general relativity. Thus, this model should provide an alternative framework for the cosmic data analysis.

In this model, there is the cosmological constant Λ from local *dS*-symmetry so that it is not just a ‘dummy’ constant at classical level as in general relativity. In fact, this model can be viewed as a kind of *dS*-gravity in a ‘special gauge’ and the 4-dimensional Riemann–Cartan manifolds should be a kind of 4-dimensional umbilical manifolds that there is local *dS*-space–time together with ‘gauged’ *dS*-algebra anywhere and anytime (see, e.g. [18,19,24]).

It is interesting that the model is renormalizable [26] with an $SO(5)$ gauge-like Euclidean action having a Riemann sphere as an instanton. Thus, quantum tunneling scenario may support

$\Lambda > 0$. For the gauge-like gravity, asymptotic freedom may indicate that the coupling constant g should be very tiny and link the cosmological constant Λ with the Planck length ℓ_P properly, since both the Λ and Planck scale as fixed points provide an infrared and an ultraviolet cut-off, respectively.

We will explain these issues in detail elsewhere.

6. Concluding remarks

In physics of the last century, symmetry, localization of symmetry and symmetry breaking play very important roles. For the cosmic scale physics without or with gravity, it should be also the case. In view of the dS special relativity and in the light of Einstein's 'Galilean regions', the PoI with maximum symmetry and its localization should still play a central role as the benchmarks of physics in the large scale.

If the universe is asymptotic to a dS -space, it should be asymptotic to a slightly closed Robertson–Walker-like dS -space, which closely relates to the BdS -space with the PoI . Therefore, the evolution of the universe also supports the PoI on the BdS -space and fix on the Beltrami systems without Einstein's 'argument in a circle'. Thus, the PoI of the dS special relativity is a benchmark of physics on the cosmic scale when gravity can be ignored.

We may require that on the space–times with gravity there should be locally the PoI with local inertial frames of full dS -symmetry anywhere and anytime. Then, the evolution of the universe can also fix on these local inertial frames. A simple model for the dS -gravity has implied these requirements.

Thus, the PoI of the dS special relativity and its localization are consistent benchmarks of physics without or with gravity in the universe.

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Yang's model as triply special relativity and the Snyder's model–de Sitter special relativity duality

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ABSTRACT

We show that if Yang's quantized space–time model is completed at both classical and quantum level, it should contain both Snyder's model, the de Sitter special relativity and their duality.

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1. Introduction

Some sixty years ago, Snyder [1] proposed a quantized space–time model by means of the projective geometry approach to the de Sitter (dS)-space of momentum with two universal constants: c and a , a scale near or at the Planck length ℓ_P . The 4-d energy–momentum was defined by the inhomogeneous projective coordinates. Then, Snyder identified the space–time coordinates' noncommutative operators \hat{x}^μ with 4-translation generators of a dS -algebra $\mathfrak{so}(1, 4)$ and other operators as angular momentum for an $\mathfrak{so}(1, 3) \subset \mathfrak{so}(1, 4)$.

Soon after, Yang [2] extended Snyder's model to the one with the third constant, the radius R of a dS universe in order to recover the translation under $R \rightarrow \infty$. Yang found an $\mathfrak{so}(1, 5)$ algebra with c, a and R in a 6-d space with Minkowski ($Mink$) signature. In Yang's algebra, there are two $\mathfrak{so}(1, 4)$ subalgebras for coordinate operators \hat{x}^μ and momentum operators \hat{p}^μ , respectively, with a common $\mathfrak{so}(1, 3)$ for angular momentum operators $\hat{l}^{\mu\nu}$. And the algebra is invariant under a Z_2 dual transformation between a, \hat{x}^μ and $\hbar/R, \hat{p}^\mu$. This is a UV–IR parameters' transformation.

Recently, the 'doubly special relativity' or the 'deformed special relativity' (DSR) has been proposed [3]. There is also a universal constant κ near the Planck energy, related to \hbar/a in Snyder's model in addition to c . In the sense with one more universal constant a near or at the Planck length ℓ_P in addition to c , Snyder's

model might be regarded as the earliest DSR and some of DSR models can be given as the generalization of Snyder's model [4]. Soon, the 'triply special relativity' (TSR) [5–7] has also been proposed under a deformed symmetry with one more universal constant, the universe radius $R \sim \Lambda^{-1/2}$. Later, it is found that the Lie algebra form of the deformed symmetry is just Yang's algebra in the 6-d space [8]. Thus, Yang's model might be regarded as a TSR.

The projective geometry approach to dS -space is basically equivalent to the Beltrami model of dS -space (BdS). It is important to emphasize that the Beltrami coordinates or inhomogeneous projective ones, without the antipodal identification for preserving orientation, play an important role in analogy with the $Mink$ -coordinates in a $Mink$ -space. Namely, in the Beltrami atlas, particles and light signals move along the time-like or null geodesics being straight world-lines with constant coordinate velocities, respectively. Among these systems, the properties are invariant under the fractionally linear transformations with a common denominator of dS -group. If these motions and systems could be regarded as of inertia without gravity, there should be the principle of inertia in dS/AdS -space–time, respectively.

In fact, just as weakening the fifth axiom leads to non-Euclidean geometry, giving up Einstein's Euclidean assumption on the rest rigid ruler and clock in special relativity leads to two other kinds of special relativity on the dS/AdS -space–time with radius R . They are based on the principle of inertia and the postulate of universal constants (c, R) on an almost equal footing with the special relativity on $Mink$ -space–time of $R \rightarrow \infty$ [9–17].

On the other hand, it is interesting to see [18,19] that in terms of the Beltrami model of dS -space [9,11], there is an important one-to-one correspondence between Snyder's quantized

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space–time model [1] as a DSR [3,4] and the dS special relativity [9–12]. Actually, the dS special relativity can be regarded and simply formulated as a space–time-counterpart of Snyder's model for dS -space of momentum so long as the constant \hbar/a in Snyder's model as dS -radius of momentum near the Planck scale is replaced by R as radius of dS -space–time. Inspired by the correspondence, the Snyder's model– dS special relativity duality as a UV–IR duality is proposed [18,19].

Since Snyder's quantized 4-d space–time model is on a 4-d dS -space of momentum, if Yang's model is really generalized Snyder's with the third universal constant R , it should also be back to 4 dimensions. But, how to realize Yang's model in 4 dimensions completely? In his very short Letter, Yang did not answer the question. Recall that there are three 4-d maximally symmetric space–times with maximum symmetries of ten generators, which are just the $Mink/dS/AdS$ -space with $ISO(1,3)/SO(1,4)/SO(2,3)$ invariance, respectively. Thus, it is impossible to realize Yang's $\mathfrak{so}(1,5)$ algebra with fifteen generators on one space of 4 dimensions in the sense of Riemann geometry and Lie symmetry. The TSR realization of Yang's $\mathfrak{so}(1,5)$ algebra gives a tentative 4-d realization. But, it is in terms of a deformed algebra with non-commutative geometry. The fact that there are two $\mathfrak{so}(1,4)$ subalgebras with a common homogeneous Lorentz algebra $\mathfrak{so}(1,3)$ in Yang's $\mathfrak{so}(1,5)$ suggests another kind of realization: A pair of dS -spaces of 4 dimensions with a dual relation.

In this Letter we show that if Yang's model can be completed with such a kind of 4-d realizations at both classical and quantum level, this complete Yang model should contain both Snyder's quantized space–time model, the dS special relativity and their duality.

This Letter is arranged as follows. In Section 2, we first recall and complete Yang's model with a UV–IR dual invariance in a 6-d dimensionless $Mink$ -space at both classical and quantum level. Then, in Section 3, we show that the two $\mathfrak{so}(1,4)$ subalgebras in the complete Yang model relevant to the space–time coordinate operators \hat{x}^μ and the momentum operators \hat{p}^μ are the same as Snyder's $\mathfrak{so}(1,4)$ algebra of quantized space–time and the algebra for 'quantized' energy, momentum, and angular momentum in a dS -space of space–time, respectively. We also present a way to get Snyder's model, the dS special relativity and their duality from the Yang model. Finally, we end with some concluding remarks.

2. A complete Yang model and a UV–IR duality

Under Yang's $\mathfrak{so}(1,5)$ algebra, there is an invariant quadratic form of signature -4 [2] in a 6-d dimensionless $Mink$ -space $\mathcal{M}^{1,5}$. Then, the metric in $\mathcal{M}^{1,5}$ reads

$$d\chi^2 = \eta_{AB} d\zeta^A d\zeta^B, \quad A, B = 0, \dots, 5, \quad (2.1)$$

where $\eta_{AB} = \text{diag}(+, -, -, -, -, -)$. The dimensionless canonical 'momentum' conjugate to the dimensionless 'coordinate' ζ^A can be introduced as $N_A = \eta_{AB} \frac{d\zeta^B}{d\chi}$. Thus, there is a 12-d phase space (\mathcal{M}, Ω) with a symplectic structure Ω and the non-vanishing basic Poisson bracket in (ζ^A, N_A) : $\{\zeta^A, N_B\} = -\delta_B^A$. Obviously, the dimensionless 6-'angular momentum' $\mathcal{L}^{AB} := \zeta^A N^B - \zeta^B N^A$ as the classical counterpart of Yang's operators (see below) forms an $\mathfrak{so}(1,5)$ algebra under Poisson bracket:

$$\{\mathcal{L}^{AB}, \mathcal{L}^{CD}\} = \eta^{AD} \mathcal{L}^{BC} + \eta^{BC} \mathcal{L}^{AD} - \eta^{AC} \mathcal{L}^{BD} - \eta^{BD} \mathcal{L}^{AC}. \quad (2.2)$$

Under canonical quantization, in the 'coordinate' representation with $\hat{N}_A = i \frac{\partial}{\partial \zeta^A}$, $[\hat{\zeta}^A, \hat{N}_B] = -i \delta_B^A$, they become operators $\hat{\mathcal{L}}^{AB}$ forming the algebra under Lie bracket. Now, the following

operators are just the operators in Yang's model [2] up to some redefined coefficients

$$\begin{aligned} \hat{x}_0 &= ia \left(\zeta^5 \frac{\partial}{\partial \zeta^0} + \zeta^0 \frac{\partial}{\partial \zeta^5} \right) = a \hat{\mathcal{L}}^{50}, \\ \hat{x}_i &= ia \left(\zeta^5 \frac{\partial}{\partial \zeta^i} - \zeta^i \frac{\partial}{\partial \zeta^5} \right) = -a \hat{\mathcal{L}}^{5i}, \\ \hat{p}_0 &= \frac{i\hbar}{R} \left(\zeta^4 \frac{\partial}{\partial \zeta^0} + \zeta^0 \frac{\partial}{\partial \zeta^4} \right) = \frac{\hbar}{R} \hat{\mathcal{L}}^{40}, \\ \hat{p}_i &= \frac{i\hbar}{R} \left(\zeta^4 \frac{\partial}{\partial \zeta^i} - \zeta^i \frac{\partial}{\partial \zeta^4} \right) = -\frac{\hbar}{R} \hat{\mathcal{L}}^{4i}, \\ \hat{M}_i &= i\hbar \left(\zeta^0 \frac{\partial}{\partial \zeta^i} + \zeta^i \frac{\partial}{\partial \zeta^0} \right) = -\hbar \hat{\mathcal{L}}^{0i}, \\ \hat{L}_i &= i\hbar \epsilon_i^{jk} \left(\zeta_j \frac{\partial}{\partial \zeta^k} \right) = \frac{\hbar}{2} \epsilon_{ijk} \hat{\mathcal{L}}^{jk}, \\ \hat{\psi} &= i \frac{a}{R} \left(\zeta^5 \frac{\partial}{\partial \zeta^4} - \zeta^4 \frac{\partial}{\partial \zeta^5} \right) = \frac{a}{R} \hat{\mathcal{L}}^{45}, \end{aligned} \quad (2.3)$$

with $\epsilon_{123} = \epsilon_1^{23} = 1$ and $\zeta_j = \eta_{jA} \zeta^A$. They form Yang's $\mathfrak{so}(1,5)$ algebra as follows:

$$\begin{aligned} [\hat{p}^\mu, \hat{p}^\nu] &= i\hbar R^{-2} \hat{\mathcal{L}}^{\mu\nu}, & [\hat{\mathcal{L}}^{\mu\nu}, \hat{p}^\rho] &= i\hbar (\eta^{\nu\rho} \hat{p}^\mu - \eta^{\mu\rho} \hat{p}^\nu), \\ \hat{p}^\mu &= \eta^{\mu\nu} \hat{p}_\nu, & \hat{\mathcal{L}}^{\mu\nu} &= \hbar \hat{\mathcal{L}}^{\mu\nu}, \\ [\hat{x}^\mu, \hat{x}^\nu] &= i\hbar^{-1} a^2 \hat{\mathcal{L}}^{\mu\nu}, & [\hat{\mathcal{L}}^{\mu\nu}, \hat{x}^\rho] &= i\hbar (\eta^{\nu\rho} \hat{x}^\mu - \eta^{\mu\rho} \hat{x}^\nu), \\ \hat{x}^\mu &= \eta^{\mu\nu} \hat{x}_\nu, \\ [\hat{x}^\mu, \hat{p}^\nu] &= i\hbar \eta^{\mu\nu} \hat{\psi}, & [\hat{\psi}, \hat{x}^\mu] &= -ia^2 \hbar^{-1} \hat{p}^\mu, \\ [\hat{\psi}, \hat{p}^\mu] &= i\hbar R^{-2} \hat{x}^\mu, & [\hat{\psi}, \hat{\mathcal{L}}^{\mu\nu}] &= 0, \end{aligned}$$

together with an $\mathfrak{so}(1,3)$ for the 4-d angular momentum operators.

It is clear that there are two $\mathfrak{so}(1,4)$ for coordinate operators \hat{x}^μ and momentum operators \hat{p}^ν , respectively, with a common $\mathfrak{so}(1,3)$ for $\hat{\mathcal{L}}^{\mu\nu}$. It is also clear that in Yang's algebra with respect to the 6-d 'angular momentum' there is a $Z_2 = \{e, r|r^2 = e\}$ dual transformation with

$$r: a \rightarrow \frac{\hbar}{R}, \quad \hat{x}^\mu \rightarrow \hat{p}^\mu, \quad \hat{\psi} \rightarrow -\hat{\psi}. \quad (2.4)$$

Since a is near or equal to the Planck length ℓ_P and R is the radius of a dS universe, the invariance under the Z_2 dual transformation is a UV–IR duality.

3. The Snyder's model– dS special relativity duality from the Yang model

3.1. Snyder's model from the Yang model

Snyder considered a homogeneous quadratic form $-\eta^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 := \eta^{AB} \eta_A \eta_B < 0$, partially inspired by Pauli. It is a model via homogeneous (projective) coordinates of a 4-d momentum space of constant curvature, a dS -space of momentum. In fact, it can also be started from a dS -hyperboloid \mathcal{H}_a in a 5-d $Mink$ -space of momentum with radius $1/a$

$$\mathcal{H}_a: \eta^{AB} \eta_A \eta_B = -\frac{\hbar^2}{a^2}, \quad ds_a^2 = \eta^{AB} d\eta_A d\eta_B. \quad (3.1)$$

Snyder's inhomogeneous projective momentum is almost the same as the momentum in Beltrami coordinates. In order to preserve orientation, the antipodal identification should not be taken so that the Beltrami atlas should contain at least eight patches to

cover the hyperboloid (see, e.g., [9]). In the patch U_{4+} , $\eta_4 > 0$, Snyder's Beltrami momentum read

$$q_\mu = \frac{\hbar \eta_\mu}{a \eta_4}. \quad (3.2)$$

Now the metric in the patch becomes $ds_a^2 = g_a^{\mu\nu} dq_\mu dq_\nu$ with

$$g_a^{\mu\nu} = \sigma^{-1}(q) \eta^{\mu\nu} + \frac{a^2}{\hbar^2} q^\mu q^\nu \sigma^{-2}(q),$$

$$\sigma(q) = 1 - \frac{a^2}{\hbar^2} q^\nu q_\nu > 0, \quad (3.3)$$

where $q^\mu = \eta^{\mu\nu} q_\nu$. Along geodesic that is the great 'circle' on \mathcal{H}_a , the space-time 'coordinates' and angular momentum are conserved

$$x_a^\mu = R \sigma^{-1}(q) \frac{dq^\mu}{ds_a} = \text{const.},$$

$$l_a^{\mu\nu} = R \left(q^\mu \frac{dq^\nu}{ds_a} - q^\nu \frac{dq^\mu}{ds_a} \right) = \text{const.} \quad (3.4)$$

Importantly, from these conserved Killing observables and $q_0 = E$ as energy,¹ it follows an important identity

$$\frac{dE}{dq_i} = \text{const.} \quad (3.5)$$

It would mean that there is some 'wave packet' moving with constant 'group velocity'. Namely, a law of inertia-like in space of momentum hidden in Snyder's model [19].

Regarding such a 'wave packet' as an object in the space of momentum, a 8-d phase space $(\mathcal{M}_a, \omega_a)$ can be constructed and locally there are Snyder's momentum q_μ as canonical momentum and the conjugate variables X^μ as canonical coordinates $(q_\mu, X^\mu = R g_a^{\mu\nu} dq_\nu / ds_a)$ with a symplectic structure ω_a and basic Poisson brackets $\{q_\mu, X^\nu\}_a = -\delta_\mu^\nu$, $\{q_\mu, q_\nu\}_a = 0$, $\{X^\mu, X^\nu\}_a = 0$. Then Snyder's space-time 'coordinates' x_a^μ and angular momentum $l_a^{\mu\nu}$ can be expressed in terms of these canonical variables (q_μ, X^μ) . And it is straightforward to show that they form an $\mathfrak{so}(1, 4)$ under Poisson bracket.

In a momentum representation of the canonical quantization, the operators of Snyder's 'coordinates' and angular momentum are just ten Killing vectors of the model, up to some coefficients,

$$\hat{x}_a^i := i\hbar \left[\frac{\partial}{\partial q_i} - \frac{a^2}{\hbar^2} q^i q_\mu \frac{\partial}{\partial q_\mu} \right] = \hat{\mathcal{L}}_a^{4i}, \quad (3.6)$$

$$\hat{x}_a^0 := i\hbar \left[\frac{\partial}{\partial q_0} - \frac{a^2}{\hbar^2} q^0 q_\mu \frac{\partial}{\partial q_\mu} \right] = \hat{\mathcal{L}}_a^{40}. \quad (3.7)$$

Together with 'boost' $\hat{M}_{ai} = \hat{x}_a^i q_0 + \hat{x}_a^0 q_i =: \hat{l}_a^{0i} = \hat{\mathcal{L}}_a^{0i}$ and '3-angular momentum' $\hat{L}_{ai} = -\frac{1}{2} \epsilon_{ijk} (\hat{x}_a^j q^k - \hat{x}_a^k q^j) =: \frac{1}{2} \epsilon_{ijk} \hat{l}_a^{jk} = \frac{1}{2} \epsilon_{ijk} \hat{\mathcal{L}}_a^{jk}$, they are the components of 5-d angular momentum $\hat{\mathcal{L}}_a^{AB}$ and form an $\mathfrak{so}(1, 4)$ algebra:

$$[\hat{x}_a^i, \hat{x}_a^j] = i\hbar^{-1} a^2 \hat{l}_a^{ij}, \quad [\hat{x}_a^0, \hat{x}_a^i] = i\hbar^{-1} a^2 \hat{l}_a^{0i},$$

$$[\hat{L}_{ai}, \hat{L}_{aj}] = i\hbar \epsilon_{ijk} \hat{L}_{ak}, \quad [\hat{M}_{ai}, \hat{M}_{aj}] = i\hbar \hat{l}_a^{ij},$$

$$\epsilon_{12}^3 = -1, \quad \text{etc.} \quad (3.8)$$

Obviously, Snyder's quantized space-time 'coordinates' $\mathfrak{so}(1, 4)$ algebra is the same as the coordinate $\mathfrak{so}(1, 4)$ subalgebra of Yang's $\mathfrak{so}(1, 5)$. But, the operators of canonical coordinates \hat{X}^μ are still commutative.

In order to get Snyder's model from the complete Yang model, we consider a dimensionless $dS_5 \cong \mathcal{H} \subset \mathcal{M}^{1,5}$:

$$\mathcal{H}: \quad \eta_{AB} \zeta^A \zeta^B = -\frac{R^2}{a^2}. \quad (3.9)$$

Take a subspace \mathcal{I}_1 of $\mathcal{H} \subset \mathcal{M}^{1,5}$ as an intersection

$$\mathcal{I}_1 = \mathcal{H}|_{\zeta^4=0}: \quad \mathcal{H} \cap \mathcal{P}|_{\zeta^4=0} \subset \mathcal{M}^{1,5}, \quad (3.10)$$

where $\mathcal{P}|_{\zeta^4=0}$ is a hyperplane defined by $\zeta^4 = 0$. Introduce dimensional coordinates

$$\eta_\mu = \frac{\hbar}{R} \zeta^\mu, \quad \eta_4 = \frac{\hbar}{R} \zeta^5, \quad (3.11)$$

then the subspace \mathcal{I}_1 becomes \mathcal{H}_a (3.1) with metric (3.3) related to the metric (2.1) restricted on \mathcal{I}_1

$$ds_a^2 = \frac{\hbar^2}{R^2} d\chi^2|_{\mathcal{I}_1}. \quad (3.12)$$

Thus, the $\mathcal{A}, \mathcal{B} \neq 4$ components of Yang's 6-d 'angular momentum' \mathcal{L}^{AB} consist of a 5-d angular momentum which is identical to the 5-d angular momentum \mathcal{L}_a^{AB} in Snyder's model. Furthermore, Yang's operators of coordinates, angular momentum $\{\hat{x}_a^\mu, \hat{l}_a^{\mu\nu}\}$ and their algebra are the same as Snyder's operators $\{\hat{x}_a^\mu, \hat{l}_a^{\mu\nu}\}$ for the Killing observables and their algebra. Therefore, the complete Yang model really contain Snyder's model as a sub-model.

3.2. The Beltrami model of dS special relativity from the Yang model

On a dS -space-time with radius R as a hyperboloid embedded in a $(1+4)$ -d Mink-space

$$\mathcal{H}_R: \quad \eta_{AB} \xi^A \xi^B = -R^2, \quad ds_R^2 = \eta_{AB} d\xi^A d\xi^B, \quad (3.13)$$

a free particle with mass \hbar/a may move uniformly along a great 'circle' defined by a conserved 5-d angular momentum

$$\frac{d\mathcal{L}_R^{AB}}{ds_R} = 0, \quad \mathcal{L}_R^{AB} := \frac{\hbar}{a} \left(\xi^A \frac{d\xi^B}{ds_R} - \xi^B \frac{d\xi^A}{ds_R} \right), \quad (3.14)$$

with an Einstein-like formula for mass \hbar/a

$$-\frac{1}{2R^2} \mathcal{L}_R^{AB} \mathcal{L}_{RAB} = \frac{\hbar^2}{a^2}, \quad \mathcal{L}_{RAB} = \eta_{AC} \eta_{BD} \mathcal{L}_R^{CD}. \quad (3.15)$$

The conserved momentum and angular momentum of the particle can be defined as

$$\frac{dp_R^\mu}{ds} = 0, \quad p_R^\mu = \frac{\hbar}{Ra} \left(\xi^4 \frac{d\xi^\mu}{ds_R} - \xi^\mu \frac{d\xi^4}{ds_R} \right), \quad (3.16)$$

$$\frac{dl_R^{\mu\nu}}{ds} = 0, \quad l_R^{\mu\nu} = \frac{\hbar}{a} \left(\xi^\mu \frac{d\xi^\nu}{ds_R} - \xi^\nu \frac{d\xi^\mu}{ds_R} \right). \quad (3.17)$$

And the Einstein-like formula becomes

$$p_R^\mu p_{R\mu} - \frac{1}{2R^2} l_R^{\mu\nu} l_{R\mu\nu} = \frac{\hbar^2}{a^2}, \quad (3.18)$$

where $p_{R\mu} = \eta_{\mu\nu} p_R^\nu$ and $l_{R\mu\nu} = \eta_{\mu\rho} \eta_{\nu\sigma} l_R^{\rho\sigma}$.

In a Beltrami atlas of the BdS -space-time [9], the Beltrami coordinates read in the patch U_{4+} , $\xi^4 > 0$,

$$y^\mu = R \frac{\xi^\mu}{\xi^4}, \quad \xi^4 \neq 0 \quad (3.19)$$

and the metric becomes $ds_R^2 = g_{R\mu\nu} dy^\mu dy^\nu$ with

$$g_{R\mu\nu} = \sigma^{-1}(y) \eta_{\mu\nu} + R^{-2} y_\mu y_\nu \sigma^{-2}(y),$$

$$\sigma(y) = 1 - R^{-2} y^\nu y_\nu > 0, \quad (3.20)$$

¹ c is set 1 in the Letter.

where $y_\mu = \eta_{\mu\nu}y^\nu$. For the free particle with mass \hbar/a along geodesic that is the great ‘circle’ on \mathcal{H}_R , its motion becomes uniform motion with constant coordinate velocity. In fact, its momentum and angular momentum

$$p_R^\mu = \frac{\hbar}{a}\sigma^{-1}(y)\frac{dy^\mu}{ds_R}, \quad l_R^{\mu\nu} = \frac{\hbar}{a}\left(y^\mu\frac{dy^\nu}{ds_R} - y^\nu\frac{dy^\mu}{ds_R}\right), \quad (3.21)$$

are constants. This leads to the law of inertia for the particle:

$$v^i := \frac{dy^i}{dt} = \text{const.} \quad (3.22)$$

For the particle, there is an associated phase space $(\mathcal{M}_R, \omega_R)$ and locally there are Beltrami coordinates as the canonical coordinates and the covariant 4-momentum as canonical momentum $(y^\mu, P_\mu = \frac{\hbar}{a}g_{\mu\nu}dy^\nu/ds_R)$ with a symplectic structure and basic Poisson brackets in the patch $\{y^\mu, P_\nu\}_R = -\delta_\nu^\mu$, $\{y^\mu, y^\nu\}_R = 0$, $\{P_\mu, P_\nu\}_R = 0$. Now, the conserved Killing momentum and angular momentum of the particle can be expressed in terms of the canonical variables and form an $\mathfrak{so}(1, 4)$ algebra under the Poisson bracket.

In a coordinate representation of the canonical quantization, the operators of these conserved Killing Beltrami momentum and angular momentum are just ten Killing vectors of the model up to some coefficients forming an $\mathfrak{so}(1, 4)$ under Lie bracket

$$[\hat{p}_R^\mu, \hat{p}_R^\nu] = \frac{i\hbar}{R^2}\hat{l}_R^{\mu\nu}, \quad [\hat{l}_R^{\mu\nu}, \hat{p}_R^\rho] = i\hbar(\eta^{\nu\rho}\hat{p}_R^\mu - \eta^{\mu\rho}\hat{p}_R^\nu), \quad (3.23)$$

together with an $\mathfrak{so}(1, 3)$ for angular momentum operators $\hat{l}_R^{\mu\nu}$. This is the same as the momentum subalgebra of the Yang model.

It is remarkable that the conserved Killing Beltrami momentum leads to the law of inertia in the patch and it holds globally in the atlas patch by patch. In fact, the dS special relativity can be set up based on the principle of inertia and the postulate of universal constants, the speed of light c and the dS -radius R [9].

In order to show that there is indeed the BdS -model of the dS special relativity from the complete Yang model, let us consider another subspace \mathcal{L}_2 of $\mathcal{H} \subset \mathcal{M}^{1,5}$ (3.9) of the Yang model as an intersection

$$\mathcal{L}_2 = \mathcal{H}|_{\zeta^5=0}: \mathcal{H} \cap \mathcal{P}|_{\zeta^5=0} \subset \mathcal{M}^{1,5}, \quad (3.24)$$

where $\mathcal{P}|_{\zeta^5=0}$ is a hyperplane defined by $\zeta^5 = 0$. Introduce dimensional coordinates

$$\xi^\mu = a\zeta^\mu, \quad \xi^4 = a\zeta^4, \quad (3.25)$$

then \mathcal{L}_2 becomes the dS -hyperploid \mathcal{H}_R (3.13) and its metric (3.20) becomes the metric (2.1) restricted on \mathcal{L}_2

$$ds_R^2 = a^2 d\chi^2|_{\mathcal{L}_2}. \quad (3.26)$$

It is also straightforward now to find that the $\mathcal{A}, \mathcal{B} \neq 5$ components of the 6-d ‘angular momentum’ operators $\hat{\mathcal{L}}^{\mathcal{A}\mathcal{B}}$ in the Yang model consist of a 5-d angular momentum which is just the angular momentum operators $\hat{\mathcal{L}}_R^{\mathcal{A}\mathcal{B}}$ in the dS special relativity. And Yang’s momentum, angular momentum operators $\hat{p}^\mu, \hat{l}^{\mu\nu}$ in (2.3) and their subalgebra are just the Killing Beltrami momentum, angular momentum operators $\hat{p}_R^\mu, \hat{l}_R^{\mu\nu}$ in the Beltrami model of the dS special relativity. Thus, the complete Yang model really contain the BdS -model of dS special relativity as a sub-model.

3.3. The Snyder’s model– dS special relativity duality in the Yang model

It is important to see [19] that between Snyder’s model and the dS special relativity, there is also a $\mathbb{Z}_2 = \{e, s \mid s^2 = e\}$ dual exchange with

$$s: x_a^\mu \rightarrow p_R^\mu, \quad a \rightarrow \frac{\hbar}{R}. \quad (3.27)$$

This is also a UV–IR exchange. And it is isomorphic to the \mathbb{Z}_2 duality in Yang’s model (2.4).

The Snyder’s model– dS special relativity duality contains some other contents. One is that the cosmological constant Λ should be a fundamental constant in the Nature like c , G and \hbar . This is already indicated in Yang’s model as long as $R = (3/\Lambda)^{1/2}$ is taken.

Thus, not only both Snyder’s model and the dS special relativity are sub-models in the complete Yang model, but their \mathbb{Z}_2 duality transformations are contained in that of the \mathbb{Z}_2 duality in the Yang model as well.

4. Concluding remarks

The above ‘surgery’ for two subspaces \mathcal{L}_2 and \mathcal{L}_2 of a dimensionless $\mathcal{H} \subset \mathcal{M}^{1,5}$ (3.9) in the Yang model shows that both Snyder’s model and the BdS -model of dS special relativity are really sub-physics of the complete Yang model. And the UV–IR duality in the Yang model is just the one-to-one exchange of the Snyder model– dS special relativity duality.

It is quite possible that there are some other physical implications and/or relations with other dualities, such as the T-duality and S-duality, if a and R may have other identifications.

It should be mentioned that a Yang-like model with an $\mathfrak{so}(2, 4)$ symmetry on a dimensionless $(2, 4)$ -d flat space $\mathcal{M}^{2,4}$ can be set up and all similar issues for Snyder’s model and the dS special relativity or for an anti-Snyder’s model on an AdS -space of momentum and the AdS special relativity can also be realized started with a dimensionless $AdS_5 \cong \mathcal{H}$ or its boundary $\partial(AdS_5) \cong \mathcal{N} \subset \mathcal{M}^{2,4}$.

Since Yang’s algebra is just the Lie algebra form of the deformed algebra in TSR, which is a generalization of DSR, it should be explored from the point of view in our approach what are the relations with the other DSR models and the TSR. It seems that the duality exists between other DSR models and dS -space–time since some DSR models can be realized as Snyder’s model in different coordinates on the dS -space of momentum, the corresponding coordinates on the dS -space–time may also be taken. But, this may cause some issues due to the non-inertial effects from the viewpoint of the dS special relativity.

Finally, we would like to emphasize that the complete Yang model should be regarded as a theory of the special relativity based on the principle of inertia in the both space–time and space of momentum as well as the postulate on three universal constants c , ℓ_P and R .

All above issues and related topics should be further studied.

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