Modeling, Analysis, and Control of Networked Evolutionary Games

—A Semi-tensor Product Approach

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Outline of Presentation

I. An Introduction to Game Theory

☞ Game Theory

Figure 1: John von Neumann

[1] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944.

☞ Non-Cooperative Game

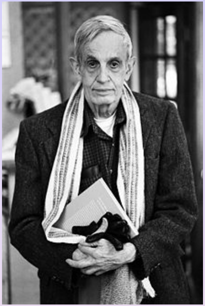


Figure 2: John Forbes Nash Jr.

[2] J. Nash, Non-cooperative game, *The Annals of Mathematics*, Vol. 54, No. 2, 286-295, 1951.

☞ Cooperative Game
(Winner of Nobel Prize in Economics 2012 with Roth)

Figure 3: Lloyd S. Shapley

[3] D. Gale, L.S. Shapley, Colle admissions and the stability of marriage, Vol. 69, American Math. Monthly, 9-15, 1962.

☞ Normal Form Games

Definition 1.1

A normal game $G = (N, \mathcal{S}, c)$:

(i) **Player**: $N = \{1, 2, \cdots, n\}$.

(ii) **Strategy**:

$$\mathcal{S}_i = \{1, 2, \cdots, k_i\}, \quad i = 1, \cdots n;$$

Situation (Profile):$\mathcal{S} = \prod_{i=1}^{n} \mathcal{S}_i$.

(iii) **Payoff function**:

$$c_j(s) : \ \mathcal{S} \to \mathbb{R}, \quad j = 1, \cdots, n. \tag{1}$$

Payoff:

$$c = \{c_1, \cdots, c_n\}.$$

☞ Nash Equilibrium

Definition 1.2

In a normal game G, a situation

$$s = (x_1^*, \cdots, x_n^*) \in \mathcal{S}$$

is a Nash equilibrium if

$$c_j(x_1^*, \cdots, , x_j^*, \cdots, x_n^*) \geq c_j(x_1^*, \cdots, x_j, \cdots, x_n^*) \\ j = 1, \cdots, n. \tag{2}$$

Example 1.3

Consider a game G with two players: P_1 and P_2:

- Strategies of P_1: $\mathcal{D}_2 = \{1, 2\}$;
- Strategies of P_2: $\mathcal{D}_3 = \{1, 2, 3\}$.

Table 1: Payoff bi-matrix

$P_1 \backslash P_2$	1	2	3
1	2, 1	3, 2	6, 1
2	1, 6	2, 3	5, 5

Nash Equilibrium is $(1, 2)$.

☞ Dynamic Games

Assumptions:

(i) finitely or infinitely repeated:

$$G \to G^N, \quad \text{or} \quad G \to G^\infty$$

(ii) Dynamics of strategies:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), \cdots, x_1(1), \cdots, x_n(1)) \\ x_2(t+1) = f_2(x_1(t), \cdots, x_n(t), \cdots, x_1(1), \cdots, x_n(1)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), \cdots, x_1(1), \cdots, x_n(1)), \end{cases} \tag{3}$$

where $x_i \in \mathcal{D}_{k_i}$, and $f_i : \prod_{j=1}^n \mathcal{D}_{k_j}^t \to \mathcal{D}_{k_i}$, $i = 1, \cdots, n$.

☞ Optimizations

Table 2: Categorization

Players\ Objectives	1	≥ 2
1	Opr. Research	Multi-obj. Decision
≥ 2	Cooper. Game	Non-cooper. Game

[4] J.M. Bilbao, *Cooperative Games on Combinatorial Structures*, Kluwer Acad. Pub., Boston, 2000.

☞ Cooperative Game

Definition 1.4

A transferable utility game G consists of three ingredients:

(i) n players $N := \{p_1, \cdots, p_n\} = \{1, \cdots, n\}$;

(ii) subsets $\{S | S \in 2^N\}$, each S is called a coalition; $S = \emptyset$ is empty coalition, $S = N$ is complete coalition.

(iii) $v : 2^N \to \mathbb{R}$ is called the characteristic function; $v(S)$ is the worth of S, (which means the profit (cost: $c : 2^N \to \mathbb{R}$) of coalition S).

$$v(\emptyset) = 0.$$

Example 1.5 (Glove Game)

Consider a game G with $P = \{p_1, p_2, \cdots, p_n\}$:

$$R = \{p_i \in P | p_i \ \ has\ a\ right\ hand\ glove\}$$

$$L = \{p_i \in P | p_i \ \ has\ a\ left\ hand\ glove\}$$

Let $S \in 2^P$. A singe glove (0.01), a pair of gloves (1), then:

$$v(S) = \min\{|S \cap L|, |S \cap R|\} + 0.01\left[n - 2\min\{|S \cap L|, |S \cap R|\}\right].$$

☞ Normal Form

$$(N, v) = (N, \{\mathcal{X}_i\}, \{P_i\}).$$

The characteristic function is:

$$v(S) = \max_{x \in \mathcal{X}_S^*} \min_{y \in \mathcal{X}_{N-S}^*} \sum_{i \in S} E_i(x, y).$$

☞ Super-additivity

Theorem 1.6

Let v be the characteristic function of a cooperative game. $\Gamma = (N, \{\mathcal{X}_i\}, \{P_i\})$. Then for $R, T \in 2^N$, $R \cap T = \emptyset$, we have

$$v(R) + v(T) \leq v(R \cup T).$$

Remark: Zero-sum (constant sum) game satisfies:

$$v(R) + v(T) = v(R \cup T), \quad \forall R \in 2^N.$$

☞ Imputation

Definition 1.7

Given a cooperative game $G = (N, v)$.

- $x \in \mathbb{R}^n$ is called an imputation, if

$$x_i \geq v(\{i\}), \quad i = 1, \cdots, n, \tag{4}$$

$$\sum_{i=1}^{N} x_i = v(N). \tag{5}$$

II. Networked Evolutionary Game

☞ What is NEG?

Definition 2.1

A networked evolutionary game, denoted by $((N,E),G,\Pi)$, consists of

(i) a network (graph) (N,E);

(ii) an FNG, G, such that if $(i,j)\in E$, then i and j play FNG with strategies $x_i(t)$ and $x_j(t)$ respectively;

(iii) a local information based strategy updating rule.

☞ Network Graph

Definition 2.2

1. (N, E) is called a graph, where N is the set of nodes and $E \subset N \times N$ is the set of edges.
2. $$U_d(i) = \{j | \text{there is a path connecting } i,\ j \text{ with leng } \leq d\}$$
3. If $(i, j) \in E$ implies $(j, i) \in E$ the graph is undirected, otherwise, it is directed.

Definition 2.3

A network is homogeneous network, if each node has same degree (for undirected graph)/ in-degree and out-degree(for directed graph).

Example 2.4

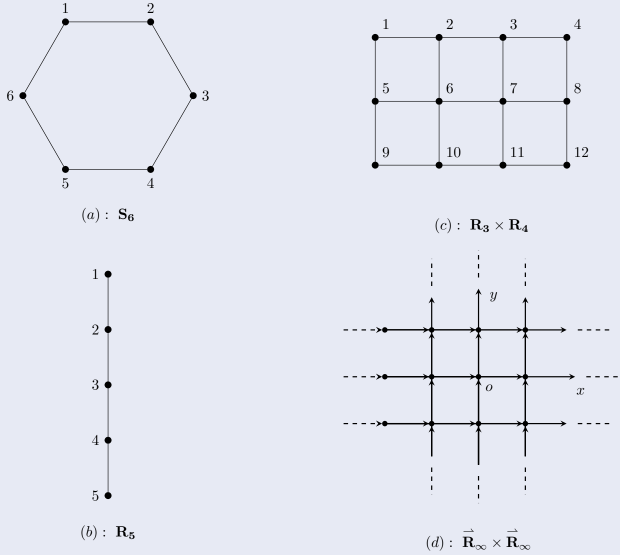


Figure 4: Some Standard Networks

☞ Fundamental Network Game

Definition 2.5

(i) A normal game with two players is called a fundamental network game (FNG), if

$$S_1 = S_2 := S_0 = \{1, 2, \cdots, k\}.$$

(ii) An FNG is symmetric, if

$$c_{1,2}(x, y) = c_{2,1}(y, x), \quad \forall x, y \in S_0.$$

☞ Overall Payoff

$$c_i(t) = \frac{1}{|U(i)| - 1} \sum_{j \in U(i) \backslash i} c_{ij}(t), \quad i \in N. \tag{6}$$

☞ Strategy Updating Rule

Definition 2.6

A strategy updating rule (SUR) for an NEG, denoted by Π, is a set of mappings:

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \big| j \in U(i)\}\right), \quad t \geq 0, \quad i \in N. \tag{7}$$

Remark 2.7

1. f_i could be a probabilistic mapping;
2. When the network is homogeneous, f_i, $i \in N$, are the same.

Example 2.7

- $\Pi - I$: *Unconditional Imitation with fixed priority*:

$$j^* = \mathrm{argmax}_{j \in U(i)} c_j(x(t)), \tag{8}$$

$\Rightarrow$

$$x_i(t+1) = x_{j^*}(t). \tag{9}$$

In non-unique case:

$$\mathrm{argmax}_{j \in U(i)} c_j(x(t)) := \{j_1^*, \cdots, j_r^*\},$$

set priority:

$$j^* = \min\{\mu | \mu \in \mathrm{argmax}_{j \in U(i)} c_j(x(t))\}. \tag{10}$$

$\Rightarrow$ Deterministic k-valued dynamics.

Example 2.7(cont'd)

- $\Pi - II$: *Unconditional Imitation with equal probability for best strategies*.

$$x_i(t+1) = x_{j^*_\mu}(t), \quad \text{with } p^i_\mu = \frac{1}{r}, \quad \mu = 1, \cdots, r. \tag{11}$$

$\Rightarrow$ Probabilistic k-valued dynamics.

- $\Pi - III$: *Simplified Femi Rule*. Randomly choose a neighborhood $j \in U(i)$.

$$x_i(t+1) = \begin{cases} x_j(t), & c_j(x(t)) > c_i(x(t)) \\ x_i(t), & \text{Otherwise.} \end{cases} \tag{12}$$

$\Rightarrow$ Probabilistic k-valued dynamics.

III. Semi-tensor Product Approach to Logical Dynamics

☞ Notations:

Set of Actions:

- $\mathcal{D}_k = \{1, 2, \cdots, k\}$;
- $\Delta_k = \{\delta_k^i | i = 1, 2, \cdots, k\}$, where $\delta_k^i = \mathrm{Col}_i(I_k)$.

$$i \sim \delta_k^i, \quad i = 1, 2, \cdots, k.$$

Logical Matrix:

- $$L = \left[\delta_k^{i_1} \ \delta_k^{i_2} \ \cdots \ \delta_k^{i_m}\right],$$

  Briefly,

  $$L = \delta_k \left[i_1 \ i_2 \ \cdots \ i_m\right].$$

- The set of $k \times m$ logical matrices is denoted as $\mathcal{L}_{k \times m}$.

$$A_{m\times n} \times B_{p\times q} = ?$$

☞ Tensor Product:

Let $A = (a_{ij})$. Then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

☞ Semi-tensor Product:

Definition 3.1

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \mathrm{lcm}(n, p).$$

Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \tag{13}$$

☞ Principle Comments

- When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
- When $n = rp$, denote it by $A \succ_r B$;
  when $rn = p$, denote it by $A \prec_r B$.
  These two cases are called the multi-dimensional case, which is particularly important in applications.
- STP keeps almost all the major properties of the conventional matrix product unchanged.

☞ Examples

Example 3.2

1. Let $X = \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 & -1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 7 & 0 \end{bmatrix}.$$

2. Let $X = \begin{bmatrix} -1 & 2 & 1 & -1 & 2 & 3 \end{bmatrix}^T$ and $Y = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 2 + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot (-2) = \begin{bmatrix} -3 \\ -6 \end{bmatrix}.$$

Example 3.2 (Continued)

3. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned} A \ltimes B &= \begin{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}. \end{aligned}$$

☞ Matrix Expression of Logical Functions

Vector Form of Logical Variables

Definition 3.3

(i) Assume $x \in \mathcal{D}_k$, its vector form is defined as $\vec{x} = \delta_k^x$.

(ii) $L \in \mathcal{M}_{k\times n}$ is called a logical matrix, if $Col(L) \in \Delta_k$, that is,

$$L = \left[\delta_k^{i_1}, \delta_k^{i_2}, \cdots, \delta_k^{i_n}\right].$$

Briefly,

$$L = \delta_k\left[i_1, i_2, \cdots, i_n\right].$$

(iii) The set of $k \times n$ logical matrices is denoted by $\mathcal{L}_{k\times n}$.

☞ Matrix Expression of Logical Functions (continued)

Theorem 3.4

Let $y \in \mathcal{D}_{k_0}$ and $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$, and

$$y = f(x_1, \cdots, x_n). \tag{14}$$

Then there exists a unique matrix $M_f \in \mathcal{L}_{k_0 \times k}$ $(k = \prod_{i=1}^{n} k_i)$ such that in vector form

$$y = M_f \ltimes_{i=1}^{n} x_i := M_f x, \tag{15}$$

where $x = \ltimes_{i=1}^{n} x_i$. M_f is called the structure matrix of f, and (15) is the algebraic form of (14).

☞ Matrix Expression of Logical Mapping

Let $x_i, y_j \in \mathcal{D}_k$, $i = 1, \cdots, n$, $j = 1, \cdots, m$, and $F : \mathcal{D}_k^n \to \mathcal{D}_k^m$ be

$$y_j = f_j(x_1, \cdots, x_n), \quad j = 1, \cdots, m. \tag{16}$$

Then in vector form we have

$$y_j = M_j x, \quad j = 1, \cdots, m. \tag{17}$$

Theorem 3.5

F can be expressed as

$$y = M_F x. \tag{18}$$

where $y = \ltimes_{j=1}^{m} y_j$, and

$$M_F = M_1 * M_2 * \cdots * M_m \in \mathcal{L}_{2^m \times 2^n}. \tag{19}$$

Khatri-Rao Product: Let $A \in \mathcal{M}_{p\times m}$, $B \in \mathcal{M}_{q\times m}$. Then

$$M * N = \left[\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N) \cdots \mathrm{Col}_m(M) \ltimes \mathrm{Col}_m(N)\right].$$

An Example

Example 2.6

There are three persons.

- A said: “B is a liar!”
- B said: “C is a liar!”
- C said: “A and B both are liars!”

Who is the liar?

Set P: A is honest; Q: B is honest; R: C is honest.
The logical expression is

$$(P \leftrightarrow \neg Q) \wedge (Q \leftrightarrow \neg R) \wedge (R \leftrightarrow \neg P \wedge \neg Q) = 1.$$

Its matrix form is

$$L(P,Q,R) = M_c M_c (M_e P M_n Q)(M_e Q M_n R)(M_e R M_c M_n P M_n Q)$$

We can calculate the canonical form of $L(P,Q,R)$ as

$$L(P,Q,R) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} PQR = \delta_2^1.$$

Only if $P = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $R = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then L is true, which means that only B is honest.

☞ Evolutionary Game

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ x_2(t+1) = f_2(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \end{cases} \tag{20}$$

where $x_i \in \mathcal{D}_{k_i}$, and $f_i : \prod_{j=1}^n \mathcal{D}_{k_j} \to \mathcal{D}_{k_i}$, $i = 1, \cdots, n$.

☞ Algebraic Form

$$x(t+1) = L_F x(t); \quad x \in \mathcal{D}_k, \tag{21}$$

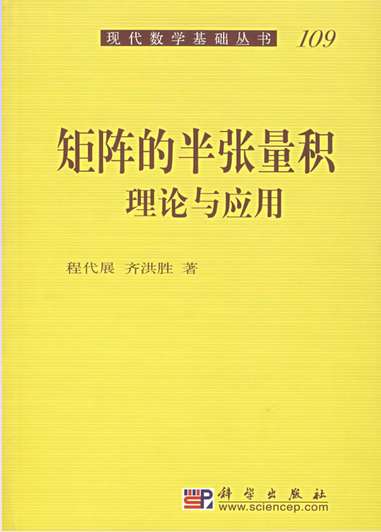
where

$$L_F \in \mathcal{L}_{k \times k},$$

and

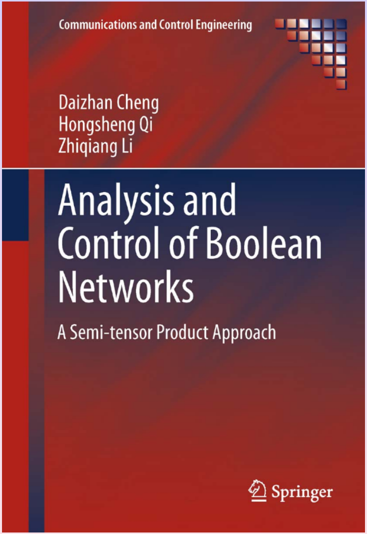
$$k = \prod_{j=1}^n k_j.$$

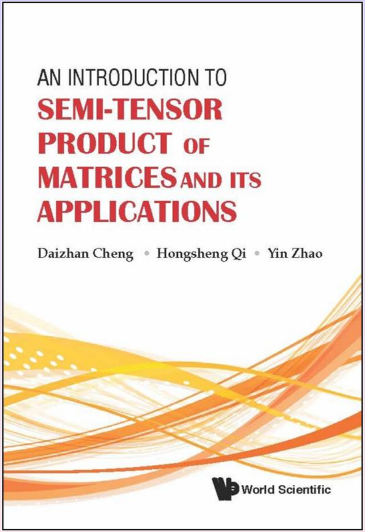
☞ Reference Book for STP



☞ Application to Power Systems

☞ Applications to Boolean Networks etc.





IV. Model of Networked Evolutionary Games

☞ Fundamental Evolutionary Equation

Recall SUR (7):

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \big| j \in U(i)\}\right), \quad t \geq 0, \quad i \in N.$$

Since $c_j(t)$ depends on $x_k(t)$, $k \in U(j)$, it follows that $x_i(t+1)$ depends on $x_j(t)$, $j \in U_2(i)$. That is, we can rewrite (7) as

$$x_i(t+1) = f_i(\{x_j(t) \big| j \in U_2(i)\}), \quad i \in N. \tag{22}$$

Remark 4.1

(i) Using the SUR, the f_i, $i \in N$ can be determined. Then (22) is called the FEE.

(ii) For a homogeneous network all f_i are the same.

☞ Calculating FEE

Example 4.2

Consider Rock - Scissors - Cloth on R_3. The payoff bi-matrix is:

Table 3: Payoff Bi-matrix (Rock-Scissors-Cloth)

$P_1 \backslash P_2$	$R = 1$	$S = 2$	$C = 3$
$R = 1$	$(0,\ 0)$	$(1,\ -1)$	$(-1,\ 1)$
$S = 2$	$(-1,\ 1)$	$(0,\ 0)$	$(1,\ -1)$
$C = 3$	$(1,\ -1)$	$(-1,\ 1)$	$(0,\ 0)$

Assume the strategy updating rule is $\Pi - I$:

Example 4.2 (cont'd)

Table 4: Payoffs $\rightarrow$ Dynamics

Profile	111	112	113	121	122	123
C_1	0	0	0	1	1	1
C_2	0	1/2	-1/2	-1	-1/2	0
C_3	0	-1	1	1	0	-1
f_1	1	1	1	1	1	1
f_2	1	1	3	1	1	1
f_3	1	1	3	1	2	2
Profile	131	132	133	211	212	213
C_1	-1	-1	-1	-1	-1	-1
C_2	1/2	1	0	1	0	1/2
C_3	0	-1	1	-1	1	0
f_1	1	1	1	3	3	3
f_2	1	1	3	3	2	3
f_3	1	1	3	3	2	3

Example 4.2 (cont'd)

Profile	221	222	223	231	232	233
C_1	0	0	0	1	1	1
C_2	-1/2	0	1/2	0	-1	-1/2
C_3	1	0	-1	-1	1	0
f_1	2	2	2	2	2	2
f_2	1	2	2	2	2	2
f_3	1	2	2	3	2	3
Profile	311	312	313	321	322	323
C_1	1	1	1	-1	-1	-1
C_2	-1/2	0	-1	0	1/2	1
C_3	0	-1	1	1	0	-1
f_1	3	3	3	2	2	2
f_2	3	3	3	1	2	2
f_3	1	1	3	1	2	2

Example 4.2 (cont'd)

Profile	331	332	333
C_1	0	0	0
C_2	1/2	-1/2	0
C_3	-1	1	0
f_1	3	3	3
f_2	3	2	3
f_3	3	2	3

Example 4.2 (cont'd)

Identifying $1 \sim \delta_3^1$, $2 \sim \delta_3^2$, $3 \sim \delta_3^3$, we have the vector form of each f_i as

$$x_i(t+1) = f_i(x_1(t), x_2(t), x_3(t)) = M_i x_1(t) x_2(t) x_3(t), \quad i = 1, 2, 3, \tag{23}$$

where

$$\begin{aligned} M_1 &= \delta_3[1\ 1\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 2\ 2\ 2\ 3\ 3\ 3]; \\ M_2 &= \delta_3[1\ 1\ 3\ 1\ 1\ 1\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 1\ 2\ 2\ 3\ 2\ 3]; \\ M_3 &= \delta_3[1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3]. \end{aligned}$$

Example 4.2 (cont'd)

Assume the strategy updating rule is $\Pi - II$:
Since player one and player 3 have no choice, f_1 and f_3 are the same as in Π is BNS. That is,

$$M_1' = M_1, \quad M_3' = M_3.$$

Consider player 2, who has two choices: either choose 1 or choose 3, and each choice has probability 0.5. Using similar procedure, we can finally figure out f_2 as:

Example 4.2 (cont'd)

$$
M_2' = \left[\begin{array}{cccccccccccccc}
1 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & 0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & 1
\end{array}\right]
$$

Now the evolution dynamics becomes a probabilistic 3-valued logical network. (to be completed!)

☞ FEE for Asymmetric Game

Example 4.3

Consider Boxed Pigs Game. P_1: smaller pig, P_2 bigger pig. The payoffs are shown in Table 5.

Table 5: Payoff Bi-matrix for the Boxed Pigs Game

$P_1 \backslash P_2$	P	W
P	$(2,\ 4)$	$(0,\ 6)$
W	$(5,\ 1)$	$(0,\ 0)$

Example 4.3(cont'd)

Next, assume there are 4 pigs, labeled 1, 2, 3 and 4, in which Pig 1 is the smallest pig, Pig 3 is the biggest one, and Pig 2 and Pig 4 are mid-size pigs. The network is shown in Figure 5.

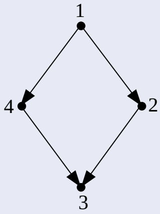


Figure 5: The Boxed Pigs Game over a Uniformed Network

Example 4.3(cont'd)

By comparing the payoffs and using $\Pi - II$, we can obtain that

$$\begin{aligned} & x_1(t+1) = f_1(x_1(t), x_2(t), x_3(t), x_4(t)) \\ & = \delta_2[1\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2]x(t) \\ & := M_1 x(t), \end{aligned} \tag{24}$$

where $x(t) = \ltimes_{i=1}^{4} x_i(t)$.

$$M_1 = \delta_2[1\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2].$$

Example 4.3(cont'd)

$$
\begin{aligned}
x_2(t+1) &= f_2(x_1(t), x_2(t), x_3(t), x_4(t)) \\
&= \begin{cases}
f_2^1 = \delta_2[1,1,2,2,2,2,2,2,2,1,2,2,1,2,2,2]x(t), \\
\quad p_2^1 = 0.25 \\
f_2^2 = \delta_2[1,1,2,2,2,2,2,2,2,1,2,2,2,2,2,2]x(t), \\
\quad p_2^2 = 0.25 \\
f_2^3 = \delta_2[1,1,2,2,2,2,2,2,2,2,2,2,1,2,2,2]x(t), \\
\quad p_2^3 = 0.25 \\
f_2^4 = \delta_2[1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2]x(t), \\
\quad p_2^4 = 0.25
\end{cases} \\
&:= M_2 x(t),
\end{aligned}
\tag{25}
$$

$$
M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.5 & 1 & 1 & 0.5 & 1 & 1 & 1 \end{bmatrix}.
$$

Example 4.3(cont'd)

Similarly, we have

$$x_3(t+1) = f_3(x_1(t), x_2(t), x_3(t), x_4(t)) := M_3 x(t), \tag{26}$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0.5 & 1 & 1 & 0.5 & 1 & 1 & 1 \end{bmatrix}.$$

$$x_4(t+1) = f_4(x_1(t), x_2(t), x_3(t), x_4(t)) := M_4 x(t), \tag{27}$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0.5 & 1 & 1 & 0.5 & 1 & 1 & 1 \end{bmatrix}.$$

V. Analysis of Networked Evolutionary Games

☞ Two Deleting Operators

Lemma 5.1

Assume $X \in \Upsilon_p$ and $Y \in \Upsilon_q$.

- Front-Maintaining Operator:

$$D_f^{p,q} = \delta_p[\underbrace{1 \cdots 1}_{q} \ \underbrace{2 \cdots 2}_{q} \ \cdots \ \underbrace{p \cdots p}_{q}],$$

then

$$D_f^{p,q} XY = X. \tag{28}$$

Lemma 5.1(cont'd)

Assume $X \in \Upsilon_p$ and $Y \in \Upsilon_q$.

- Rear-Maintaining Operator:

$$D_r^{p,q} = \delta_q[\underbrace{\underbrace{12\cdots q}\ \underbrace{12\cdots q}\ \cdots\ \underbrace{12\cdots q}}_{p}],$$

then

$$D_r^{p,q} XY = Y. \tag{29}$$

☞ From FEE to Evolutionary Dynamics

Algorithm 5.2

Assume an NEG is on S_n, with its FEE as

$$x_i(t+1) = M_i x_{i-2}(t) x_{i-1}(t) x_i(t) x_{i+1}(t) x_{i+2}(t). \qquad (30)$$

Using Lemma 5.1, we have

$$\begin{array}{rcl} x_i(t+1) & = & M_i D_r^{k^{i-3},k^5} x_1(t) x_2(t) \cdots x_{i+2}(t) \\ & = & M_i D_r^{k^{i-3},k^5} D_f^{k^{i+2},k^{n-i-2}} \ltimes_{j=1}^n x_j(t) \\ & := & \tilde{M}_i x(t), \end{array} \qquad (31)$$

where $x(t) = \ltimes_{j=1}^n x_j(t)$.

Algorithm 5.2(cont'd)

The evolutionary dynamics has the following form

$$x_i(t+1) = M_i x(t), \quad i = 1, \cdots, n. \tag{32}$$

The Overall dynamics as

$$x(t+1) = M_G x(t), \tag{33}$$

where $M_G \in \mathcal{L}_{k^n \times k^n}$ is determined by

$$M_G = M_1 * M_2 * \cdots * M_n. \tag{34}$$

☞ Basic Structure

Theorem 5.3

Consider a k-valued logical dynamic network

$$x(t+1) = Lx(t), \tag{35}$$

where $x(t) = \prod_{i=1}^{n} x_i(t)$, $L \in \mathcal{L}_{k^n \times k^n}$. Then

- δ_k^i is its fixed point, if and only if the diagonal element ℓ_{ii} of L equals to 1. It follows that the number of equilibriums of (35), denoted by N_e, is

$$N_e = \mathrm{tr}(L). \tag{36}$$

Theorem 5.3(cont'd)

- The number os length s cycles, N_s, is inductively determined by

$$\begin{cases} N_1 = N_e \\ N_s = \dfrac{\mathrm{tr}(L^s) - \sum\limits_{t \in \mathcal{P}(s)} tN_t}{s}, \quad 2 \leq s \leq k^n. \end{cases} \tag{37}$$

Note that in (37) $\mathcal{P}(s)$ is the set of proper factors of s. For instance, $\mathcal{P}(6) = \{1, 2, 3\}$, $\mathcal{P}(125) = \{1, 5, 25\}$.

Example 5.4

Recall Example 4.2 (Rock - Scissors - Cloth).

- Consider the case when $\Pi - I$ is used: Then we have the evolutionary dynamics as

$$x(t+1) = M_G x(t), \tag{38}$$

where

$$\begin{aligned} M_G &= M_1 * M_2 * M_3 \\ &= \delta_{27}[1\ 1\ 9\ 1\ 2\ 2\ 27\ 23\ 27\ 1\ 1\ 9\ 10 \\ &\quad 14\ 14\ 15\ 14\ 15\ 25\ 25\ 29\ 10\ 14\ 14\ 27\ 23\ 27]. \end{aligned} \tag{39}$$

Example 5.4(cont'd)

$$M_G^k = \delta_{27}[1\ 1\ 27\ 1\ 1\ 1\ 27\ 14\ 27\ 1\ 1\ 27\ 1\ 14\ 14\ 14\ 14\ 14\ 27\ 27\ 27\ 1\ 14\ 14\ 27\ 14\ 27], \quad k \geq 2,$$

We can figure out that:

- if $x(0) \in \{\delta_{27}^1, \delta_{27}^2, \delta_{27}^4, \delta_{27}^5, \delta_{27}^6, \delta_{27}^{10}, \delta_{27}^{11}, \delta_{27}^{13}, \delta_{27}^{22}\}$, then $x(\infty) = x(2) = \delta_{27}^1 \sim (1, 1, 1)$;
- if $x(0) \in \{\delta_{27}^8, \delta_{27}^{14}, \delta_{27}^{15}, \delta_{27}^{16}, \delta_{27}^{17}, \delta_{27}^{18}, \delta_{27}^{23}, \delta_{27}^{24}, \delta_{27}^{26}\}$, then $x(\infty) = x(2) = \delta_{27}^{14} \sim (2, 2, 2)$;
- if $x(0) \in \{\delta_{27}^3, \delta_{27}^7, \delta_{27}^9, \delta_{27}^{12}, \delta_{27}^{19}, \delta_{27}^{20}, \delta_{27}^{21}, \delta_{27}^{25}, \delta_{27}^{27}\}$, then $x(\infty) = x(2) = \delta_{27}^{27} \sim (3, 3, 3)$.

So the network converges to one of three uniformed strategy cases with equal probability.

Example 5.4(cont'd)

- Consider the other case when $\Pi - II$ is used: we have the transition matrix as

$$M_G = M_1 * M_2' * M_3. \tag{40}$$

Then the dynamics of NEG is

$$x(t+1) = M_G x(t). \tag{41}$$

(Here M_G is also skipped.) We can show that

$$\begin{aligned} M_G^k &= \delta_{27}[1\ 1\ 27\ 1\ 1\ 1\ 27\ 14\ 27\ 1\ 1\ 27\ 1 \\ &\quad 14\ 14\ 14\ 14\ 14\ 27\ 27\ 27\ 1\ 14\ 14\ 27\ 14\ 27], \\ &\quad k \geq 16. \end{aligned}$$

Same as $\Pi - I$ but converges much slower.

VI. Control of Networked Evolutionary Games

☞ Control NEG

Definition 6.1

Let $((N,E),G,\Pi)$ be an NEG,

$$N = X \cup W, \qquad X \cap W = \emptyset.$$

Then $(X \cup W, E), G, \Pi)$ is called a control NEG, if the strategies for nodes in W, denoted by $w_j \in W$, $j = 1, \cdots, |W|$, can be assigned at each moment $t \geq 0$. Moreover, $x \in X$ is called a state and $w \in W$ is called a control.

☞ Controllability & Stabilization

Definition 6.2

- A state x_d is said to be $T > 0$ step reachable from $x(0) = x_0$, if there exists a sequence of controls $w_0, \cdots, w_{T-1}$ such that $x(T) = x_d$. The set of T step reachable states is denoted as $R_T(x_0)$;
- The reachable set from x_0 is defined as

$$R(x_0) := \cup_{t=1}^{\infty} R_t(x_0).$$

Definition 6.2(cont'd)

- A state x_e is said to be stabilizable from x_0, if there exist a control sequence $w_0, \cdots, w_\infty$ and a $T > 0$, such that the trajectory from x_0 converges to x_e, precisely, $x(t) = x_e$, $t \geq T$. x_e is stabilizable, if it is stabilizable from $\forall x_0 \in \mathcal{D}_k^n$.

☞ Analysis of Dynamics

Assume $X = \{x_1, \cdots, x_n\}$ and $W = \{w_1, \cdots, w_m\}$, and we set $x = \ltimes_{i=1}^{n} x_i$ and $w = \ltimes_{j=1}^{m} w_j$, where $x_i,\ w_j \in \Delta_k \sim \mathcal{D}_k$ and $k = |S_0|$.

For each $w \in \Delta_{k^m}$ we have a (control-depending) strategy transition matrix (STM) M_w.

Define:

$$M\left(w = \delta_{k^m}^{i}\right) := M_i, \quad i = 1, 2, \cdots, k^m. \tag{42}$$

☞ Controlled Trajectory

The set of control-depending STM is denoted $\mathcal{M}_W$.

Let $x(0)$ be the initial state. Driven by control sequence

$$w(0) = \delta_{k^m}^{i_0},\ w(1) = \delta_{k^m}^{i_1},\ w(2) = \delta_{k^m}^{i_2},\ \cdots.$$

Then the trajectory will be

$$x(1) = M_{i_0}x(0),\ x(2) = M_{i_1}M_{i_0}x(0),\ x(3) = M_{i_2}M_{i_1}M_{i_0}x(0),\ \cdots.$$

☞ Main Results

Theorem 6.3

Consider a control NEG $(X \cup W, E), G, \Pi)$, with $|X| = n$, $|W| = m$, $|S_0| = k$.

- x_d is reachable from x_0, if and only if there exists a sequence $\{M_0, M_1, \cdots, M_{T-1}\} \subset \mathcal{M}_W$, $T \le k^n$, such that

$$x_d = M_{T-1} M_{T-2} \cdots M_1 M_0 x_0. \quad (43)$$

- x_d is stabilizable from x_0, if and only if (i) x_d is reachable from x_0 and there exists at least one $M^* \in \mathcal{M}_W$, such that x_d is a fixed point of M^*.

An immediate consequence of Theorem 6.3 is the following:

Corollary 6.4

For any $x_0 \in \mathcal{D}_k^n$, the reachable set satisfies

$$R(x_0) \subset \cup_{M \in \mathcal{M}_W} \mathrm{Col}(M). \tag{44}$$

Example 6.5

Consider a game $((N, E), G, \Pi)$, where (i) $N = (X \cup W)$, where $X = \{x_1, x_2, x_3\}$, $W = \{w\}$, the network graph is shown in Figure 66:

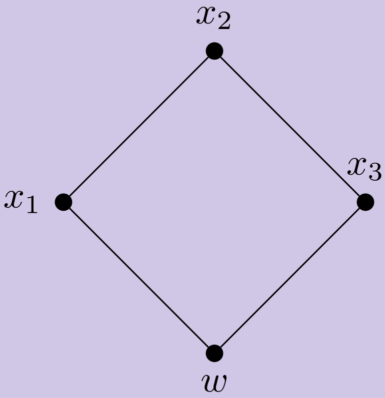


Figure 6: Control of BK-game

Example 6.5(cont'd)

(ii) G is Benoit-Krishna Game with

$$S_0 = \{1(D): \text{ Deny}, \;\; 2(W): \text{ Waffle}, \;\; 3(C): \text{ Confess}\}.$$

Payoffs:

Table 6: Payoff Table (Benoit-Krishna)

$P_1 \backslash P_2$	$D = 1$	$W = 2$	$C = 3$
$D = 1$	$(10,\ 10)$	$(-1,\ -12)$	$(-1,\ 15)$
$W = 2$	$(-12,\ -1)$	$(8,\ 8)$	$(-1,\ -1)$
$C = 3$	$(15,\ -1)$	$(8,\ 1)$	$(0,\ 0)$

Example 6.5(cont'd)

(iii) $\Pi = \Pi - I$:
This model can be explained as follows. There is a game of three $\{x_1, x_2, x_3\}$.

- x_1 is the head, who is able to contact x_2 and x_3.
- w is a detective, who sneaked in and is able to contact only x_2 and x_3.
- The purpose of w is to let all x_i to confess.

Example 6.5(cont'd)

First, we calculate the control-depending strategy transition matrix by letting $w = \delta_3^i$, $i = 1, 2, 3$ respectively. Then we have

$$\begin{aligned}
M(w = \delta_3^1) \quad = M_1 = \quad & \delta_{27}[1, 1, 9, 1, 1, 9, 27, 27, 27 \\
& 1, 1, 9, 1, 14, 18, 27, 7, 27, \\
& 25, 25, 27, 25, 26, 27, 27, 27, 27] \\
M(w = \delta_3^2) \quad = M_2 = \quad & \delta_{27}[1, 1, 9, 1, 5, 3, 27, 27, 27 \\
& 1, 11, 18, 13, 14, 14, 27, 14, 14, \\
& 25, 26, 27, 19, 14, 14, 27, 14, 27] \\
M(w = \delta_3^3) \quad = M_3 = \quad & \delta_{27}[21, 21, 27, 21, 24, 27, 27, 27, 27 \\
& 21, 1, 27, 24, 14, 14, 27, 14, 27, \\
& 27, 27, 27, 27, 14, 27, 27, 27, 27].
\end{aligned} \tag{45}$$

Example 6.5(cont'd)

$$
\begin{array}{ll}
R(\delta_{27}^{1}) = \delta_{27}\{1, 21, 27\} & R(\delta_{27}^{2}) = \delta_{27}\{1, 21, 27\} \\
R(\delta_{27}^{3}) = \delta_{27}\{9, 27\} & R(\delta_{27}^{4}) = \delta_{27}\{1, 21, 27\} \\
R(\delta_{27}^{5}) = \delta_{27}\{1, 5, 14, 24, 21, 27\} & R(\delta_{27}^{6}) = \delta_{27}\{1, 3, 9, 21, 27\} \\
R(\delta_{27}^{7}) = \delta_{27}\{27\} & \cdots \\
\vdots & \\
\cdots & R(\delta_{27}^{27}) = \delta_{27}\{27\}.
\end{array}
$$

Example 6.5(cont'd)

There are two common fixed points:

- $$x_e^1 = \delta_{27}^{14} = \delta_3^2 \ltimes \delta_3^2 \ltimes \delta_3^2;$$
- $$x_e^2 = \delta_{27}^{27} = \delta_3^3 \ltimes \delta_3^3 \ltimes \delta_3^3;$$

So the overall system is not stabilizable.

But any $x(0) \in \Delta_{27} \setminus \left\{\delta_{27}^{14}\right\}$, can be stabilized to $x_e^2 = \delta_{27}^{27}$ via a proper control sequence.

Example 6.5(cont'd)

For example, when $x(0) = \delta_{27}^{6} = \delta_3^1 \ltimes \delta_3^2 \ltimes \delta_3^3$, we can drive it to x_e^2 by any one of the following control sequences:

(i) $w(0) = \delta_3^3$, then the trajectory will be $x(1) = M_3 x(0) = \delta_{27}^{27}$;

(ii) $w(0) = \delta_3^2$, $w(1) = \delta_3^3$, then the trajectory will be $x(1) = M_2 x(0) = \delta_{27}^{9}, x(2) = M_3 M_2 x(0) = \delta_{27}^{27}$;

(iii) $w(0) = \delta_3^1$ and $w(1)$ can choose any one of $\delta_3^1, \delta_3^2, \delta_3^3$, then the trajectory will be $x(1) = M_1 x(0) = \delta_{27}^{9}$, and $x(2) = M_1 M_1 x(0) = \delta_{27}^{27}$, or $x(2) = M_2 M_1 x(0) = \delta_{27}^{27}$ or $x(2) = M_3 M_1 x(0) = \delta_{27}^{27}$.

VII. Conclusion

☞ What we did?

A rigorous mathematical frame of Networked Evolutionary Game (NEG) and Control Networked Evolutionary Game is presented. It contains the followings:

- Fundamental Evolutionary Equation (FEE) is proposed, which is computable.
- Using FEE, Evolutionary Dynamics of (NEGs) is constructed.
- The properties of NEG is analyzed via FEE and/or Evolutionary Dynamics.
- Controllability and Stabilizability of NGGs are investigated. Necessary and sufficient conditions are obtained.

☞ What else we can do?

- Applied to large scale networks.
- Various Control Problems for NEG.
- Consensus.
- Network stability strategy.
- Applications to (i) Biosystem; (ii) Economical Systems; (iii) Social Systems; etc.

Reference

[5] D. Cheng, F. He, H. Qi, T. Xu, F. He, Modeling, analysis and control of networked evolutionary games, http://lsc.amss.ac.cn/ dcheng/preprint/NTGAME02.pdf (submitted for pub).

Thank you for your attention!

Question?