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Permanence of stability for a class of system of differential equations with two delays

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The paper studies a class of a system of linear retarded differential difference equations with several parameters. It presents some sufficient conditions under which no stability changes for an equilibrium point occurs. Application of these results is given. October, 2006 ICMC-USP

1. INTRODUCTION

In this paper, our objective is to begin the study of the changes of the stability for a system of linear retarded differential difference equations with two delays.

Let us consider the n-dimensional system of linear retarded differential difference equations with two delays:

$$\dot{x}(t) = \alpha\, Ax(t) + \beta\, Bx(t-\tau) + \gamma\, Cx(t-\mu) \tag{1}$$

$(\,\dot{}= \frac{d}{dt})$, where we assume that τ, μ are non-negative real numbers, the parameters α, β, γ are real numbers and A, B, C are real $n \times n$ matrices.

In this work, we study a particular class of such system, it is assumed that the matrices A, B, C are simultaneously triangulares. The general case will be the subject of our next study.

We present some results that give us sufficient conditions about these parameters under which no stability switch occurs. As a consequence of these theorems, we can say that the stability of the trivial solution for the system (1) is determined only by α, when we have this parameter sufficiently large. We also introduce a study for the case $|\alpha|$ small.

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Systems of delay differential equations have been investigated in many contexts. For example, in [6], the dynamical behavior of a two neuron netlet of excitation and inhibition with a transmission delay is investigated.

We can use linear systems of the form (1) to study the stability of equilibria for a n-dimensional system of autonomous retarded functional differential equations:

$$\dot{x}(t) = f(x(t), x(t-\tau), x(t-\mu)), \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and are such that solutions to initial value problems exist and are continuous.

System (2) appears in many applications, for example, see [5]. In this paper, they study a particular case of this equation, for $n = 2$, using the Nyquist criterion on the characteristic equation.

The stability for the two delay equation (2) also has been investigated by many authors. See, for instance, [1], [9],[7], [8], [10].

Bélair and Campbell, in [2], considered the retarded differential difference equation

$$\dot{x}(t) = f_1(x(t-T_1)) + f_2\,(x(t-T_2)),$$

where the functions $f_i(u) = -A_i \tanh(u)$, $i = 1, 2$(A_i are positive constants), to analyse the influence of multiple negative feedback loops on the stability of a single-action mechanism.

Now, we are going to study the stability for the solution $x = 0$ of the equation (1) using the works by Cooke and van den Driessche[4] and Boese[3].

The Section 2 is dedicated to study a relationship between a particular type of the system (1) and a first-order delay differential equation which has the form:

$$\dot{x}(t) = ax(t) + bx(t-\tau) + cx(t-\mu), \tag{3}$$

with a, b, c complex numbers. In this section, we also analyse the characteristic equation associated to (3) and in the next section, we present our main results that give us conditions to have unchanged stability for this equation. Such theorems are proved in the Section 4.

One application of all these results is introduced in the Section 5. In this section, we present some parameters that claim the stability of the origin for a linear system of differential equations with two delays.

2. THE CHARACTERISTIC EQUATION

In this paper, we require that the matrices A, B, C cited in (1) are simultaneously triangulares. The general case, much more complex, will be the subject of our next study.

Observe that a necessary condition to have simultaneously triangularity is that each matrix should be triangular. It is not necessary that the matrices A, B, C should be commutative, however this condition is sufficient to occur the triangularity. This fact is illustrated with an example in the Section 5.

We suppose that there is a basis $\mathcal{B}= \{u_1, u_2, ..., u_n\}$ of $\mathcal{C}^n$ so that all the matrices A, B, C are triangulares.

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Therefore, let us consider that exists an invertible matrix P of order n such that:

$$\begin{aligned} P^{-1}AP &= triang(\alpha_1, ..., \alpha_n) = A_1 \\ P^{-1}BP &= triang(\beta_1, ..., \beta_n) = B_1 \\ P^{-1}CP &= triang(\gamma_1, ..., \gamma_n) = C_1. \end{aligned}$$

Note that the characteristic equation associated to (1) is:

$$p(s) = \det(sI - \alpha A - \beta B e^{-\tau s} - \gamma C e^{-\mu s}) = 0.$$

Then,

$$\begin{aligned} p(s) &= \det(sPP^{-1} - \alpha P A_1 P^{-1} - \beta e^{-\tau s} P B_1 P^{-1} - \gamma e^{-\mu s} P C_1 P^{-1}) \\ &= \det\left[P(sI - \alpha A_1 - \beta e^{-\tau s} B_1 - \gamma e^{-\mu s} C_1)P^{-1}\right] \\ &= \det P . \det(sI - \alpha A_1 - \beta e^{-\tau s} B_1 - \gamma e^{-\mu s} C_1) . \det P^{-1} \\ &= \det(sI - \alpha A_1 - \beta e^{-\tau s} B_1 - \gamma e^{-\mu s} C_1) \\ &= \prod_{i=1}^{n} (s - \alpha\alpha_i - \beta e^{-\tau s}\beta_i - \gamma e^{-\mu s}\gamma_i) = 0. \end{aligned}$$

Thus, if we are interested on the study of the local stability for the trivial solution(i.e. the zero solution) of (1), we have to analyse the existence of $s \in \mathcal{C}$ so that

$$s - \alpha - \beta e^{-\tau s} - \gamma e^{-\mu s} = 0, \tag{4}$$

with α, β, $\gamma \in \mathcal{C}$.

Observe that the last equation is the characteristic function of a first-order delay differential equation which has the form:

$$\dot{x}(t) = \alpha x(t) + \beta x(t - \tau) + \gamma x(t - \mu), \tag{5}$$

therefore, we can relate the stability of the trivial solution to our system with the stability of this solution for such equations.

Throughout this paper, we refer to the stability of a differential equations as the stability of its trivial solution.

Let us begin to analyse the characteristic equation (4).

If we define the functions $P(s) = s - \alpha - \gamma e^{-\mu s}$, $Q(s) = -\beta$, the characteristic equation (4) takes the form:

$$P(s) + Q(s)\, e^{-\tau s} = 0. \tag{6}$$

To study the equation (6), we use the following result that can be found in [4] and [3]:

THEOREM 2.1. *Consider the equation* $P(\lambda)+Q(\lambda)\,e^{-\lambda\tau}=0,\ P(\lambda)$ *and* $Q(\lambda)$ *are analytic functions in* $\Re(\lambda)>0$, *and satisfy the following conditions:*

(i) $P(\lambda)$ *and* $Q(\lambda)$ *have no common imaginary root;*
(ii) $\overline{P(-iy)}=P(iy),\ \overline{Q(-iy)}=Q(iy)$ *for real* y;
(iii) $P(0)+Q(0)\neq 0$;
(iv) $\limsup\{|Q(\lambda)|/|P(\lambda)| : |\lambda|\to\infty,\ \Re(\lambda)\geq 0\}<1$;
(v) $F(y)=|P(iy)|^2-|Q(iy)|^2$ *for real* y *has at most a finite number of real zeros.*

Then the following statements are true:

(a) *If* $F(y)=0$ *has no positive roots, then no stability switch may occur.*
(b) *If* $F(y)=0$ *has at least one positive root and each of them is simple, then as* τ *increases, a finite number of stability switches may occur and eventually the considered equation becomes unstable.*

Remark 2. 1. In this work, we only analyse the case that α,β,γ are real constants. The other case, when α,β,γ are complex numbers, will be done in a forthcoming paper.

Suppose that $\beta\neq 0,\ \ \alpha+\beta+\gamma\neq 0$ and we examine all the hypothesis described in the Theorem 2.1. First of all, let us show that the items (i) to (iv) are satisfied.

(i) The fact that $\beta\neq 0$ implies $Q(s)\neq 0\ \forall s$, and then, $P(s)$ and $Q(s)$ have no common imaginary roots.

(ii) Observe that $P(iy) = iy-\alpha-\gamma\ e^{-\mu iy}$, therefore, we get $\overline{P(-iy)}=P(iy)$. Also, we have $Q(iy)=-\beta=\overline{Q(-iy)}$.

(iii) We have that $P(0)=-\alpha-\gamma$ and $Q(0)=-\beta$. There is, $P(0)+Q(0)=-(\alpha+\beta+\gamma)\neq 0$.

(iv) It it easy to see that

$$\limsup\frac{|Q(s)|}{|P(s)|}=\frac{|\beta|}{|s-\alpha-\gamma e^{-\mu s}|}=0:|s|\to\infty,\ \Re(s)\geq 0.$$

Now, we examine the item $v)$ through the next proposition.
We see that

$$\begin{aligned}|P(iy)|^2&=|iy-\alpha-\gamma(\cos\mu y-i\sin\mu y)|^2\\&=|-\alpha-\gamma\cos\mu y+i(y+\gamma\ \sin\mu y)|^2\\&=(\alpha+\gamma\cos\mu y)^2+(y+\gamma\sin\mu y)^2\\&=\alpha^2+\gamma^2+y^2+2\alpha\gamma\cos\mu y+2y\gamma\sin\mu y.\end{aligned}$$

Now, let we define $F(y)=|P(iy)|^2-|Q(iy)|^2$, and so

$$F(y)=\alpha^2+\gamma^2+y^2-\beta^2+2\alpha\gamma\cos\mu y+2y\gamma\sin\mu y. \tag{7}$$

Then, to verify when the item (v) of the Theorem 2.1 is true, we need to search for conditions under which $F(y)$ has at most a finite number of real zeros. This has been done in our next result.

PROPOSITION 2.1. *The function $F(y)$ defined by (7) has only a finite number of real roots.*

Proof. Observe that

$$F(y) = 0 \Leftrightarrow y^2 + (2\gamma\,\sin\mu y)\,y + (\alpha^2 + \gamma^2 - \beta^2 + 2\alpha\gamma\cos\mu y) = 0.$$

Note that a necessary condition for the existence of a real root y from $F(y)$ is that the discriminant Δ should be greater than zero, where

$$\Delta = -4(\gamma^2\cos^2\mu y + \alpha^2 - \beta^2 + 2\alpha\gamma\cos\mu y).$$

Therefore, the necessary condition is:

$$(\gamma\cos\mu y + \alpha - \beta)\,(\gamma\cos\mu y + \alpha + \beta) \leq 0.$$

Thus, we can have two cases:
Case 1: If $\gamma > 0$, the necessary condition is that:

$$\frac{-\alpha+\beta}{\gamma} \leq \cos\mu y \leq \frac{-\alpha-\beta}{\gamma}, \text{ if } \ \beta < 0$$

or

$$\frac{-\alpha-\beta}{\gamma} \leq \cos\mu y \leq \frac{-\alpha+\beta}{\gamma}, \text{ if } \beta > 0.$$

Case 2: If $\gamma < 0$, the necessary condition is that:

$$\frac{-\alpha-\beta}{\gamma} \leq \cos\mu y \leq \frac{-\alpha+\beta}{\gamma}, \text{ if } \ \beta < 0$$

or

$$\frac{-\alpha+\beta}{\gamma} \leq \cos\mu y \leq \frac{-\alpha-\beta}{\gamma}, \text{ if } \beta > 0.$$

Note that

$$F(y) = 0 \text{ if and only if } y = -\gamma\sin\mu y \pm \sqrt{\beta^2 - (\gamma\cos\mu y + \alpha)^2}.$$

Consider the functions $g_i : D(g_i) \subset \mathbb{R} \to \mathbb{R},\ i = 1, 2$, such that:

1. $g_1(y) = -\gamma\sin\mu y + \sqrt{\beta^2 - (\gamma\cos\mu y + \alpha)^2}$,

2. $g_2(y) = -\gamma \sin \mu y - \sqrt{\beta^2 - (\gamma \cos \mu y + \alpha)^2}$.

Then, F has real zeros if and only if $g_1(y)$ or $g_2(y)$ intersects $f(y) = y$.

Now, we analyse the domain of the function g_i, $i = 1, 2$.

Let us suppose that $\gamma > 0$. We have seen that if we define $a = a(\alpha, \beta, \gamma) = \dfrac{-\alpha - |\beta|}{|\gamma|}$ and $b = b(\alpha, \beta, \gamma) = \dfrac{-\alpha + |\beta|}{|\gamma|}$, then we have $D(g_i) = \{y; \quad a \leq \cos \mu y \leq b, \quad a < 1, \quad b > -1\}$, $\forall i = 1, 2$.

Consider that $|\beta| - |\gamma| > 0$.

1. If α, β, γ satisfy $|\gamma| - |\beta| \leq \alpha \leq |\beta| - |\gamma|$, then $a \leq -1$ and $b \geq 1$, that is, $D(g_i) = \mathbb{R}$.

2. If $\alpha > |\beta| - |\gamma|$, then $a < -1$ and $b < 1$. In this case, there are real numbers $b_0 \in (0, \dfrac{\pi}{\mu})$, $b_1 \in (\dfrac{\pi}{\mu}, \dfrac{2\pi}{\mu})$ such that $cos(\mu b_0) = cos(\mu b_1) = b$. And then, the domain is $D(g_i) = \cup_{k \in \mathbb{Z}} [b_0 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]$.

3. If $\alpha < |\gamma| - |\beta|$, then $a > -1$ and $b > 1$. Observe that there are a_0, a_1, real numbers, $a_0 \in (0, \dfrac{\pi}{\mu})$, $a_1 \in (\dfrac{\pi}{\mu}, \dfrac{2\pi}{\mu})$, such that $cos(\mu a_0) = cos(\mu a_1) = a$. Thus, the domain is

$$D(g_i) = \cup_{k \in \mathbb{Z}} \{[k.\frac{2\pi}{\mu}, a_0 + k.\frac{2\pi}{\mu}] \cup [a_1 + k.\frac{2\pi}{\mu}, (1+k)\frac{2\pi}{\mu}]\}.$$

Now, suppose that $|\gamma| - |\beta| > 0$.

1. If α, β, γ satisfy $|\beta| - |\gamma| < \alpha < -|\beta| + |\gamma|$, then $a > -1$ and $b < 1$. In this case, there are real numbers, a_0, b_0, belong to the interval $(0, \dfrac{\pi}{\mu})$, a_1, b_1 in the interval $(\dfrac{\pi}{\mu}, \dfrac{2\pi}{\mu})$, $b_0 < a_0 < a_1 < b_1$ such that $cos(\mu a_0) = cos(\mu a_1) = a$ and $cos(\mu b_0) = cos(\mu b_1) = b$. And then, the domain of g_i is given by: $D(g_i) = \cup_{k \in \mathbb{Z}} \{[b_0 + k.\dfrac{2\pi}{\mu}, a_0 + k.\dfrac{2\pi}{\mu}] \cup [a_1 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]\}$.

2. If α satisfies $\alpha \geq |\gamma| - |\beta|$, therefore $a \leq -1$ and $b < 1$. As we have studied this case, $D(g_i) = \cup_{k \in \mathbb{Z}} [b_0 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]$.

3. If $\alpha \leq |\beta| - |\gamma|$, thus $a > -1$ and $b \geq 1$ and, of course, we have that the domain is $D(g_i) = \cup_{k \in \mathbb{Z}} \{[k.\dfrac{2\pi}{\mu}, a_0 + k.\dfrac{2\pi}{\mu}] \cup [a_1 + k.\dfrac{2\pi}{\mu}, (1+k)\dfrac{2\pi}{\mu}]\}$.

Consider that β, γ such that $|\beta| = |\gamma|$.

1. If $\alpha > 0$, we have that $a < -1$ and $b < 1$, then $D(g_i) = \cup_{k \in \mathbb{Z}} [b_0 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]$.

2. If $\alpha < 0$, then $a > -1$ and $b > 1$; thus, we have seen that the domain of g_i is $D(g_i) = \cup_{k \in \mathbb{Z}} \{[k.\dfrac{2\pi}{\mu}, a_0 + k.\dfrac{2\pi}{\mu}] \cup [a_1 + k.\dfrac{2\pi}{\mu}, (1+k)\dfrac{2\pi}{\mu}]\}$.

3. if $\alpha = 0$, $a = -1$ and $b = 1$, then, $D(g_i) = \mathbb{R}$.

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Now, we suppose that $\gamma < 0$. We have seen that if we define the numbers $a = a(\alpha, \beta, \gamma) = \dfrac{\alpha - |\beta|}{|\gamma|}$ and $b = b(\alpha, \beta, \gamma) = \dfrac{\alpha + |\beta|}{|\gamma|}$, then the domain is $D(g_i) = \{y;\ \ a \leq \cos \mu y \leq b,\ \ a < 1,\ \ b > -1\}$, $\forall i = 1, 2$.

Consider that $|\beta| - |\gamma| > 0$.

1. If α, β, γ satisfy $|\gamma| - |\beta| \leq \alpha \leq |\beta| - |\gamma|$, then $a \leq -1$ and $b \geq 1$, that is, $D(g_i) = \mathbb{R}$.
2. If $|\beta| - |\gamma| < \alpha < |\beta| + |\gamma|$, thus $a > -1$ and $b > 1$. In this case, we saw that $D(g_i) = \cup_{k\in\mathbb{Z}}\{[k.\dfrac{2\pi}{\mu}, a_0 + k.\dfrac{2\pi}{\mu}] \cup [a_1 + k.\dfrac{2\pi}{\mu}, (1+k)\dfrac{2\pi}{\mu}]\}$.
3. If $-|\beta| - |\gamma| < \alpha < |\gamma| - |\beta|$, therefore $a < -1$ and $b < 1$. And then, we have $D(g_i) = \cup_{k\in\mathbb{Z}}[b_0 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]$.

Now, suppose that $|\beta| - |\gamma| < 0$.

1. If α, β, γ satisfy $|\beta| - |\gamma| < \alpha < -|\beta| + |\gamma|$, then $a > -1$ and $b < 1$. In this case, $D(g_i) = \cup_{k\in\mathbb{Z}}\{[b_0 + k.\dfrac{2\pi}{\mu}, a_0 + k.\dfrac{2\pi}{\mu}] \cup [a_1 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]\}$.
2. If we take α such that $|\gamma| - |\beta| \leq \alpha < |\beta| + |\gamma|$, thus $a > -1$ and $b \geq 1$. As we studied this case, $D(g_i) = \cup_{k\in\mathbb{Z}}\{[k.\dfrac{2\pi}{\mu}, a_0 + k.\dfrac{2\pi}{\mu}] \cup [a_1 + k.\dfrac{2\pi}{\mu}, (1+k)\dfrac{2\pi}{\mu}]\}$.
3. If $-|\beta| - |\gamma| < \alpha \leq |\beta| - |\gamma|$, then $a \leq -1$ and $b < 1$ and of course, the domain is $D(g_i) = \cup_{k\in\mathbb{Z}}[b_0 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]$.

Consider that β, γ such that $|\beta| = |\gamma|$.

1. If $\alpha > 0$, we have that $a > -1$ and $b > 1$; therefore, the domain of g_i is $D(g_i) = \cup_{k\in\mathbb{Z}}\{[k.\dfrac{2\pi}{\mu}, a_0 + k.\dfrac{2\pi}{\mu}] \cup [a_1 + k.\dfrac{2\pi}{\mu}, (1+k)\dfrac{2\pi}{\mu}]\}$.
2. If $\alpha < 0$, then $a < -1$ and $b < 1$; thus $D(g_i) = \cup_{k\in\mathbb{Z}}[b_0 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]$.
3. if we take $\alpha = 0$, that is, $a = -1$ and $b = 1$; then, $D(g_i) = \mathbb{R}$.

To summarize, if we have α, β, γ such that $-|\beta| - |\gamma| < \alpha < |\beta| + |\gamma|$, we get four types of domain for g_i:

Type 1: $D(g_i) = \mathbb{R}$. This case holds for parameters which satisfy:

1.1 $|\beta| - |\gamma| > 0$, $|\gamma| - |\beta| \leq \alpha \leq |\beta| - |\gamma|, \gamma \neq 0$.
1.2 $|\beta| - |\gamma| = 0$, $\alpha = 0, \gamma \neq 0$.

Type 2: $D(g_i) = \cup_{k\in\mathbb{Z}}[b_0 + k.\dfrac{2\pi}{\mu}, b_1 + k.\dfrac{2\pi}{\mu}]$, $b_0 \in (0, \dfrac{\pi}{\mu})$, $b_1 \in (\dfrac{\pi}{\mu}, \dfrac{2\pi}{\mu})$, $cos(\mu b_0) = cos(\mu b_1) = b$, where $b = \dfrac{-\alpha + |\beta|}{|\gamma|}$, if $\gamma > 0$ or $b = \dfrac{\alpha + |\beta|}{|\gamma|}$, if $\gamma < 0$. This case is true for the parameters:

2.1 $\gamma > 0$, $|\beta| - |\gamma| > 0$, $|\beta| - |\gamma| < \alpha < |\beta| + |\gamma|$.

2.2 $\gamma > 0$, $|\beta| - |\gamma| < 0$, $-|\beta| + |\gamma| \leq \alpha < |\beta| + |\gamma|$.
2.3 $\alpha.\gamma > 0$, $|\beta| - |\gamma| = 0$.
2.4 $\gamma < 0$, $|\beta| - |\gamma| > 0$, $-|\beta| - |\gamma| < \alpha < |\gamma| - |\beta|$.
2.5 $\gamma < 0$, $|\beta| - |\gamma| < 0$, $-|\beta| - |\gamma| < \alpha \leq |\beta| - |\gamma|$.

Type 3: $D(g_i) = \cup_{k\in\mathbb{Z}}\{[k.\frac{2\pi}{\mu}, a_0+k.\frac{2\pi}{\mu}]\cup[a_1+k.\frac{2\pi}{\mu}, (1+k)\frac{2\pi}{\mu}]\}, a_0 \in (0, \frac{\pi}{\mu}), a_1 \in (\frac{\pi}{\mu}, \frac{2\pi}{\mu})$, $cos(\mu a_0) = cos(\mu a_1) = a$, where we have $a = \frac{-\alpha - |\beta|}{|\gamma|}$, if $\gamma > 0$ or $a = \frac{\alpha - |\beta|}{|\gamma|}$, if $\gamma < 0$. This case is true for the parameters:

3.1 $\gamma > 0$, $|\beta| - |\gamma| > 0$, $-|\beta| - |\gamma| < \alpha < |\gamma| - |\beta|$.
3.2 $\gamma > 0$, $|\beta| - |\gamma| < 0$, $-|\beta| - |\gamma| < \alpha \leq |\beta| - |\gamma|$.
3.3 $\alpha.\gamma < 0$, $|\beta| - |\gamma| = 0$.
3.4 $\gamma < 0$, $|\beta| - |\gamma| > 0$, $|\beta| - |\gamma| < \alpha < |\beta| + |\gamma|$.
3.5 $\gamma < 0$, $|\beta| - |\gamma| < 0$, $|\gamma| - |\beta| \leq \alpha < |\beta| + |\gamma|$.

Type 4: $D(g_i) = \cup_{k\in\mathbb{Z}}\{[b_0 + k.\frac{2\pi}{\mu}, a_0 + k.\frac{2\pi}{\mu}] \cup [a_1 + k.\frac{2\pi}{\mu}, b_1 + k.\frac{2\pi}{\mu}]\}$, a_0, a_1, b_0, b_1 as we defined before. This case is hold for the parameters:

4.1 $\gamma > 0$, $|\beta| - |\gamma| < 0$, $|\beta| - |\gamma| < \alpha < -|\beta| + |\gamma|$.
4.2 $\gamma < 0$, $|\beta| - |\gamma| < 0$, $|\beta| - |\gamma| < \alpha < -|\beta| + |\gamma|$.

Let us go back to the question of the existence of real zeros for the functions F defined before. So, we have studied that $D\ (g_i) = \mathbb{R}$ or $D(g_i)$ is a union of disjoint closed intervals.

Observe that the functions g_i are periodic with period $2\pi/\mu$ and are continuous, then they are bounded on $D\ (g_i) \cap [\,0,\ 2\pi/\mu]$. So, there exists $k \in \mathbb{Z}^+$ so that the intersection points of g_i with $f(y) = y,\ y \geq 0$ belong to the interval $[\,0,\ k.\frac{2\pi}{\mu}]$.

We must prove that there exists only a finite number of intersection points between the functions g_i and $f(y) = y$ on the interval $[\,0, k.2\pi/\mu]$, which implies that $F(y),\ y \geq 0$ has only a finite number of real roots.

Note that the functions $g_i(y),\ i = 1, 2$ are differentiable in every real y such that $cos(\mu y) \neq \frac{-\alpha + \beta}{\gamma}$ and $cos(\mu y) \neq \frac{-\alpha - \beta}{\gamma}$.

Therefore, we only need to prove that there are a finite number of intersection points y between the functions g_i, $i = 1, 2$ and f in $[0, k.2\pi/\mu]$ such that g_i are differentiable in y.

If the functions $g_i(y)$ are differentiable in y, then

$$g_i'(y) = -\gamma\mu\cos\mu y \pm \frac{\gamma\mu\sin\mu y\,(\gamma\cos\mu y + \alpha)}{\sqrt{\beta^2 - (\gamma\cos\mu y + \alpha)^2}},\quad i = 1, 2.$$

Observe that if there exists an infinity number of intersection points between the functions $g_i(y)$, $i = 1, 2$ and $f(y)$ on $[\,0, k.2\pi/\mu]$, then there will be an infinity number of points y such that $g_i'(y) = 1$ on $[\,0, k.2\pi/\mu]$. Therefore, we search for points y such that $g_i'(y) = 1$.

Let $\gamma \cos \mu y = x$. By the expression of the derivative of g_i, we have that

$$(1+\mu x)^2 = \frac{\gamma^2 \mu^2 \sin^2 \mu y (x+\alpha)^2}{\beta^2 - (x+\alpha)^2} \Longrightarrow (1+\mu x)^2 = \frac{\mu^2(\gamma^2 - x^2)(x+\alpha)^2}{\beta^2 - (x+\alpha)^2}$$

which implies that

$$2\mu x^3 + x^2(-\beta^2\mu^2 + 1 + 4\alpha\mu + \mu^2\gamma^2) + x(-2\mu\beta^2 + 2\alpha + 2\mu\alpha^2 + 2\alpha\mu^2\gamma^2) - \beta^2 + \alpha^2 + \alpha^2\mu^2\gamma^2 = 0.$$

Of course, we obtain at most three real roots x for the above equation. But $\cos \mu y = \frac{x}{\gamma}$, and then there exists at most six real y on $[0, 2\pi/\mu]$ such that $g_i'(y) = 1, \ i = 1, 2$. In this manner, we showed that there is at most $7k$ real intersection points y between $g_i(y)$ and $f(y) = y$ on $[0, 2k\pi/\mu]$.

Hence, we have proved that there exists only a finite number of intersection points between the functions g_i and $f(y) = y$ on the interval $[0, 2k\pi/\mu]$, and therefore, $F(y)$, $y \geq 0$ has only a finite number of real roots. ■

3. MAIN RESULTS

In this section, we present some theorems about stability for our system of differential equations. All these results introduce some sufficient conditions about the parameters α, β, γ for what the equation (5) have not stability changes. The proofs of these theorems is given in the next section.

Using the analysis of the preceding section, we can state the following result:

THEOREM 3.1. *Let us suppose that the parameters α, β, γ satisfy the inequality: $|\alpha| \geq |\beta| + |\gamma|$. Then there is no stability switch for the equation (5).*

Now, we present a result which considers the parameter $|\alpha|$ small.

THEOREM 3.2. *Consider the parameters α, β, γ satisfying $|\alpha| < |\beta| + |\gamma|$ and one of the following conditions:*

(i) $\gamma > 0$, $|\beta| - |\gamma| > 0$, $|\beta| - |\gamma| < \alpha < |\beta| + |\gamma|$.
(ii) $\gamma > 0$, $|\beta| - |\gamma| < 0$, $|\beta| - |\gamma| < \alpha < -|\beta| + |\gamma|$.
(iii) $\gamma < 0$, $|\beta| - |\gamma| < 0$, $|\beta| - |\gamma| < \alpha < -|\beta| + |\gamma|$.
(iv) $\gamma > 0$, $|\beta| - |\gamma| < 0$, $-|\beta| + |\gamma| \leq \alpha < |\beta| + |\gamma|$.
(v) $\gamma < 0$, $|\beta| - |\gamma| > 0$, $-|\beta| - |\gamma| < \alpha < |\gamma| - |\beta|$.
(vi) $\gamma < 0$, $|\beta| - |\gamma| < 0$, $-|\beta| - |\gamma| < \alpha \leq |\beta| - |\gamma|$.
(vii) $\alpha.\gamma > 0$, $|\beta| - |\gamma| = 0$.

Define $b_0 \in (0, \frac{\pi}{\mu})$ such that $\cos(\mu b_0) = b$, where $b = \frac{-\alpha + |\beta|}{|\gamma|}$, if $\gamma > 0$ or $b = \frac{\alpha + |\beta|}{|\gamma|}$, if $\gamma < 0$. If $m = \sqrt{(|\beta| + |\gamma|)^2 - \alpha^2} < b_0$, there is no stability switch for the equation (5).

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3.1. Using the analysis that we have done in the proof of the Proposition 2.1, it is easy to see that the result is true for the parameters α, β, γ satisfying the inequality: $|\alpha| > |\beta| + |\gamma|$.

First of all, let us prove the theorem when α, β, γ satisfy $\beta > 0$, $\gamma > 0$, $\alpha = \beta + \gamma$.

Observe that, in this case, we can prove that $\dfrac{-\gamma - 2\beta}{\gamma} < -1$ which implies that $\gamma \cos \mu y + \alpha + \beta > 0$. Therefore, $\Delta = -4\gamma(\cos \mu y + 1)(\gamma \cos \mu y + \alpha + \beta) \leq 0$ and we have

$$\Delta = 0 \Leftrightarrow \cos \mu y + 1 = 0.$$

If y satisfies $\cos \mu y = -1$, it is easy to see that $F(y) \neq 0$. Therefore, with the preceding studies, we conclude that the function $F(y)$ defined by (7) has not real zeros.

Now, if the parameters α, β, γ is such that $\beta > 0, \gamma > 0$ and $\alpha = -\beta - \gamma$, we get that $\gamma \cos \mu y + \alpha - \beta < 0$ and $\gamma \cos \mu y + \alpha + \beta \leq 0$, that is,

$$\Delta \leq 0 \text{ and } \Delta = 0 \Leftrightarrow \cos \mu y = 1.$$

We can prove that if y satisfies this last expression, then y is not root of the function F, that is, this function has not real zeros.

Finally, the other cases follow in analogous ways. ■

Now, we consider the cases in which the parameters α, β, γ satisfy $|\alpha| < |\beta| + |\gamma|$, that is, when $F(y)$ may have real roots and stability switches for the equation (5) can occur.

Proof of Theorem 3.2. First of all, we analyse the maximum and minimum points of the functions $g_i(y)$, $i = 1, 2$ in the interval $[0, 2\pi/\mu]$. To do this, we study all their critical points.

Remember that the functions $g_i(y)$ was defined in Section 2 by:

$$g_1(y) = -\gamma \sin \mu y + \sqrt{\beta^2 - (\gamma \cos \mu y + \alpha)^2}$$

and

$$g_2(y) = -\gamma \sin \mu y - \sqrt{\beta^2 - (\gamma \cos \mu y + \alpha)^2}.$$

Suppose that $-1 \leq \dfrac{\beta - \alpha}{\gamma} \leq 1$. Then, there is $y_0 \in [0, 2\pi/\mu]$ such that $\cos(\mu y_0) = \dfrac{\beta - \alpha}{\gamma}$, that is, $g_i(y)$ is not differentiable in $y_0, \forall i = 1, 2$.

As we have $g_i(y_0) = -\gamma \sin(\mu y_0)$, thus

$$g_i(y_0) = \sqrt{\gamma^2 - (\beta - \alpha)^2} \quad \text{or} \quad g_i(y_0) = -\sqrt{\gamma^2 - (\beta - \alpha)^2}, \quad i = 1, 2.$$

In an analogous way, if α, β, γ satisfy $-1 \leq \dfrac{-\alpha-\beta}{\gamma} \leq 1$, we take $y_1 \in [0, 2\pi/\mu]$ such that $\cos(\mu y_1) = \dfrac{-\alpha-\beta}{\gamma}$, that is, $g_i(y)$ is not differentiable in $y_1, \forall i = 1, 2$, and

$$g_i(y_1) = \sqrt{\gamma^2 - (\alpha+\beta)^2} \quad \text{or} \quad g_i(y_1) = -\sqrt{\gamma^2 - (\alpha+\beta)^2}, \quad i = 1, 2.$$

Now, let $y \in D(g_1)$ (or $D(g_2)$) such that the functions g_1 (or g_2) is differentiable in this point and $g_1^{'}(y) = 0$ (or $g_2^{'}(y) = 0$). Then, we have:

$$\gamma\mu \cos \mu y = \pm \frac{\gamma\mu \sin \mu y(\alpha + \gamma \cos \mu y)}{\sqrt{\beta^2 - (\alpha + \gamma \cos \mu y)^2}}$$

which implies that

$$\cos \mu y = \pm \frac{\sin \mu y(\alpha + \gamma \cos \mu y)}{\sqrt{\beta^2 - (\alpha + \gamma \cos \mu y)^2}}.$$

Therefore, if y is such that $g_1^{'}(y) = 0$ or $g_2^{'}(y) = 0$, it must satisfy:

$$\beta \cos(\mu y) = \alpha + \gamma \cos(\mu y) \quad \text{or} \quad \beta \cos(\mu y) = -(\alpha + \gamma \cos(\mu y)).$$

In this way, if $\beta = \gamma$ and we take y such that $\cos(\mu y) = \dfrac{-\alpha}{2\beta}$ and $\sin(\mu y) = -\dfrac{\sqrt{4\beta^2 - \alpha^2}}{2\beta}$, then we obtain $g_1'(y) = 0$ and $g_1(y) = \sqrt{4\beta^2 - \alpha^2}$. And more, if we get y such that $\cos(\mu y) = \dfrac{-\alpha}{2\beta}$ and $\sin(\mu y) = \dfrac{\sqrt{4\beta^2 - \alpha^2}}{2\beta}$, we get $g_2'(y) = 0$ and $g_2(y) = -\sqrt{4\beta^2 - \alpha^2}$.

In the same case, when $\beta = -\gamma$, we get y such that $\cos(\mu y) = \dfrac{\alpha}{2\beta}$ and $\sin(\mu y) = \dfrac{\sqrt{4\beta^2 - \alpha^2}}{2\beta}$, then we have $g_1'(y) = 0$ and $g_1(y) = \sqrt{4\beta^2 - \alpha^2}$.

Furthermore, if we take a real number y such that $\cos(\mu y) = \dfrac{\alpha}{2\beta}$ and $\sin(\mu y) = -\dfrac{\sqrt{4\beta^2 - \alpha^2}}{2\beta}$, we obtain $g_2'(y) = 0$ and $g_2(y) = -\sqrt{4\beta^2 - \alpha^2}$.

Now, suppose that $\beta \neq \gamma$ or $\beta \neq -\gamma$, then, there is a real number y satisfying: $\cos \mu y = \dfrac{\alpha}{\beta - \gamma}$ or $\cos \mu y = \dfrac{-\alpha}{\beta + \gamma}$.

Let y be so that $\cos \mu y = \dfrac{\alpha}{\beta - \gamma}$, therefore $\sin \mu y = \pm \dfrac{\sqrt{(\beta-\gamma)^2 - \alpha^2}}{|\beta - \gamma|}$. We note that

$$\begin{aligned} {g'}^1_2(y) &= -\gamma\mu \cos \mu y \pm \frac{\gamma\mu \sin \mu y(\alpha + \gamma \cos \mu y)}{\sqrt{\beta^2 - (\alpha + \gamma \cos \mu y)^2}} = -\gamma\mu \cos \mu y \pm \frac{\gamma\mu\beta \cos \mu y \sin \mu y}{\sqrt{\beta^2(1 - \cos^2 \mu y)}} \\ &= -\gamma\mu \cos \mu y \pm \frac{\gamma\mu\beta \cos \mu y \sin \mu y}{|\beta| . |\sin \mu y|} = -\gamma\mu \cos \mu y\, (1 \mp \frac{\beta \sin \mu y}{|\beta \sin \mu y|}). \end{aligned}$$

So, if $\beta > 0$, we have that $g_1'(y) = 0$ implies that $\sin \mu y > 0$ and thus $\sin \mu y = \dfrac{\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$ and $g_2'(y) = 0$ follows that $\sin \mu y < 0$ and so $\sin \mu y = -\dfrac{\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$.

If $\beta < 0$, we have that $g_1'(y) = 0$ implies that $\sin \mu y = -\dfrac{\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$, and $g_2'(y) = 0$ follows that $\sin \mu y = \dfrac{\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$.

In an analogous manner, if we take the other case, that is, $\cos \mu y = -\dfrac{\alpha}{\beta+\gamma}$, we obtain the following results:

If $\beta > 0,$ we have that $g_1'(y) = 0$ implies that $\sin \mu y = -\dfrac{\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$ and $g_2'(y) = 0$ follows that $\sin \mu y = \dfrac{\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$.

If $\beta < 0,$ we have that $g_1'(y) = 0$ implies that $\sin \mu y = \dfrac{\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$ and $g_2'(y) = 0$ follows that $\sin \mu y = -\dfrac{\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$.

Let us summarize the results above.

(i) Suppose $\beta > 0$, y a critical point of g_1 such that:

(a) $\cos \mu y = \dfrac{\alpha}{\beta-\gamma}$ and $\sin \mu y = \dfrac{\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$, then

$$g_1(y) = \operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}.$$

(b) $\cos \mu y = \dfrac{-\alpha}{\beta+\gamma}$ and $\sin \mu y = -\dfrac{\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$, then

$$g_1(y) = \operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}.$$

(ii) Suppose $\beta < 0$, y a critical point of g_1 such that:

(c) $\cos \mu y = \dfrac{\alpha}{\beta-\gamma}$ and $\sin \mu y = -\dfrac{\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$, then

$$g_1(y) = -\operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}.$$

(d) $\cos \mu y = \dfrac{-\alpha}{\beta+\gamma}$ and $\sin \mu y = \dfrac{\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$, then

$$g_1(y) = -\operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}.$$

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(iii) Suppose $\beta > 0$, y a critical point of g_2 such that:

(e) $\cos\mu y = \dfrac{\alpha}{\beta-\gamma}$ and $\sin\mu y = \dfrac{-\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$, then

$$g_2(y) = -\operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}.$$

(f) $\cos\mu y = \dfrac{-\alpha}{\beta+\gamma}$ and $\sin\mu y = \dfrac{\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$, then

$$g_2(y) = -\operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}.$$

(iv) Suppose $\beta < 0$, y a critical point of g_2 such that:

(g) $\cos\mu y = \dfrac{\alpha}{\beta-\gamma}$ and $\sin\mu y = \dfrac{\sqrt{(\beta-\gamma)^2-\alpha^2}}{|\beta-\gamma|}$, then

$$g_2(y) = \operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}.$$

(h) $\cos\mu y = \dfrac{-\alpha}{\beta+\gamma}$ and $\sin\mu y = \dfrac{-\sqrt{(\beta+\gamma)^2-\alpha^2}}{|\beta+\gamma|}$, then

$$g_2(y) = \operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}.$$

As a consequence of this analysis, we obtain:
(i) If $\beta > 0$, $\cos\mu y = \dfrac{\alpha}{\beta-\gamma}$, we conclude that

$$g_1(y) = \operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}, \text{ and } g_2(y) = -\operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2},$$

(ii) If $\beta > 0$, $\cos\mu y = -\dfrac{\alpha}{\beta+\gamma}$, we conclude that

$$g_1(y) = \operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}, \text{ and } g_2(y) = -\operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2},$$

(iii) If $\beta < 0$, $\cos\mu y = \dfrac{\alpha}{\beta-\gamma}$, we conclude that

$$g_1(y) = -\operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}, \text{ and } g_2(y) = \operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2},$$

(iv) If $\beta < 0$, $\cos\mu y = -\dfrac{\alpha}{\beta+\gamma}$, we conclude that

$$g_1(y) = -\operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}, \text{ and } g_2(y) = \operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}.$$

Suppose that y_i is a critical point of g_i, $i = 1, 2$. Then, we have that:

⋄ if $\beta > 0$, $g_1(y_1)$ satisfy

$$g_1(y_1) = \operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2} \text{ or } g_1(y_1) = \operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}$$

and for $g_2(y_2)$, we have

$$g_2(y_2) = -\operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2} \text{ or } g_2(y_2) = -\operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}$$

⋄ if $\beta < 0$, $g_1(y_1)$ satisfy

$$g_1(y_1) = -\operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2} \text{ or } g_1(y_1) = -\operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2}$$

and for $g_2(y_2)$, we have

$$g_2(y_2) = \operatorname{sgn}(\beta-\gamma)\sqrt{(\beta-\gamma)^2-\alpha^2} \text{ or } g_2(y_2) = \operatorname{sgn}(\beta+\gamma)\sqrt{(\beta+\gamma)^2-\alpha^2}.$$

Now, using this study about the maximum and minimum points of the functions $g_i(y)$, $i = 1, 2$, in the interval $[0, 2\pi/\mu]$, and, of course, in $\mathbb{R}$, we present some cases that guarantee the non-existence of intersection points between the functions g_i and $f(y) = y$.

Now, let us prove the case (i) for the theorem. As we saw in the proof of the Proposition 2.1, in this case, we have that $D(g_i)$ is Type 2, that is, $D(g_i) = \cup_{k\in\mathbb{Z}}[b_0 + k.\frac{2\pi}{\mu}, b_1 + k.\frac{2\pi}{\mu}]$, with $b_0 \in (0, \frac{\pi}{\mu}), b_1 \in (\frac{\pi}{\mu}, \frac{2\pi}{\mu})$, satisfying $cos(\mu b_0) = cos(\mu b_1) = \frac{-\alpha + |\beta|}{\gamma}$.

We can see that the functions g_i are not differentiable in b_0 and b_1, that is, these numbers are critical points to the g_i and they assume the values $g_i(b_0) = -g_i(b_1) = -\sqrt{\gamma^2 - (|\beta| - \alpha)^2}, i = 1, 2$.

Now, we find the others critical points of $g_i, i = 1, 2$.

We have that $\frac{\alpha}{|\beta| - \gamma} > 1$, thus there is not y such that $\cos(\mu y) = \frac{\alpha}{|\beta| - \gamma}$. We also get $-1 < \frac{-\alpha}{|\beta| + \gamma} < 1$, therefore there exists $y_1 \in \mathbb{R}$ satisfying $\cos(\mu y_1) = \frac{-\alpha}{|\beta| + \gamma}$, thus, y_1 is a critical point of the function g_i and $g_1(y_1) = -g_2(y_1) = \sqrt{(|\beta| + \gamma)^2 - \alpha^2}$.

Therefore, using the analysis that we have done before, the maximum and minimum values of the functions $g_i, i = 1, 2$ should be in the set

$$\{\pm\sqrt{\gamma^2 - (|\beta| - \alpha)^2}, \pm\sqrt{(|\beta| + \gamma)^2 - \alpha^2}\}.$$

It is easy to see that $\sqrt{\gamma^2 - (|\beta| - \alpha)^2} < \sqrt{(|\beta| + \gamma)^2 - \alpha^2}$. Thus, if we take

$$m = \sqrt{(|\beta| + \gamma)^2 - \alpha^2} < b_0,$$

there is no intersection between the functions g_i, $i = 1$, 2 and $f(y) = y$ in $\mathbb{R}$. To illustrate this study, we present an example in the Figure 1.

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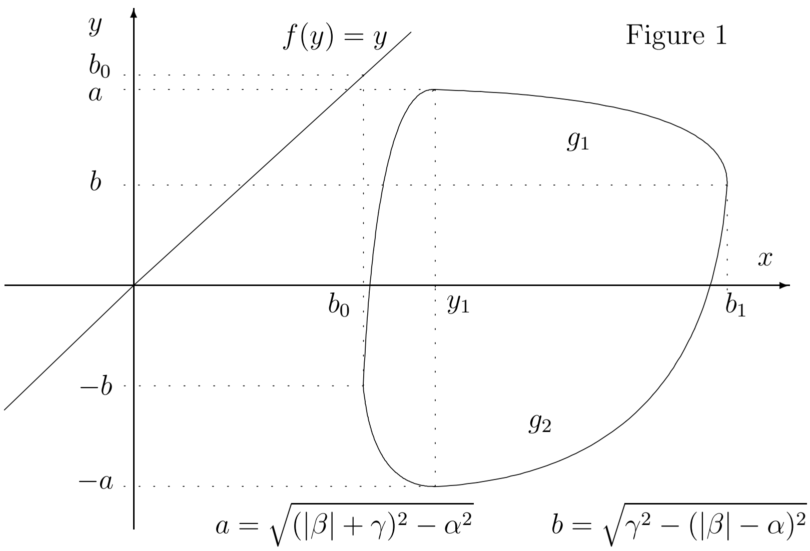


Figure 1

Thus, for this case, we proved that the function $F(y)$ defined by (7) does not have real roots. Now, we prove the theorem for the case (ii). For the other cases, the proofs are analogous.

We use the analysis about the maximum and minimum points for the functions $g_i(y)$, $i = 1, 2$, in $\mathbb{R}$ that was done for the case (i) above. It is proved that, in this case, there is no intersection points between the functions g_i and $f(y) = y$.

Observe that, we can use the proof of the Proposition 2.1 to see that, in this case, $D(g_i)$ is Type 4 and it is given by:

$$D(g_i) = \cup_{k \in \mathbb{Z}} \{[b_0 + k.\frac{2\pi}{\mu}, a_0 + k.\frac{2\pi}{\mu}] \cup [a_1 + k.\frac{2\pi}{\mu}, b_1 + k.\frac{2\pi}{\mu}]\},$$

where a_0, a_1, b_0, b_1 as we defined in this proposition.

We can see that the functions g_i are not differentiable in a_0, a_1, b_0 and b_1, that is, these real numbers are critical points to the g_i and

$$g_i(a_0) = -g_i(a_1) = -\sqrt{\gamma^2 - (\alpha + |\beta|)^2},$$

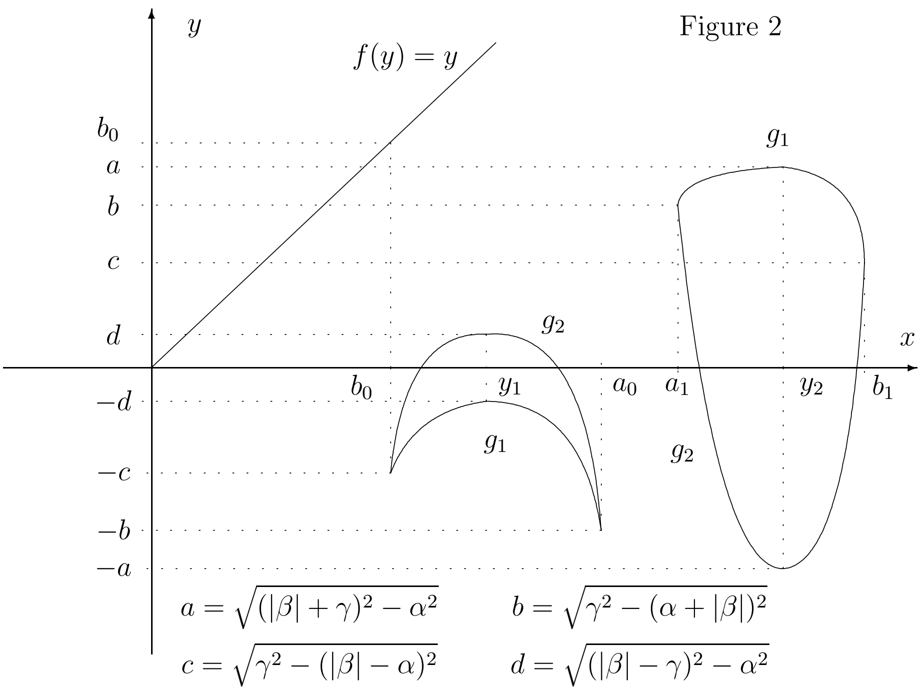
$$g_i(b_0) = -g_i(b_1) = -\sqrt{\gamma^2 - (|\beta| - \alpha)^2},\ i = 1, 2.$$

Now, we find the others critical points of $g_i, i = 1, 2$. We have that $-1 < \frac{\alpha}{|\beta| - \gamma} < 1$, thus there is $y_1 \in (0, \frac{\pi}{\mu})$ such that $\cos(\mu y_1) = \frac{\alpha}{|\beta| - \gamma}$ and $g_1(y_1) = -g_2(y_1) = -\sqrt{(|\beta| - \gamma)^2 - \alpha^2}$.

We also have $-1 < \dfrac{-\alpha}{|\beta|+\gamma} < 1$, therefore there exists $y_2 \in (\dfrac{\pi}{\mu}, \dfrac{2\pi}{\mu})$ satisfying $\cos(\mu y_2) = \dfrac{-\alpha}{|\beta|+\gamma}$, and $g_1(y_2) = -g_2(y_2) = \sqrt{(|\beta|+\gamma)^2 - \alpha^2}$. Therefore, using the analysis that we have done before, the maximum and minimum values of the functions $g_i, i = 1,2$ should be between the values

$$\pm\sqrt{\gamma^2 - (\alpha + |\beta|)^2}, \pm\sqrt{\gamma^2 - (|\beta| - \alpha)^2}, \pm\sqrt{(|\beta|+\gamma)^2 - \alpha^2} \text{ or } \pm\sqrt{(|\beta|-\gamma)^2 - \alpha^2}.$$

It can be shown that if we take $\sqrt{(|\beta|+\gamma)^2 - \alpha^2} < b_0$, all the values described above are less than b_0, thus, there is no intersection between the functions $g_i,\ i = 1,2$ and $f(y) = y$ in $\mathbb{R}$. We illustrate all this study with an example in the Figure 2.



Finally, we have proved that the function $F(y)$ defined by (7) has not real roots. ■

Remark 4. 1. With the same ideas that we have used in the proof of this theorem, one can find other parameters under which no stability changes occurs.

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5. APPLICATION

As an application of the results obtained in the previous sections, we analyse the equation:

$$\dot{x}(t) = \alpha \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} x(t) + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t-\tau) + \gamma \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} x(t-\mu) \tag{8}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$, τ, μ positive real numbers. We denote

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

Of course, A has a unique eigenvalue given by $\lambda_1 = \lambda_2 = 3$ and the correspondent eigenvectors are $v = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$, with v_1, v_2 real numbers.

Observe that the eigenvalues of B are: $\lambda_1 = 1$ and $\lambda_2 = -1$ and the correspondent eigenvectors to λ_1 are $v = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix}$ and the correspondent eigenvectors to λ_2 are $v = \begin{pmatrix} v_2 \\ -v_2 \end{pmatrix}$ v_1, v_2 real numbers.

The eigenvalues of C are: $\lambda_1 = 2$ and $\lambda_2 = 1$ and the correspondent eigenvectors to λ_1 are $v = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix}$ and the correspondent eigenvectors to λ_2 are $v = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$,$v_1, v_2$ real numbers.

So, if we consider the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$, we have that the triangular matrices A_1, B_1, C_1, are:

$$A_1 = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 5. 1. Observe that this example illustrates the fact that a family of linear operators does not need be commutative to guarantee the existence of a basis so that each operator is represented by a triangular matrix.

Using the notation of the Section 2, the characteristic equation associated to (8) is:

$$p(s) = \Pi_{i=1}^{2}(s - \alpha\alpha_i - \beta\beta_i\, e^{-\tau\, s} - \gamma\gamma_i\, e^{-\mu\, s}) = 0.$$

In our case, $\alpha_1 = \alpha_2 = 3$; $\beta_1 = -1$; $\beta_2 = 1$; $\gamma_1 = 2$; $\gamma_2 = 1$. Therefore, the characteristic equation associated to equation (8) is:

$$p(s) = (s - 3\,\alpha + \beta e^{-\tau\, s} - 2\,\gamma\, e^{-\mu\, s})(s - 3\,\alpha - \beta\, e^{-\tau\, s} - \gamma\, e^{-\mu\, s}) = 0.$$

So, we obtain

$$s - 3\,\alpha + \beta e^{-\tau\, s} - 2\,\gamma\, e^{-\mu\, s} = 0 \tag{9}$$

or

$$s - 3\,\alpha - \beta\, e^{-\tau\, s} - \gamma\, e^{-\mu\, s} = 0. \tag{10}$$

Observe that the equation (9) takes the form $P(s) + Q(s)\, e^{-\tau\, s} = 0$, if $P(s) = s - 3\,\alpha - 2\,\gamma\, e^{-\mu\, s}$ and $Q(s) = \beta$.

It follows from Theorem 3.1 that if the parameters α, β, γ satisfy $\alpha \geq \dfrac{|\beta| + 2|\gamma|}{3}$ or $\alpha \leq \dfrac{-|\beta| - 2|\gamma|}{3}$, there will be no stability switch for the equation

$$\dot{x_1}(t) = 3\,\alpha\, x_1(t) - \beta\, x_1(t - \tau) + 2\,\gamma\, x_1(t - \mu) \tag{11}$$

as τ varies.

Note that the equation (10) assumes the form $P(s) + Q(s)\, e^{-\tau\, s} = 0$, with $P(s) = s - 3\alpha - \gamma e^{-\mu\, s}$ and $Q(s) = -\beta$. Besides, by Theorem 3.1, if $\alpha \geq \dfrac{|\beta| + |\gamma|}{3}$ or $\alpha \leq \dfrac{-|\beta| - |\gamma|}{3}$, there is no stability switch for the equation

$$\dot{x_2}(t) = 3\,\alpha\, x_2(t) + \beta\, x_2(t - \tau) + \gamma\, x_2(t - \mu) \tag{12}$$

as τ varies.

Therefore, the conclusion is: if $\alpha \geq \dfrac{|\beta| + 2|\gamma|}{3}$ or $\alpha \leq \dfrac{-|\beta| - 2|\gamma|}{3}$, there is no stability switch for the equation (8) as τ varies.

Now, we take the parameters $\alpha = \dfrac{7}{10}$, $\beta = 3$ and $\gamma = 1$. Of course, these parameters satisfy the condition (i) of Theorem 3.2 and if we have $m = \sqrt{(|\beta| + 2\gamma)^2 - 9\alpha^2} \simeq 4.5 < b_0$, there is no stability switch for the equation (11). For example, we can have $\mu < 0.68$, and there is no stability switch for this equation.

Proceeding in the same manner by taking $b_0 > 3.4$, there is no stability switch for the equation (12). That is, if $\mu < 0.92$, change of stability for this equation does not occur.

So, the conclusion is: if $\mu < 0.68$, there is no stability switches for the system

$$\dot{x}(t) = \frac{7}{10}\, A\, x(t) + 3\, B\, x(t - \tau) + C\, x(t - \mu).$$

If $\mu > 0.68$, changes of stability can be occur.

Consider again the equation (8). According to the above conclusions, if $\alpha \geq \dfrac{|\beta| + 2|\gamma|}{3}$ or $\alpha \leq \dfrac{-|\beta| - 2|\gamma|}{3}$, there is no stability switch for (8) as τ varies.

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We let now take $\tau = 0$ in the equation (8) and analyse the stability of a system with one delay μ.

$$\dot{x}(t) = D\,x(t) + \gamma\,C\,x(t-\mu), \tag{13}$$

where $D = \begin{pmatrix} 4\,\alpha & \alpha+\beta \\ -\alpha+\beta & 2\,\alpha \end{pmatrix}$.

By taking the basis $\mathcal{B} = \Big\{(1,-1),\,(0,1)\Big\}$, we have that

$$(D)_B = \begin{pmatrix} 3\alpha-\beta & \alpha+\beta \\ 0 & \beta+3\alpha \end{pmatrix} \text{ and } (C)_B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this manner, the characteristic equation associated to (13) is:

$$p(s) = \left| s\,I - \begin{matrix} 3\alpha-\beta & \alpha+\beta \\ 0 & \beta+3\alpha \end{matrix} - \gamma \begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\; e^{-\mu s} \right| = (s-(3\alpha-\beta)-2\gamma e^{-\mu s})(s-(\beta+3\alpha)-\gamma e^{-\mu s}).$$

Now, we study two cases.

Case 1. Consider $s-(3\alpha-\beta)-2\gamma e^{-\mu s} = 0$, and define $P(s) = s-(3\alpha-\beta)$ and $Q(s) = -2\gamma$. By Theorem 2.1, we need to analyse the function

$$\begin{aligned} F(s) &= |P(iy)|^2 - |Q(iy)|^2 \\ &= |iy-(3\alpha-\beta)|^2 - |-2\gamma|^2 = y^2 + (3\alpha-\beta)^2 - 4\gamma^2. \end{aligned}$$

Case 1.a: Suppose that $2|\gamma| \le 3\alpha - |\beta|$. In this case,

$$2|\gamma| \le 3\alpha - |\beta| < 3\alpha - \beta \le |3\alpha-\beta|, \text{ if } \beta \neq 0.$$

Case 1.b: Suppose $2|\gamma| \le -3\alpha - |\beta|$. If $\beta > 0$, it follows that

$$2|\gamma| \le -3\alpha - \beta < -3\alpha + \beta \le |-3\alpha+\beta| = |3\alpha-\beta|.$$

When $\beta < 0$, using the fact that $\gamma \neq 0$, it follows that

$$2|\gamma| \le -3\alpha + \beta < |3\alpha - \beta|.$$

Anyway, if $\beta \neq 0$, $2|\gamma| < |3\alpha-\beta|$ implies that $(3\alpha-\beta)^2 - 4\gamma^2 > 0$. Then, $F(y) = y^2 + (3\alpha-\beta)^2 - 4\gamma^2$ has no positive roots. So, as τ varies there is no stability switch.

Case 2. Assume that $s-(\beta+3\alpha)-\gamma e^{-\mu s}=0$. In this case, we take $P(s)=s-(\beta+3\alpha)$ and $Q(s)=-\gamma$. So,

$$F(y)=|P(iy)|^2-|Q(iy)|^2=|iy-(3\alpha+\beta)|^2-\gamma^2=y^2+(3\alpha+\beta)^2-\gamma^2.$$

Case 2.a: Suppose that $2|\gamma|\leq 3\alpha-|\beta|$. Then,

$$|\gamma|\leq 2|\gamma|\leq 3\alpha-|\beta|<3\alpha+\beta\leq|3\alpha+\beta|,\ \text{with}\ \beta\neq 0.$$

Case 2.b: Suppose $2|\gamma|\leq -3\alpha-|\beta|$. Then if $\beta>0$, we have that

$$|\gamma|\leq 2|\gamma|\leq -3\alpha+\beta<-3\alpha-\beta\leq|3\alpha+\beta|.$$

If $\beta<0$, we obtain that

$$|\gamma|<2|\gamma|\leq -3\alpha-\beta\leq|3\alpha+\beta|.$$

Anyway, it follows that$|\gamma|<|3\alpha+\beta|$ implies that $(3\alpha+\beta)^2-\gamma^2>0$, and then $F(y)$ has no positive roots. So, there is no stability switch as μ varies.

Now, we analyse the stability for $\mu=0$, that is, the system is:

$$\dot{x}(t)=E\,x(t), \tag{14}$$

where $E=\begin{pmatrix} 3\alpha-\beta+2\gamma & \alpha+\beta \\ 0 & \beta+3\alpha+\gamma \end{pmatrix}$. Of course, the eigenvalues of the matrix E are

$$\lambda_1=3\alpha-\beta+2\gamma \text{ and } \lambda_2=\beta+3\alpha+\gamma.$$

We will consider the cases:

Case 1. Suppose $3\alpha-|\beta|\geq 2|\gamma|$.
Case 1.a: If $\beta>0$, $3\alpha-\beta\geq 2|\gamma|$.

Case 1.a.i: If $\gamma>0$, $\lambda_2=3\alpha+\beta+\gamma\geq 3\gamma+2\beta>0$.

Case 1.a.ii: If $\gamma<0$, then $\lambda_2=3\alpha+\beta+\gamma\geq 2\beta-\gamma>0$.

Case 1.b: If $\beta<0$, $3\alpha+\beta\geq 2|\gamma|$.

Case 1.b.i: If $\gamma>0$, $\lambda_2=3\alpha+\beta+\gamma\geq 3\gamma>0$.

Case 1.b.ii: If $\gamma<0$, $\lambda_2=3\alpha+\beta+\gamma\geq -\gamma>0$.

Anyway, note that if the parameters α,β,γ satisfy $3\alpha-|\beta|\geq 2|\gamma|$, the equation (14) is unstable.

Case 2. Suppose $3\alpha+|\beta|\leq -2|\gamma|$.

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Case 2.a: If $\beta > 0$, $3\alpha + \beta \leq -2|\gamma|$, $\lambda_1 = 3\alpha - \beta + 2\gamma \leq -2|\gamma| - 2\beta + 2\gamma = 2(\gamma - |\gamma| - \beta)$ and $\lambda_2 = 3\alpha + \beta + \gamma \leq -2|\gamma| + \gamma$.

Case 2.a.i: If $\gamma > 0$, $\lambda_1 \leq 2(\gamma - |\gamma| - \beta) = -2\beta < 0$, and $\lambda_2 \leq -\gamma < 0$.
Case 2.a.ii: If $\gamma < 0$, $\lambda_1 \leq 2(2\gamma - \beta) < 0$, and $\lambda_2 \leq 3\gamma < 0$.

Case 2.b. If $\beta < 0$, $3\alpha - \beta \leq -2|\gamma|$, $\lambda_1 = 3\alpha - \beta + 2\gamma \leq 2(\gamma - |\gamma|)$, and $\lambda_2 = 3\alpha + \beta + \gamma \leq \beta - 2|\gamma| + \beta + \gamma = 2\beta + (\gamma - 2|\gamma|)$.

Case 2.b.i: If $\gamma < 0$, $\lambda_1 \leq 4\gamma < 0$, and $\lambda_2 \leq 2\beta + 3\gamma < 0$.
Case 2.b.ii: If $\gamma > 0$, $\lambda_1 \leq 0$, and $\lambda_2 \leq 2\beta - \gamma < 0$.

Therefore, when the parameters α, β, γ satisfy $3\alpha + |\beta| \leq -2|\gamma|$, the system (14) is stable.

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